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Converse Results in the Walsh Theory of Overconvergence

A. S. Cavaretta, Jr., A. Sharma [(1)], and R. S. Varga [(2)]

(1) The work done by this author was performed at Kent State University, while he has on leave from The University of Alberta.

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Résumé. Récemment, J. Szabados a obtenu un nouveau théorème réciproque dans la théorie de la sur-convergence de Walsh, fondé sur l'Interpolation de Lagrange. Ici, nous développons un théorème réciproque similaire, fondé sur l'Interpolation d'Hermite, qui généralise le résultat de Szabados.

Abstract. Recently, J. Szabados has obtained a new converse theorem in the Walsh overconvergence theory, based on Lagrange interpolation. Here, we similarly develop a related converse theorem, based on Hermite interpolation, which generalizes Szabados' result.

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Converse Results in the Walsh Theory of Overconvergence

A. S. Cavaretta, Jr., A. Sharma [1], *and R. S. Varga* [2].

1. Introduction.

Let A_ρ denote the collection of functions analytic in $|z| < \rho$, and, as usual, let π_m denote the collection of all complex polynomials of degree at most m. For any $f(z) \in A_\rho$ with $\rho > 1$, and for any positive integer n, let $L_{n-1}(z; f)$ denote the Lagrange polynomial interpolant in π_{n-1} of $f(z)$ in the n-th roots of unity, i.e.,

$$L_{n-1}(\omega; f) = f(\omega), \tag{1.1}$$

where ω is any n-th root of unity. With $f(z) := \sum_{k=0}^{\infty} a_k z^k$ in $|z| < \rho$, and for each positive integer l, set

$$Q_{n-1,l}(z; f) := \sum_{j=0}^{l-1} \sum_{k=0}^{n-1} a_{k+jn} z^k, \tag{1.2}$$

so that $Q_{n-1,l}(z; f)$ is also an element of π_{n-1}. Then, the original and oft-cited beautiful result of J. L. Walsh [6, p. 153] on overconvergence is the case $l = 1$ of

Theorem A ([1]). For any $f(z) \in A_\rho$ with $\rho > 1$, and for any positive integer l,

$$\lim_{n\to\infty} \left\{ L_{n-1}(z; f) - Q_{n-1,l}(z; f) \right\} = 0, \text{ for all } |z| < \rho^{l+1}, \tag{1.3}$$

the convergence being uniform and geometric on any closed subset of $|z| < \rho^{l+1}$. Moreover, the result is best possible (in the sense that (1.3) is not valid at each point of $|z| = \rho^{l+1}$ for all $f(z)$ in A_ρ.

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Now Theorem A, in the terminology of approximation theory, is a *direct* theorem in the Walsh overconvergence theory, in that the assumption $f(z) \in A_\rho$ leads to the overconvergence result of (1.3). Recently, Szabados [4] obtained the following interesting *converse* theorem to Theorem A. For notation, let A_1C denote the collection of all $f(z)$ in A_1 which are continuous on $|z| = 1$.

Theorem B ([4]). Assume that $f(z) \in A_1C$. If $\rho > 1$, if l is a positive integer, and if the sequence

$$\left\{ L_{n-1}(z\,;f) - Q_{n-1,l}(z\,;f) \right\}_{n=1}^{\infty} \tag{1.4}$$

is uniformly bounded on every closed subset of $|z| < \rho^{l+1}$, then $f(z) \in A_\rho$.

It may be asked if the conclusion of Theorem 3 (namely, that $f(z) \in A_\rho$) is best possible, i.e., with the hypothesis of Theorem 3, could $f(z) \in A_{\rho'}$ where $\rho' > \rho$, in general? On considering the particular function $\hat{f}(z) := (\rho - z)^{-1}$ which, with (1.3) satisfies the hypothesis of Theorem B, one sees that $\hat{f}(z)$ is an element of A_ρ, but is clearly not an element of $A_{\rho'}$ for any $\rho' > \rho$. In this sense, Theorem B is best possible, as was remarked by Szabados [4].

There are now many known direct theorems in the Walsh overconvergence theory on the difference of interpolating polynomials (cf.[1], Rivlin [2], [5, ch. 4]). It is natural to ask if there are similar converse theorems which complement Szabados' Theorem B. Here, we show that such a converse theorem can be similarly derived for Hermite polynomial interpolation.

2. Statement of a New Result.

We first state a direct theorem for Hermite interpolation in the Walsh overconvergence theory. To fix notations, for any $f(z) \in A_\rho$ with $\rho > 1$, for a fixed positive integer r, and for every positive integer n, let $h_{rn-1}(z\,;f)$ denote the Hermite polynomial interpolant in π_{rn-1} of $f, f', \cdots, f^{(r-1)}$ in the n-th roots of unity, i.e.,

$$h_{rn-1}^{(j)}(\omega\,;f) = f^{(j)}(\omega)\,, \quad j = 0, 1, \cdots, r-1\,, \tag{2.1}$$

where ω is any n-th root of unity. Again, with $f(z) := \sum_{k=0}^{\infty} a_k z^k$ in $|z| < \rho$, and for any positive integer l, set

$$\tilde{Q}_{rn-1,l}(z\,;f) := \sum_{k=0}^{rn-1} a_k z^k + \sum_{j=1}^{l-1} \beta_{j,r}(z^n) \sum_{k=0}^{n-1} a_{k+(r+j-1)n} z^k\,, \tag{2.2}$$

where (cf. [1])

$$\beta_{j,r}(z) := \sum_{k=0}^{r-1} \binom{r+j-1}{k} (z-1)^k\,, \quad j = 1, 2, \cdots, \tag{2.3}$$

and where the last sum in (2.2) is defined here, and subsequently, to be zero when $l = 1$. Note that $\tilde{Q}_{rn-1,l}(z)$ is also in π_{rn-1}. With these notations, a direct theorem for Hermite interpolation in the Walsh overconvergence theory is

Theorem C ([1]). For any $f(z) \in A_\rho$ with $\rho > 1$, and for any positive integers r and l,

$$\lim_{n\to\infty} \left\{ h_{rn-1}(z\,;f) - \tilde{Q}_{rn-1,l}(z\,;f) \right\} = 0\,, \text{ for all } |z| < \rho^{1+(l/r)}\,, \tag{2.4}$$

the convergence being uniform and geometric on any closed subset of $|z| < \rho^{1+(l/r)}$. Moreover, the result is best possible.

A new result, a converse result to Theorem C, is the following. For notation, for each positive integer r, let $A_1C^{(r-1)}$ denote the collection of all $f(z)$ in A_1 for which $f(z), f'(z), \cdots$, and $f^{(r-1)}(z)$ are all continuous on $|z| = 1$. For any $f(z) \in A_1C^{(r-1)}$ and for any $n \geq 1$, it is evident that the interpolatory polynomials $h_{rn-1}(z\,;f)$ and $\tilde{Q}_{rn-1,l}(z\,;f)$ of (2.1) - (2.2) are well-defined.

Theorem 1. Assume that $f(z) \in A_1 C^{(r-1)}$. If $\rho > 1$, if l is a positive integer, and if the sequence

$$\left\{ h_{rn-1}(z\,;f) - \tilde{Q}_{rn-1,l}(z\,;f) \right\}_{n=1}^{\infty} \tag{2.5}$$

is uniformly bounded on every closed subset of $|z| < \rho^{1+(l/r)}$, then $f(z) \in A_\rho$.

As the special case $r=1$ of Theorem 1 reduces to Szabados' Theorem B, we remark that Theorem 1 then generalizes Theorem B.

The proof of Theorem 1 will be given in Section 3. Because it is needed in the proof of Theorem 1, we state, as in Theorem D below, a recent related result of Saff and Varga [3, Theorem 2] on Hermite interpolation in the Walsh overconvergence theory.

Theorem D ([3]). For each $f(z) \in A_\rho$, and for each pair of positive integers r and l, the sequence (2.5) can be bounded in at most $r+l-1$ distinct points in $|z| > \rho^{1+(l/r)}$.

3. Proof of Theorem 1.

With the notations from Section 2, we begin with the following result which, for $r = 1$, reduces to Lemma 1 of [4].

Lemma 1. If $f(z) := \sum_{k=0}^{\infty} a_k z^k$ *is an element of* $A_1 C^{(r-1)}$, *then for each positive integer* l,

$$h_{rn-1}(z\,;\,f) - \tilde{Q}_{rn-1,l}(z\,;\,f) = h_{rn-1}\Big(z\,;\,\sum_{k=(r+l-1)n}^{\infty} a_k z^k\Big) . \tag{3.1}$$

Proof. As $h_{rn-1}(z\,;\,f)$ of (2.1) is necessarily a linear operator which reproduces all polynomials of degree at most $rn-1$, then

$$h_{rn-1}(z\,;\,f) - h_{rn-1}\left(z\,;\,\sum_{k=(r+l-1)n}^{\infty} a_k z^k\right) = h_{rn-1}\left(z\,;\,\sum_{k=0}^{(r+l-1)n-1} a_k z^k\right)$$

$$= h_{rn-1}\left(z\,;\,\sum_{k=0}^{rn-1} a_k z^k\right) + h_{rn-1}\left(z\,;\,\sum_{k=rn}^{(r+l-1)n-1} a_k z^k\right)$$

$$= \sum_{k=0}^{rn-1} a_k z^k + \sum_{k=rn}^{(r+l-1)n-1} a_k\, h_{rn-1}(z\,;\,z^k)$$

$$= \sum_{k=0}^{rn-1} a_k z^k + \sum_{j=1}^{l-1} \sum_{k=0}^{n-1} a_{k+(r+j-1)n}\, h_{rn-1}(z\,;\,z^{k+(r+j-1)n}) .$$

It is known (cf. [1, eq. (4.4)]) that

$$h_{rn-1}(z\,;\,z^{k+(r+j-1)n}) = z^k \beta_{j,r}(z^n)\,, \text{ for } j = 1, 2, \cdots , \tag{3.2}$$

where $\beta_{j,r}(z)$ is defined in (2.3). Inserting the above identity into the previous display gives, with the definition of $\tilde{Q}_{rn-1,l}(z\,;\,f)$ in (2.2), the desired result of (3.1). □

Szabados [4] has pointed out that his special case $r = 1$ of Lemma 1 gives an elementary proof of Theorem A. We remark that Lemma 1 similarly gives an elementary proof of Theorem C. As its proof follows along the lines of the proof of Theorem 1, we omit the details.

Next, as $\beta_{j,r}(z)$ from (2.3), is in π_{r-1}, we can write

$$\beta_{j,r}(z) := \sum_{\nu=0}^{r-1} C_{\nu,r}(j) z^{\nu} , \tag{3.3}$$

where evidently

$$C_{\nu,r}(j) := \sum_{k=\nu}^{r-1} (-1)^{k-\nu} \binom{r+j-1}{k} \binom{k}{\nu}, \text{ for } \nu = 0, 1, \cdots, r-1 . \quad (3.4)$$

Lemma 2. The polynomials

$$C_{\nu,r}(x) := \sum_{k=\nu}^{r-1} (-1)^{k-\nu} \binom{x+r-1}{k} \binom{k}{\nu} = \sum_{k=\nu}^{r-1} (-1)^{k-\nu} \frac{(x+r-1)\cdots(x+r-k)}{(k-\nu)!\,\nu!}, \quad (3.5)$$

for $\nu = 0, 1, \cdots, r-1$, *form a Lagrangian basis for* π_{r-1}, *i.e., for any* $p_{r-1}(x) \in \pi_{r-1}$,

$$p_{r-1}(x) \equiv \sum_{j=0}^{r-1} p_{r-1}(j+1-r)\, C_{j,r}(x), \text{ for all } x . \quad (3.6)$$

In particular, choosing $p_{r-1}(x) \equiv 1$ *in (3.5) gives*

$$1 = \sum_{\nu=0}^{r-1} C_{\nu,r}(\lambda + l) \quad \text{for any integers } \lambda \text{ and } l . \quad (3.7)$$

Proof. It is evident from (3.5) that

$$C_{\nu,r}(x+1-r) = \frac{x(x-1)\cdots(x-\nu+1)}{\nu!} \left\{ 1 + \sum_{k=1}^{r-\nu-1} (-1)^k \frac{(x-\nu)(x-\nu-1)\cdots(x-k-\nu+1)}{k!} \right\} . \quad (3.8)$$

As the multiplier $x(x-1)\cdots(x-(\nu-1))$ in (3.8) vanishes for $x = 0, 1, \cdots, \nu-1$, then $C_{\nu,r}(j+1-r) = 0$ for $j = 0, 1, \cdots, \nu-1$, while for $x = \nu$, (3.8) gives $C_{\nu,r}(\nu+1-r) = 1$. Similarly, for $x = \nu + l$ (where $1 \le l \le r-1$), the quantity in braces in (3.8) reduces to $\left\{ 1 + \sum_{k=1}^{l} (-1)^k \frac{l(l-1)\cdots(l-(k-1))}{k!} \right\}$, which is the binomial expansion of $(1-1)^l = 0$. Thus, we have shown that

$$C_{\nu,r}(j+1-r) = \delta_{j,\nu}, \text{ for all } j = 0, 1, \cdots, r-1 .$$

Consequently, $\left\{ C_{\nu,r}(x) \right\}_{\nu=0}^{r-1}$ forms a Lagrangian basis for π_{r-1}, from which (3.6) and (3.7) directly follow. □

Proof of Theorem 1. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be any element in $A_1 C^{(r-1)}$ satisfying the hypothesis of Theorem 1, and let R be any number satisfying

$$1 < R < \rho^{1+(l/r)} . \tag{3.9}$$

Now, the boundedness hypothesis of (2.5) implies, from (3.1) of Lemma 1, that there is a constant $M(R)$ such that

$$\max_{|z|=R} \left| h_{rs-1}\left(\sum_{k=(r+l-1)s}^{\infty} a_k z^k \right) \right| \leq M(R) < \infty , \tag{3.10}$$

for any $s \geq 1$. In particular, choosing $s = 2n$ in (3.10) gives

$$\max_{|z|=R} \left| h_{2rn-1}\left(z ; \sum_{k=2(r+l-2)n} a_k z^k \right) \right| \leq M(R) . \tag{3.11}$$

Next, setting

$$h_{2rn-1}\left(z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right) =: \sum_{k=0}^{2rn-1} b_k z^k , \tag{3.12}$$

the bound from (3.11), along with Cauchy's formula, implies

$$| b_k | \leq M(R) \cdot R^{-k} , \quad k = 0, 1, \cdots, 2rn - 1 . \tag{3.13}$$

Since the set of $2n$-th roots of unity includes all n-th roots of unity, we obtain (cf. (2.1)) the identity:

$$h_{rn-1}(z ; g) = h_{rn-1}\left(z ; h_{2rn-1}(z ; g) \right) , \tag{3.14}$$

for any $g(z) \in A_1 C^{(r-1)}$. Choosing $g(z) := \sum_{k=2(r+l-1)n}^{\infty} a_k z^k$, then $g(z)$ is just $f(z)$, minus a polynomial, and is hence in $A_1 C^{(r-1)}$, for any $n \geq 1$. Using in succession the identity of (3.14), the definition of (3.12), the fact that h_{rn-1} is a linear operator which reproduces polynomials in π_{rn-1}, and the identity (3.2), we obtain the chain of equalities:

$$h_{rn-1}\left(z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right) = h_{rn-1}\left(z ; h_{2rn-1}(z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k) \right)$$

$$= h_{rn-1}\left(z ; \sum_{k=0}^{2rn-1} b_k z^k \right) = \sum_{k=0}^{rn-1} b_k z^k + \sum_{k=0}^{rn-1} b_{k+rn} h_{rn-1}(z ; z^{k+rn})$$

$$= \sum_{k=0}^{rn-1} b_k z^k + \sum_{k=0}^{n-1} \sum_{\lambda=0}^{r-1} b_{k+(r+\lambda)n} h_{rn-1}(z ; z^{k+(r+\lambda)n})$$

$$= \sum_{k=0}^{rn-1} b_k z^k + \sum_{\lambda=0}^{r-1} \beta_{\lambda+1,r}(z^n) \sum_{k=0}^{n-1} b_{k+(r+\lambda)n} z^k \quad \text{, i.e. ,}$$

$$h_{rn-1}\left(z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k\right) = \sum_{k=0}^{rn-1} b_k z^k + \sum_{\lambda=0}^{r-1} \beta_{\lambda+1,r}(z^n) \sum_{k=0}^{n-1} b_{k+(r+\lambda)n} z^k \ . \tag{3.15}$$

Now, it follows from the definition in (2.3) that

$$| \beta_{\lambda+1,r}(z^n) | \leq 2^{r+\lambda} (|z|^n + 1)^{r-1} \text{ for all } z \text{ , and all } \lambda \geq 0 \ ,$$

from which it easily follows that

$$\max_{|z|=R} | \beta_{\lambda+1,r}(z^n) | \leq 2^{2r+\lambda} R^{nr} \ , \quad \text{for all } \lambda \geq 0 \ . \tag{3.16}$$

Applying the bounds of (3.16) and (3.13) to the terms of (3.15) gives, after an easy calculation, that

$$\max_{|z|=R} \left| h_{rn-1}\left(z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right) \right| \leq n \, 2^{3r} M(R) \ . \tag{3.17}$$

This can be used as follows. By linearity again,

$$h_{rn-1}\left(z ; \sum_{k=(r+l-1)n}^{2(r+l-1)n-1} a_k z^k \right) = h_{rn-1}\left(z ; \sum_{k=(r+l-1)n}^{\infty} a_k z^k \right) - h_{rn-1}\left(z ; \sum_{k=2(r+l-1)n}^{\infty} a_k z^k \right) ,$$

so that with (3.17) and (3.10) (for the case $s = n$),

$$\max_{|z|=R} \left| h_{rn-1}\left(z ; \sum_{k=(r+l-1)n}^{2(r+l-1)n-1} a_k z^k \right) \right| \leq \left(n \, 2^{3r} + 1 \right) M(R) \ . \tag{3.18}$$

Using in succession again the linearity of the operator h_{rn-1} , the identity of (3.2), and (3.4), we obtain

$$h_{rn-1}\left(z ; \sum_{k=(r+l-1)n}^{2(r+l-1)n-1} a_k z^k \right) = \sum_{k=0}^{n-1} \sum_{\lambda=0}^{r+l-2} a_{k+(r+\lambda+l-1)n} h_{rn-1}(z ; z^{k+(r+\lambda+l-1)n})$$

$$= \sum_{k=0}^{n-1} \sum_{\lambda=0}^{r+l-2} a_{k+(r+\lambda+l-1)n} z^k \beta_{l+\lambda}(z^n)$$

$$= \sum_{k=0}^{n-1} \sum_{\nu=0}^{r-1} z^{k+\nu n} \sum_{\lambda=0}^{r+l-2} C_{\nu,r}(\lambda + l)\, a_{k+(r+\lambda+l-1)n} \ .$$

Applying Cauchy's formula and the bound of (3.18) to the above expression gives

$$\left| \sum_{\lambda=0}^{r+l-2} C_{\nu,r}(\lambda + l)\, a_{k+(r+\lambda+l-1)n} \right| \le \frac{(n\, 2^{3r}+1)M(R)}{R^{k+\nu n}} \ , \tag{3.19}$$

for all $k = 0, 1, \cdots, n-1$; $\nu = 0, 1, \cdots, r-1$.

Suppose we set

$$\sum_{\lambda=0}^{r+l-2} C_{\nu,r}\ (\lambda + l)\ a_{k+(r+\lambda+l-1)n} =: \mu_{k,\nu,n} \ , \tag{3.20}$$

for $k = 0, 1, \cdots, n-1$; $\nu = 0, 1, \cdots, r-1$, where from (3.19),

$$| \mu_{k,\nu,n} | \le \frac{(n\, 2^{3r}+1)M(R)}{R^{k+\nu n}} \ . \tag{3.21}$$

On summing both sides of (3.20) with respect to ν and using the identity of (3.7), we can write

$$\sum_{j=r+l-1}^{2(r+l-1)-1} a_{k+jn} = \sum_{\nu=0}^{r-1} \mu_{k,\nu,n} \ ,$$

so that

$$\left| \sum_{j=r+l-1}^{2(r+l-1)-1} a_{k+jn} \right| \le \sum_{\nu=0}^{r-1} | \mu_{k,\nu,n} | \ .$$

Applying the upper bound of (3.21) then gives

$$\left| \sum_{j=r+l-1}^{2(r+l-1)-1} a_{k+jn} \right| \le \frac{r(n\, 2^{3r}+1)M(R)}{R^{k}} \ , \tag{3.22}$$

for all $k = 0, 1, \cdots, n-1$, all $n \ge 1$.

We now state a result which is implicit in the work of Szabados [4].

Lemma 3 ([4]). If $g(z) = \sum_{k=0}^{\infty} \alpha_k z^k$ *is an element of* $A_1 C$ *, and if, for each positive integer* s *and each* R *with* $1 < R < \rho^{l+1}$ *, there is a constant* $M(R)$ *such that*

$$| \sum_{j=s}^{2s-1} \alpha_{k+jn} | \leq \frac{(2n+1)M(R)}{R^k}, \; for\ all\ k = 0, 1, \cdots, n-1, \; all\ n \geq 1, \tag{3.23}$$

then

$$\overline{\lim_{n \to \infty}} | \alpha_n |^{1/n} \leq \begin{cases} R^{-1/2}, & if\ s = 1; \\ R^{-(3s^2+1)}, & if\ s > 1 \end{cases} < 1 . \tag{3.24}$$

Lemma 3 can be applied as follows. As $f(z)$, by hypothesis an element in $A_1C^{(r-1)}$, is necessarily in A_1C, and as (3.22) holds, then (3.24) of Lemma 3 with $s = r + l - 1$ gives that

$$\overline{\lim_{n \to \infty}} | a_n |^{1/n} < 1 . \tag{3.25}$$

This last inequality ensures, as in [4], that $f(z)$ can be analytically continued from $|z| \leq 1$ into a larger circle. Let $\overline{\rho} > 1$ be the *maximal* radius for which $f(z)$ is analytic in $|z| < \overline{\rho}$, so that $f(z)$ has a singularity on $|z| = \overline{\rho}$. But, by Theorem D, the sequence (2.6) can be bounded in at most $r + l - 1$ distinct points in $|z| > \overline{\rho}^{1+(l/r)}$. As the hypothesis of Theorem 1 ensures that this sequence is uniformly bounded on every closed subset of $|z| < \rho^{1+(l/r)}$, it is evident that $\rho \leq \overline{\rho}$, showing that $f(z) \in A_\rho$. □

To conclude, we mention some open questions. It would be interesting to see if similar converse results hold for lacunary interpolation in the roots of unity, or for Rivlin's case [2] of l_2 - convergence. Moreover, the above proof of Theorem 1 depends on the use of Saff and Varga's Theorem D. Is it possible to prove Theorem 1 *without* the use of Theorem D?

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