OVERRELAXATION APPLIED TO IMPLICIT ALTERNATING DIRECTION METHODS

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§1. Introduction

Let $A = (a_{i,j})$ be a real non-singular nxn matrix with $a_{i,i} \neq 0$, $1 \leq i \leq n$. If we seek to solve the matrix equation

$$(1) \qquad A\underline{x} = \underline{k},$$

then the Young-Frankel [1,2] successive overrelaxation iterative method is defined by

$$(2) \qquad x_i^{(\ell+1)} = x_i^{(\ell)} + \omega \left\{ \sum_{j<i} b_{i,j} x_j^{(\ell+1)} + \sum_{j>i} b_{i,j} x_j^{(\ell)} + g_i - x_i^{(\ell)} \right\},$$

$$1 \leq i \leq n,$$

where the matrix $B = (b_{i,j})$, and the vector $\underline{g}$ are defined by

$$(2') \qquad b_{i,j} = \begin{Bmatrix} -a_{i,j}/a_{i,i} & , & i \neq j \\ 0 & , & i = j \end{Bmatrix}, \ g_i = k_i/a_{i,i}, \ 1 \leq i, j \leq n.$$

For suitable choice of the relaxation parameter ω, Young [1] established the rapid convergence of (2) for a large class of matrix problems satisfying his definition of property (A). It was later shown that property (A) for the matrix A implies that the matrix B of (2') is, in the terminology of Romanovsky [3], cyclic of index 2, and the Young-Frankel successive overrelaxation method was generalized [4] to cases where the matrix B of (2') is cyclic of index $p \geq 2$.

Since the introduction of the Young-Frankel successive overrelaxation method, several variants of a new implicit alternating direction method [6, 7, 8] have appeared, and for the model problem, the numerical solution of the Dirichlet problem for the unit square on a discrete uniform mesh of side h, these new iterative methods are asymptotically an order of magnitude faster than the successive overrelaxation method, as $h \to 0$. More precisely, the rate of convergence of the successive overrelaxation method with optimum relaxation factor is, for the model problem, proportional to h, while the new implicit alternating direction methods, with optimum acceleration parameters, have rates of convergence proportional to $|\ln h|^{-1}$ for the model problem, as $h \to 0$. Unfortunately, these new methods have been demonstrated to be rigorously faster essentially only for the case of the Dirichlet problem for a rectangle [5], since these methods depend upon the ability to split the matrix A into

$$(3) \qquad A = H + V$$

where H and V are symmetric and positive definite, and

$$(4) \qquad HV = VH.$$

Obviously, this implies that A is also symmetric and positive definite. For a large class of matrix equations arising from five-point numerical approximations of self-adjoint elliptic differential equations of the form

$$\Sigma\,(x,y)\;u\,(x,y) - \operatorname{div}\{D\,(x,y)\;\operatorname{grad}\,u\,(x,y)\} \;=\; s\,(x,y), \qquad (5)$$

matrix equations can be derived [5, 8] so that A = H + V, and all matrices are symmetric and positive definite. It is the condition of (4) which is difficult in general to fulfill.

We seek to construct an iterative method, with assumptions weaker than that of (4), whose rate of convergence for the numerical solution of the Dirichlet problem on a uniform mesh of side h in a non-rectangular region is asymptotically $h^{1/2}$ as $h \to 0$. We shall accomplish this by combining the attractive features of both the Peaceman-Rachford iterative method [6], one particular variant of the implicit alternating direction methods, and the successive overrelaxation iterative method as applied to matrices of cyclic index greater than 2.

§2. Cyclic Matrices

Let M be a real rxr matrix. If there exists an rxr permutation matrix Λ such that

$$(5)\qquad \Lambda M \Lambda^{-1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & M_1 \\ M_2 & 0 & 0 & & 0 & 0 \\ 0 & M_3 & 0 & & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & M_p & 0 \end{bmatrix}, \qquad (6)$$

where the diagonal submatrices are square, we say [3, 9] that M is <u>cyclic of index p</u>. Such matrices are the basis for Young's property (A) and its generalizations. In fact, if M is in the block form of (5), it is also <u>consistently ordered</u> in the sense of [4]. If we seek to solve the matrix equation

$$(6)\qquad \underline{u} = M\,\underline{u} + \underline{f}, \qquad (7)$$

then, assuming that M is block form of (5) and that the vectors $\underline{u}$ and $\underline{f}$ are partitioned into vector components U_j, F_j, $1 \leq j \leq p$, according to the partitioning of the block form of (5), the successive overrelaxation iterative method applied to (6) is

$$(7)\qquad \begin{cases} U_1^{(\ell+1)} = U_1^{(\ell)} + \omega\left\{M_1\, U_p^{(\ell)} + F_1 - U_1^{(\ell)}\right\} \\ U_j^{(\ell+1)} = U_j^{(\ell)} + \omega\left\{M_j\, U_{j-1}^{(\ell+1)} + F_j - U_j^{(\ell)}\right\}, \quad 2 \leq j \leq p. \end{cases} \qquad (8)$$

We now consider the Peaceman-Rachford iterative method [6]. With (3), we write (1) as two equations

$$(8)\qquad \begin{cases} (H + \rho I)\,\underline{x} = \underline{k} + (\rho I - V)\,\underline{x} \\ (V + \rho I)\,\underline{x} = \underline{k} + (\rho I - H)\,\underline{x}. \end{cases} \qquad (9)$$

The matrices H and V, in addition to being symmetric and positive definite, are each, after a suitable similarity transformation by a permutation matrix, tridiagonal matrices. The Peaceman-Rachford iterative method is defined by

$$(9)\qquad \begin{cases} (H + \rho_i\, I)\,\underline{x}_i^{*} = \underline{k} + (\rho_i\, I - V)\,\underline{x}_i \\ (V + \rho_i\, I)\,\underline{x}_{i+1} = k + (\rho_i\, I - H)\,\underline{x}_i^{*}, \end{cases} \qquad (10)$$

where the constants ρ_i are positive acceleration factors. The matrices $H + \rho_i\, I$ and $V + \rho_i\, I$ are symmetric and positive definite tridiagonal matrices, and thus the matrix equations of (9) can be efficiently solved by means of the Gauss elimination method [6].

The concept of a cyclic matrix can be profitably applied to the study of the Peaceman-Rachford iterative method. We write the equations of (8) in the block form

$$(10) \qquad \begin{pmatrix} H + \rho I & V - \rho I \\ H - \rho I & V + \rho I \end{pmatrix} \begin{pmatrix} \underline{x} \\ \underline{x} \end{pmatrix} = \begin{pmatrix} \underline{k} \\ \underline{k} \end{pmatrix}, \qquad \text{(11)}$$

where $\left(\frac{\underline{x}}{\underline{x}}\right)$ and $\left(\frac{\underline{k}}{\underline{k}}\right)$ are column vectors with 2n components. Since the square diagonal submatrices $H + \rho I$ and $V + \rho I$ are non-singular, (10) is equivalent to

$$(11) \qquad \begin{pmatrix} \underline{x} \\ \underline{x} \end{pmatrix} = \begin{pmatrix} 0 & (H + \rho I)^{-1} (\rho I - V) \\ (V + \rho I)^{-1} (\rho I - H) & 0 \end{pmatrix} \begin{pmatrix} \underline{x} \\ \underline{x} \end{pmatrix} \qquad \text{(12)}$$

$$+ \begin{pmatrix} (H + \rho I)^{-1} \underline{k} \\ (V + \rho I)^{-1} \underline{k} \end{pmatrix}.$$

If

$$(12) \qquad B_\rho \equiv \begin{pmatrix} 0 & (H + \rho I)^{-1} (\rho I - V) \\ (V + \rho I)^{-1} (\rho I - H) & 0 \end{pmatrix}, \quad \underline{u} \equiv \begin{pmatrix} \underline{x} \\ \underline{x} \end{pmatrix}, \qquad \text{(13)}$$

then (11) is exactly of the form (6), where B_ρ is cyclic of index 2. Applying the Gauss-Seidel iterative method to (6), the special case $\omega = 1$ of (7), to the matrix equation (6) with $B_\rho \equiv M$, we have

$$(13) \qquad \begin{cases} \underline{x}_1^{(\ell+1)} = (H + \rho I)^{-1} (\rho I - V)\, \underline{x}_2^{(\ell)} + (H + \rho I)^{-1} \underline{k} \\ \underline{x}_2^{(\ell+1)} = (V + \rho I)^{-1} (\rho I - H)\, \underline{x}_1^{(\ell+1)} + (V + \rho I)^{-1} \underline{k}, \end{cases}$$

or equivalently

$$(14) \qquad \begin{cases} (H + \rho I)\, \underline{x}_1^{(\ell+1)} = (\rho I - V)\, \underline{x}_2^{(\ell)} + \underline{k} \\ (V + \rho I)\, \underline{x}_2^{(\ell+1)} = (\rho I - H)\, \underline{x}_1^{(\ell+1)} + \underline{k}. \end{cases} \qquad \text{(14')}$$

This, except for notation, is equivalent to (9) for the case in which all the parameters ρ_i are constant. Thus, the Peaceman-Rachford iterative method with parameters ρ_i all constant is just the Gauss-Seidel iterative method applied to a matrix which is cyclic of index 2. Generalizing, if the Peaceman-Rachford iterative scheme uses q parameters ρ_i cyclically, then we consider

$$\underline{u} = B_\rho(q)\,\underline{u} + \underline{g}\,, \tag{15}$$

where

$$B_\rho(q) = \begin{bmatrix} 0 & 0 & \cdots & 0 & (H+\rho_1 I)^{-1}(\rho_1 I - V) \\ (V+\rho_1 I)^{-1}(\rho_1 I - H) & 0 & & 0 & 0 \\ 0 & (H+\rho_2 I)^{-1}(\rho_2 I - V) & & 0 & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & \cdots & (V+\rho_q I)^{-1}(\rho_q I - H) & 0 \end{bmatrix}. \tag{16}$$

Thus, $B_\rho(q)$ is cyclic of index $2q$, and $\underline{u}$ is a vector with $2n\cdot q$ components. It is easily shown that the Gauss-Seidel iterative method applied to (15) is equivalent to the Peaceman-Rachford iterative method with q parameters ρ_i used cyclically. The special case $\rho_1 = \rho_2 = \cdots = \rho_q = \rho$ is admitted in the preceding formulas.

It is interesting to point out that the Gauss-Seidel $\mathcal{L}(\rho)$ matrix derived from (13) is

$$\mathcal{L}(\rho) = \begin{pmatrix} 0 & (H+\rho I)^{-1}(\rho I - V) \\ 0 & (V+\rho I)^{-1}(\rho I - H)(H+\rho I)^{-1}(\rho I - V) \end{pmatrix}, \tag{17}$$

(18)

so that we may write (13) in the form

$$\begin{pmatrix} \underline{x}_1^{(\ell+1)} \\ \underline{x}_2^{(\ell+1)} \end{pmatrix} = \mathcal{L}(\rho) \begin{pmatrix} \underline{x}_1^{(\ell)} \\ \underline{x}_2^{(\ell)} \end{pmatrix} + \begin{pmatrix} \underline{g}_1 \\ \underline{g}_2 \end{pmatrix} \tag{13'}$$

The eigenvalues of $\mathcal{L}(\rho)$ are thus zeros and the eigenvalues of the matrix $T_\rho \equiv (V+\rho I)^{-1}(\rho I - H)(H+\rho I)^{-1}(\rho I - V)$, which we shall call the Peaceman-Rachford matrix.

We seek to obtain rates of convergence of order $h^{1/2}$, for the Dirichlet problem on a uniform mesh, as follows. The matrix $B_\rho(2)$, which depends on but one parameter ρ, is cyclic of index 4 and consistently ordered. By choosing the parameter ρ so as to minimize the spectral radius of $B_\rho(2)$, we then apply the successive overrelaxation theory of [4] to the matrix equation (15), with $B_\rho(2)$ replacing $B_\rho(q)$. This introduces a second parameter ω. The conditions under which successive overrelaxation can be rigorously applied to (15) are weaker than the condition of (4), and includes (4) as a special case.

§3. One Parameter Acceleration Methods

Both the successive overrelaxation iterative method, with relaxation factor ω, and the Peaceman-Rachford iterative method, with a single parameter ρ, can be considered as one parameter iterative methods. It is of interest to compare their rates of convergence. The spectral radius of the Peaceman-Rachford matrix

$$(18) \qquad T_\rho = (V+\rho I)^{-1}(\rho I - H)(H+\rho I)^{-1}(\rho I - V)$$

can be estimated as follows. Since H and V are symmetric and positive definite matrices, let $0 < \alpha_1 \leq x \leq \beta_1$, $0 < \alpha_2 \leq x \leq \beta_2$ be the smallest intervals containing all the eigenvalues of H and V, respectively. We now consider the matrix $\tilde{T}_\rho$ defined by

$$(19) \qquad \tilde{T}_\rho = (V+\rho I)\, T_\rho\, (V+\rho I)^{-1} = (\rho I - H)(H+\rho I)^{-1}(\rho I - V)(V+\rho I)^{-1}.$$

If $\|\underline{x}\| \equiv \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$, for $\rho > 0$, let

$$(20) \qquad \sup_{\|\underline{x}\|=1} \|(\rho I - V)(V+\rho I)^{-1}\underline{x}\| \equiv \Lambda_1(\rho).$$

Since V is symmetric and positive definite, then $\underline{x} = \sum_{k=1}^{n} c_k \underline{\epsilon}_k$, where the $\underline{\epsilon}_k$ form an orthonormal basis for our n-dimensional vector space over the complex numbers, and $V\underline{\epsilon}_k = \tau_k \underline{\epsilon}_k$. Thus

$$(\rho I - V)(V+\rho I)^{-1}\underline{x} = \sum_{k=1}^{n} c_k \left(\frac{\rho - \tau_k}{\rho + \tau_k}\right) \underline{\epsilon}_k .$$

Since $\max_k \left(\frac{\rho - \tau_k}{\rho + \tau_k}\right)^2 \leq \max_{\alpha_2 \leq t \leq \beta_2} \left(\frac{\rho - t}{\rho + t}\right)^2 = \max\left\{\left(\frac{\rho-\alpha_2}{\rho+\alpha_2}\right)^2 ; \left(\frac{\rho - B_2}{\rho+\beta_2}\right)^2\right\}$,

and $\sum_{k=1}^{n} |c_k|^2 = 1$ from $\|\underline{x}\| = 1$, it follows that

$$(21) \qquad \Lambda_1(\rho) \leq \max\left\{ \left|\frac{\rho - \alpha_2}{\rho + \alpha_2}\right| ; \left|\frac{\rho - \beta_2}{\rho + \beta_2}\right| \right\} \equiv \tilde{\Lambda}_1(\rho) < 1.$$

Similarly, if

$$(20') \qquad \sup_{\|\underline{x}\| = 1} \|(\rho I - H)(H + \rho I)^{-1} \underline{x}\| \equiv \Lambda_2(\rho),$$

then

$$(21') \qquad \Lambda_2(\rho) \leq \max\left\{ \left|\frac{\rho - \alpha_1}{\rho + \alpha_1}\right| ; \left|\frac{\rho - \beta_1}{\rho + \beta_1}\right| \right\} \equiv \tilde{\Lambda}_2(\rho) < 1.$$

It follows now that $\|\tilde{T}_\rho \underline{x}\| \leq \Lambda_1(\rho) \cdot \Lambda_2(\rho)$, and since $\tilde{T}_\rho$ and T_ρ have the same eigenvalues,

$$(22) \qquad \bar{\mu}(T_\rho) \leq \Lambda_1(\rho) \cdot \Lambda_2(\rho) \leq \tilde{\Lambda}_1(\rho) \cdot \tilde{\Lambda}_2(\rho) < 1.$$

Thus, $\bar{\mu}(T\ )$, the spectral radius* of the matrix T , is less than unity, showing that T is a convergent matrix [5]. If

$$(23) \qquad \min_{\rho > 0} \tilde{\Lambda}_1(\rho) \cdot \tilde{\Lambda}_2(\rho) = \tilde{\Lambda}_1(\rho_0) \cdot \tilde{\Lambda}_2(\rho_0),$$

it can be shown, with $\rho_1 \equiv \sqrt{\alpha_1 \beta_1}$ and $\rho_2 \equiv \sqrt{\alpha_2 \beta_2}$, that

$$(24) \qquad \max\{\rho_1, \rho_2\} \geq \rho_0 \geq \min\{\rho_1, \rho_2\}.$$

For the numerical solution of the Dirichlet problem on the unit square with uniform mesh spacing $h = 1/N$, the five point approximation

$$(25) \qquad 4u_{i,j} - (u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}) = 0 \quad , \quad 1 \leq i,j \leq N-1,$$

to Laplace's equation leads to a system of $(N-1)^2$ linear equations. It can be verified that the spectral radius of the corresponding Jacobi matrix B of (2') is

*If C is an arbitrary nxn complex matrix, then $\bar{\mu}(C) \equiv \max_k |\lambda_k|$, where the λ_k are eigenvalues of C.

$$(26) \qquad \bar{\mu}(B) = \cos(\pi h).$$

Assuming that the matrix A of (1) is consistently ordered, the formula for the optimum relaxation factor ω_b is

$$(27) \qquad \omega_b = \frac{2}{1+\sqrt{1-\bar{\mu}^2(B)}} = \frac{2}{1+\sin(\pi h)}.$$

If $R[\mathcal{L}_\omega] = -\ln \bar{\mu}(\mathcal{L}_\omega)$ is the rate of convergence for the successive overrelaxation method, it is known [1] that $\bar{\mu}(\mathcal{L}_{\omega_b}) = (\omega_b - 1)$, and asymptotically

$$(28) \qquad R[\mathcal{L}_{\omega_b}] = 2\pi h + O(h^2), \quad h \to 0.$$

For this model problem, $\alpha_1 = \alpha_2 = 2[1 - \cos(\pi h)]$, $\beta_1 = \beta_2 = 2[1 + \cos(\pi h)]$, and $\rho_1 = \rho_2 = 2 \sin(\pi h)$. From (24), $\rho_0 = 2 \sin(\pi h)$, and from (22), we derive

$$(29) \qquad \bar{\mu}(T_{\rho_0}) \leq \left[\frac{1 + \cos(\pi h) - \sin(\pi h)}{1 + \cos(\pi h) + \sin(\pi h)}\right]^2 = \omega_b - 1.$$

It can actually be shown that $\min\limits_{\rho} \bar{\mu}(T_\rho) = \bar{\mu}(T_{\rho_0}) = \omega_b - 1$, and thus

$$(30) \qquad \max_{\rho} R[T_\rho] = R[T_{\rho_0}] = R[\mathcal{L}_{\omega_b}] = \max_{\omega} R[\mathcal{L}_\omega].$$

We conclude that, for the model problem, the Peaceman-Rachford iterative method, with a single optimum parameter ρ, is exactly as fast as the successive overrelaxation method with optimum relaxation parameter ω. Thus, from (28), both of these one parameter methods give rates of convergence proportional to h, the mesh spacing, as $h \to 0$.

We now consider the numerical solution of the Dirichlet problem for a closed set of points Ω consisting of the (connected) union of a finite number of unit squares with sides parallel to the coordinate axes in the plane. We denote by $\bar{\Omega}$ the smallest (in area) square consisting of unit squares such that $\bar{\Omega} \supseteq \Omega$. If we impose a uniform mesh of side $h = 1/N$ over Ω and $\bar{\Omega}$, and approximate Laplace's equation as in (25), then the eigenvalue bounds α_i, β_i and $\bar{\alpha}_i$, $\bar{\beta}_i$ for Ω and $\bar{\Omega}$ respectively satisfy

$$(30) \qquad \begin{cases} 0 < \bar{\alpha}_i \leq \alpha_i \leq \beta_i \leq \bar{\beta}_i & , \ i = 1, 2, \\ \sqrt{\bar{\alpha}_i \bar{\beta}_i} \leq \sqrt{\alpha_i \beta_i} & , \ i = 1, 2, \end{cases} \qquad (31)$$

where $\bar{\alpha}_1 = \bar{\alpha}_2$, $\bar{\beta}_1 = \bar{\beta}_2$. Let $\bar{\mu}(T_\rho(\Omega))$ and $\bar{\mu}(T_\rho(\bar{\Omega}))$ denote respectively the spectral radii of the Peaceman-Rachford matrix with a single parameter for the numerical solution of the Dirichlet problem over Ω and $\bar{\Omega}$. Applying the previous results of the model problem to $\bar{\Omega}$, we conclude that

$$(31) \qquad \bar{\mu}(T_{\rho_0}(\bar{\Omega})) = \left(\frac{\bar{\beta}_1 - \sqrt{\bar{\alpha}_1 \bar{\beta}_1}}{\bar{\beta}_1 + \sqrt{\bar{\alpha}_1 \bar{\beta}_1}} \right)^2 = \min_\rho \bar{\mu}(T_\rho(\bar{\Omega})),$$ (32)

where $\rho_0 = \sqrt{\bar{\alpha}_1 \bar{\beta}_1} = \sqrt{\bar{\alpha}_2 \bar{\beta}_2}$. Because of the inequalities of (30), we have, for Ω, that $\tilde{\Lambda}_1(\rho_0) = \left(\frac{\beta_1 - \rho_0}{\beta_1 + \rho_0} \right) \leq \left(\frac{\bar{\beta}_1 - \rho_0}{\bar{\beta}_1 + \rho_0} \right)$, and similarly, $\tilde{\Lambda}_2(\rho_0) \leq \left(\frac{\bar{\beta}_2 - \rho_0}{\bar{\beta}_2 + \rho_0} \right)$. Since $\bar{\beta}_1 = \bar{\beta}_2$, then with (22) and (31), we have

$$(32) \qquad \bar{\mu}(T_{\rho_0}(\Omega)) \leq \bar{\mu}(T_{\rho_0}(\bar{\Omega})),$$ (33)

and thus

$$(32') \qquad \max_\rho R(T_\rho(\Omega)) \geq \max_\rho R(T_\rho(\bar{\Omega})) = R(T_{\rho_0}(\bar{\Omega})).$$ (33')

We conclude that the optimum rate of convergence, for the numerical solution of the Dirichlet problem, of the Peaceman-Rachford iterative method with a single parameter ρ is at least at fast for Ω as it is for $\bar{\Omega}$. Young and Ehrlich observed this numerically in [11]. What is more important, however, is that for either Ω or $\bar{\Omega}$, the rate of convergence of the optimized Peaceman-Rachford method with a single parameter is at least of order h, as $h \to 0$.

§4. Results of the p-Cyclic Theory and Applications

We briefly summarize the pertinent results of [4]. Let B be a consistently ordered real nxn matrix which is cyclic of index $p \geq 2$. If $1 > \bar{\mu}(B) > 0$, let ω_b be the unique positive value (less than $p/(p-1)$) satisfying

$$(33) \qquad \omega_b^p = \left(\frac{p}{\bar{\mu}(B)} \right)^p (p-1)^{1-p} (\omega_b - 1).$$ (34)

From the mapping $\mu(Z) = \frac{1}{\omega_b}\left\{ Z + \frac{\omega_b - 1}{Z^{p-1}} \right\}$, let $S_p(\bar{\mu}(B))$ be the image of the circle $|Z| = \left|(\omega_b - 1)(p - 1)\right|^{1/p}$. For $p = 2$, S_p is the interval $-\bar{\mu}(B) \leq x \leq \bar{\mu}(B)$. For $p > 2$, S_p is a Jordan curve whose interior contains the origin $\mu = 0$. Let $\tilde{S}_p(\bar{\mu}(B))$ denote the set of points μ consisting of S_p and its interior. If $\mathcal{L}_\omega$ is the overrelaxation matrix derived from B where ω is the relaxation factor, then [4] if the eigenvalues μ of B lie in $\tilde{S}_p(\bar{\mu}(B))$,

$$(34)\qquad \begin{aligned} \bar{\mu}\left(\mathcal{L}_{\omega_b}\right) &= (\omega_b - 1)(p - 1) \\ \bar{\mu}\left(\mathcal{L}_\omega\right) &> \bar{\mu}\left(\mathcal{L}_{\omega_b}\right) \quad \text{for } \omega \neq \omega_b. \end{aligned}$$ (35)

Now let $T_p(\bar{\mu}(B))$ be the set of points μ^p where $\mu \in S_p(\bar{\mu}(B))$. For $p = 2$, T_2 is the interval $0 \leq x \leq \bar{\mu}^2\, B$. But for $p > 2$, T_p not only contains the interval $0 \leq x \leq \bar{\mu}^p(B)$, but non-real points as well. Since the relationship [4] between the eigenvalues μ of B and the eigenvalues λ of $\mathcal{L}_1$, the Gauss-Seidel matrix, is $\lambda = \mu^p$, then the results of (34) hold if the eigenvalues of $\mathcal{L}_1$ lie in $T_p(\bar{\mu}(B))$. From [4], we have the following asymptotic relationship

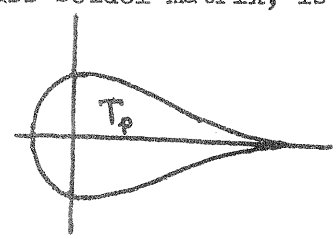

Figure 1.

$$(35)\qquad R\left(\mathcal{L}_{\omega_b}\right) \sim \left(\frac{2p}{p-1}\right)^{1/2} \left[R\left(\mathcal{L}_1\right)\right]^{1/2}, \quad \bar{\mu}(B) \to 1-.$$ (36)

With these facts, we come to the application of the above theory to the iterative method which combines successive overrelaxation and the Peaceman-Rachford iterative method. In §2, we showed that the Peaceman-Rachford iterative method with two parameters ρ_1 and ρ_2 was equivalent to applying the Gauss-Seidel iterative method to a particular 4-cyclic matrix. With $\rho_1 = \rho_2 = \rho$, let $\mathcal{L}_1(\rho)$ denote the Gauss-Seidel matrix whose spectral radius is $\bar{\mu}^4(\rho)$. With the above discussion, we have the

__Theorem.__ If the eigenvalues of $\mathcal{L}_1(\rho)$ lie in $T_4(\bar{\mu}(\rho))$, then the optimum relaxation factor ω_b satisfies (33) with $p = 4$, and

$$\bar{\mu}\left(\mathcal{L}_{\omega_b}(\rho)\right) = 3\,(\omega_b - 1).$$

__Corollary.__ If H and V satisfy (4), then the eigenvalues of $\mathcal{L}_1(\rho)$ lie in $T_4(\bar{\mu}(\rho))$

for all $\rho > 0$, and the conclusions of the above theorem are valid.

The proof of the corollary follows from the fact that the eigenvalues of $\mathcal{L}_1(\rho)$ are just zeros and the eigenvalues of T_ρ^2. But if H and V commute, the eigenvalues of $\mathcal{L}_1(\rho)$ are necessarily non-negative real numbers for every $\rho > 0$, and we can apply the results of the theorem above.

Finally, this new iterative method, which depends on two acceleration parameters ρ, ω, has a rate of convergence of order $h^{1/2}$, as $h \to 0$, for certain problems. To see this, we obtained a rate of convergence of order h for the Dirichlet problem of §3 <u>without</u> the assumption of (4), by selecting the parameter ρ properly. Assuming the weaker conditions of the Theorem, we can apply the asymptotic results of (35), and conclude that our two parameter iterative method with optimum parameters is of order $h^{1/2}$ as $h \to 0$., for the solution of the Dirichlet problem in the plane.

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