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k-Step Iterative Methods for Solving Nonlinear Systems of Equations

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Summary. Let Φ: $\mathbb{C}^n \to \mathbb{C}^n$ be Fréchet differentiable, and let the equation

$$x = \Phi(x) \tag{1}$$

have at least one fixed point. We consider k-step stationary iterative methods

$$y_m := \mu_0 \Phi(y_{m-1}) + \mu_1 y_{m-1} + \ldots + \mu_k y_{m-k}, \qquad m \geqq k, \tag{2}$$

with $\mu_0 + \mu_1 + \ldots + \mu_k = 1$. Using results for an affine mapping Φ: $\mathbb{C}^n \to \mathbb{C}^n$, it is proven that (2) may converge locally even in cases where the usual iteration $x_m = \Phi(x_{m-1})$ belonging to (1) diverges. These results are extended to nonstationary methods of type (2) and to "cyclic" mappings.

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§ 1. Introduction

Let Φ: $\mathbb{C}^n \to \mathbb{C}^n$ be a Fréchet differentiable function having at least one fixed point x, i.e., there exists an $x \in \mathbb{C}^n$ satisfying

$$x = \Phi(x). \tag{1.1}$$

In analogy with certain *k-step stationary iterative methods* for linear systems, which have been investigated by Kublanovskaya [7], Niethammer and Varga [10–12] and others, one may consider iterations of the form

$$y_m := \mu_0 \Phi(y_{m-1}) + \mu_1 y_{m-1} + \ldots + \mu_k y_{m-k}, \qquad m = k, k+1, \ldots, \tag{1.2}$$

where $y_0, y_1, \ldots, y_{k-1}$ are given starting vectors in $\mathbb{C}^n$ and where $\mu_0, \mu_1, \ldots, \mu_k$ are fixed complex numbers which are assumed to satisfy

$$\mu_0 + \mu_1 + \ldots + \mu_k = 1, \qquad \mu_0 \neq 0, \ \mu_k \neq 0. \tag{1.3}$$

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The above guarantees that the limit of any convergent sequence $\{y_m\}$, generated by (1.2), is a fixed point of Φ. We remark that iterations of the form (1.2) have been considered by Gekeler [2].

If we let $\tilde{y}_m := \Phi(y_{m-1})$, then y_m in (1.2) is the result of an averaging process of the vectors $\tilde{y}_m, y_{m-1}, \ldots, y_{m-k}$ with weights $\mu_0, \ldots, \mu_k$. "Averaging iterations" of a similar type have been considered by various authors (cf. the review paper of Mann [8]). Their relationship to the iterative methods of the form (1.2) is described in Gutknecht and Kaiser [6]. In general, all preceding iterates $y_0, \ldots, y_m$ are used in these iterations. As pointed out in Mann [8], the aim is "the construction of generalized iteration procedures which sometimes succeed where ordinary iteration fails". The usual setting is where one considers a compact convex set E in a Banach space with Φ a continous mapping of E into E. Under the assumption that Φ is "*nonexpansive*", i.e., $\|\Phi(x)-\Phi(y)\| \leqq \|x-y\|$ for all $x, y \in E$, various results on global convergence in E can be proven. However, since these global convergence results for nonexpansive mappings extend the well-known results for strictly contractive mappings only slightly, there usefulness is very limited; they are not applicable in many important problems arising in applications.

We restrict ourselves here to the finite-dimensional Banach space $\mathbb{C}^n$, but we only assume that there exists a fixed point x of the continous Fréchet differentiable mapping Φ such that 1 is not an eigenvalue of $\Phi'(x)$; this implies that there exists a neighborhood U of x such that $I-\Phi$ is injective in U, i.e., x is the unique fixed point in U. Moreover, we only aim at local convergence. But this enables us to treat *expansive* operators, a fact quite important for practical applications. The main idea is as follows: Associated with iteration (1.2), there is a new operator $\boldsymbol{\Phi}: \mathbb{C}^{kn} \to \mathbb{C}^{kn}$, depending on Φ and the parameters $\mu_0, \ldots, \mu_k$, such that $\mathbf{x} := (x, \ldots, x) \in \mathbb{C}^{kn}$ is a fixed point of $\boldsymbol{\Phi}$, and such that the iteration (1.2) is the *ordinary iteration*

$$\mathbf{y}_m := \boldsymbol{\Phi}(\mathbf{y}_{m-1}) \tag{1.4}$$

for $\boldsymbol{\Phi}$. For a suitable choice of the parameters, it is often possible to make $\boldsymbol{\Phi}$ strongly contractive at $\mathbf{x}$, when Φ is expansive at x. When Φ is itself contractive, the iteration (1.2) may still be very useful because of a better rate of convergence.

In this paper, we first give in §2 a short account of part of the linear theory. Then, in §3, by applying Ostrowski's Theorem [15, Thm. 10.1.3; 16, Thm. 22.1] and the corresponding result on the rate of convergence [15, Thm. 10.1.4], we easily relate the nonlinear iteration (1.2) to a linear one, thus obtaining, as in Gekeler [2], a local convergence theorem from the (global) convergence theorem of the linear theory. In §4, this local convergence result is extended to *nonlinear asymptotically stationary k-step iterative methods* of the form

$$y_m := \mu_0^{(m)} \Phi(y_{m-1}) + \mu_1^{(m)} y_{m-1} + \ldots + \mu_k^{(m)} y_{m-k}, \quad m = k, k+1, \ldots \tag{1.5}$$

with

$$\mu_0^{(m)} + \mu_1^{(m)} + \ldots + \mu_k^{(m)} = 1, \quad \mu_0^{(m)} \neq 0, \quad m = k, k+1, \ldots, \tag{1.6a}$$

and with

$$\mu_l^{(m)} \to \mu_l, \quad l = 0, \ldots, k, \quad \mu_0 \neq 0, \ \mu_k \neq 0. \tag{1.6b}$$

In §5, assuming that Φ is a contraction and that the coefficients μ_l (or $\mu_l^{(m)}$, respectively) are nonnegative real, we prove the global convergence of iterations (1.2) and (1.5).

In §6, we study the application of these nonlinear iterative methods to weakly cyclic (of index ν) systems of equations.

Some of the results given in this paper, in particular the convergence theorems in §§4–6, are generalizations of results of Gutknecht [4] who, for a special function Φ appearing in Theodorsen's integral equation for conformal maps, studied appropriately chosen nonlinear 2-step methods. On the other hand, in Gutknecht and Kaiser [6] some of our results are extended to a Banach space setting.

Finally, in §7, we include examples of applications of the foregoing theory.

§ 2. Background Material from the Linear Theory

Our local convergence results are heavily based on known convergence theorems for the linear version of iteration (1.2), i.e., the k-step formula

$$y_m := \mu_0(Ty_{m-1} + c) + \mu_1 y_{m-1} + \ldots + \mu_k y_{m-k}, \qquad m = k, k+1, \ldots, \qquad (\mu_0 + \mu_1 + \ldots + \mu_k = 1) \tag{2.1}$$

where T is an $n \times n$ matrix whose spectrum $\sigma(T)$ does not contain 1, so that the linear system

$$x = Tx + c$$

has a unique solution x. Associated with (2.1) is the rational function

$$p(z) := \frac{\mu_0 z}{1 - \mu_1 z - \ldots - \mu_k z^k} \tag{2.2}$$

which will be used to characterize the convergence behavior of (2.1).

The appropriate measure for the asymptotic (linear) rate of convergence of an iteration method, such as (1.2) or (2.1), is the linear root-convergence factor

$$\kappa := \sup_{y_0, \ldots, y_{k-1}} \overline{\lim_{m \to \infty}} \|y_m - x\|^{1/m}, \tag{2.3}$$

where the starting values $y_0, \ldots, y_{k-1}$ are restricted to values for which convergence takes place (Ortega and Rheinboldt [15, Def. 9.2.1]). The root-convergence factor of a fixed convergent sequence, $\overline{\lim}_{m \to \infty} \|y_m - x\|^{1/m}$, is independent of the norm used in $\mathbb{C}^n$ [15, Thm. 9.2.2], and it is always at most as great as the quotient-convergence factor $\overline{\lim}_{m \to \infty} [\|y_{m+1} - x\| / \|y_m - x\|]$, which is norm-dependent [15, Thm. 9.3.1].

The linear k-th order difference equation (2.1) can be transformed as usual into an equivalent first-order difference equation

$$\mathbf{y}_m := \mathbf{T}\mathbf{y}_{m-1} + \mu_0 \mathbf{c}, \qquad m = k, k+1, \ldots, \tag{2.4}$$

where $\mathbf{T}$ is a $kn \times kn$ matrix, given by

$$\mathbf{T} := \mathbf{T}(T, p) := \begin{bmatrix} \mu_0 T + \mu_1 I & \mu_2 I & \dots & \mu_{k-1} I & \mu_k I \\ I & 0 & & 0 & 0 \\ 0 & I & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \end{bmatrix}, \tag{2.5}$$

and where

$$\mathbf{y}_m := \begin{bmatrix} y_m \\ y_{m-1} \\ \vdots \\ y_{m-k+1} \end{bmatrix}, \quad \mathbf{c} := \begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{2.6}$$

If (2.1) converges to $x = (I - T)^{-1} c$, then (2.4) converges to $\mathbf{x} := (x, x, \dots, x)^T$. Now, as is well known in the convergent case [20] (and in fact is readily seen from the Jordan canonical form of $\mathbf{T}$), the linear root-convergence factor of method (2.4), i.e.,

$$\tilde{\kappa} := \sup_{\mathbf{y}_{k-1}} \overline{\lim_{m \to \infty}} \|\mathbf{y}_m - \mathbf{x}\|^{1/m}, \tag{2.7}$$

is equal to the spectral radius $\rho(\mathbf{T})$ of $\mathbf{T}$. Here again, $\tilde{\kappa}$ does not depend on the norm chosen in $\mathbb{C}^{kn}$. In particular, if we choose the norm such that

$$\|\mathbf{y}_m - \mathbf{x}\| = \max_{0 \le j \le k-1} \|y_{m-j} - x\|$$

(where on the right-hand-side $\|\cdot\|$ is any norm in $\mathbb{C}^n$), then it is clear that $\tilde{\kappa}$ is equal to the rate κ defined by (2.3) for method (2.1) (a detailed proof was given by Voigt [21, Thm. 2.3]). We restate the above in the form of

Lemma 1. *The linear root-convergence factor κ of the k-step formula* (2.1) *is equal to the spectral radius of* $\mathbf{T}(T, p)$.

On the other hand, the spectrum of $\mathbf{T}(T, p)$ is related to the spectrum of T by the following well-known result (cf., e.g., Bittner [1], Rjabenki and Filippow [18]).

Lemma 2. *If τ_i $(i = 1, \dots, n)$ are the eigenvalues of the matrix T, then the eigenvalues of* $\mathbf{T}(T, p)$ *of* (2.5) *are the nk zeros of the n polynomials $q_i(\lambda) := q(\tau_i; \lambda)$, where*

$$q(\tau; \lambda) := \mu_k + \mu_{k-1} \lambda + \dots + \mu_1 \lambda^{k-1} + \mu_0 \lambda^{k-1} \tau - \lambda^k, \tag{2.8}$$

which is related to the function p of (2.2) *through*

$$q(\tau; \lambda) = \mu_0 \lambda^{k-1} \left[\tau - \frac{1}{p(1/\lambda)} \right]. \tag{2.9}$$

(Multiple eigenvalues appear according to their algebraic multiplicity.)

Of course, it is unrealistic to assume that the eigenvalues of T are known. (In fact, in this case, a nonstationary 1-step Richardson iteration converging in at most n steps is well known [20].) However, it is reasonable to suppose that we know a priori some *set* S containing the spectrum of T.

Conversely, we may pose the following questions: Given a k-step formula (2.1), can there be determined a set S in $\mathbb{C}$ such that $\rho(\mathbf{T})<1$ if $\sigma(T)\subseteq S$? In addition, can there be determined a subset S_η of S such that $\rho(\mathbf{T})\leqq 1/\eta<1$ if $\sigma(T)\subseteq S_\eta$?

Indeed, with

$$D_\eta:=\{z\in\mathbb{C}:\ |z|<\eta\},\qquad \bar{D}_\eta:=\{z\in\mathbb{C}:\ |z|\leqq\eta\},\tag{2.10}$$

and with p as in (2.2), let us define the point set

$$S(p):=\bar{\mathbb{C}}\setminus\{1/p(\bar{D}_1)\}=\{z\in\mathbb{C}:\ 1/z\notin p(\bar{D}_1)\},\tag{2.11}$$

and, for $\eta>1$, its subsets

$$S_\eta(p):=\bar{\mathbb{C}}\setminus\{1/p(D_\eta)\}=\{z\in\mathbb{C}:\ 1/z\notin p(D_\eta)\}.\tag{2.12}$$

(Without further assumptions on p it may happen that $S(p)$ is empty.) Then, the following theorem answers the questions posed above:

Theorem 1. *Let κ be the root-convergence factor* (2.3) *of the k-step formula* (2.1), *and let p be the associated function* (2.2). *Then there holds:*

a) *If $\sigma(T)\subseteq S(p)$, then $\kappa<1$ and the iterative method* (2.1) *converges for arbitrary starting values $y_0,\ldots,y_{k-1}$.*

b) *If $\sigma(T)\subseteq S_\eta(p)$ for some $\eta>1$, then*

$$\kappa\leqq 1/\eta,\tag{2.13}$$

and if there is at least one eigenvalue of T on the boundary of $S_\eta(p)$, equality holds in (2.13).

Proof. We omit the proof of part a) since it is similar to the following proof of the first half of part b). By virtue of Lemma 1 and Lemma 2, inequality (2.13) will be established if we can show that

$$q(\tau;\lambda)\neq 0\quad\text{if}\quad \tau\in S_\eta(p)\quad\text{and if}\quad |\lambda|>1/\eta,\tag{2.14}$$

since this implies that all eigenvalues of $\mathbf{T}(T,p)$ lie in $\bar{D}_{1/\eta}$, i.e., $\rho(\mathbf{T}(T,p))\leqq 1/\eta$. Now, in fact, by the definition of $S_\eta(p)$, we have $1/p(1/\lambda)\notin S_\eta(p)$ if $|\lambda|>1/\eta$, i.e., $1/\lambda\in D_\eta$. Hence, (2.14) is an immediate consequence of relation (2.9).

We next assume that τ_1 is an eigenvalue of T on the boundary of $S_\eta(p)$. Because $1/p$ is rational, interior points of D_η are mapped in interior points of $(1/p)(D_\eta)$, the complement of $S_\eta(p)$. Therefore, there is a ζ_1 on the boundary of D_η such that $\tau_1=1/p(\zeta_1)$ and, by (2.9), $q(\tau_1;1/\zeta_1)=0$, i.e., $\lambda_1=1/\zeta_1$ is an eigenvalue of $\mathbf{T}(T,p)$ having modulus $1/\eta$. This implies $\kappa=1/\eta$, as claimed. □

We can ask whether the regions $S(p)$ and $S_\eta(p)$ can be described in terms of $\mu_0,\ldots,\mu_k$. For this, let the function p of (2.2) be holomorphic and univalent in some neighborhood of the closed unit disk $\bar{D}_1$ (i.e., in the notation of [12], let p be an *Euler function*), and let us call

$$\hat{\eta} := \hat{\eta}(p) := \sup\{\eta > 1 \colon p \text{ is univalent in } D_\eta\} \tag{2.15}$$

its *maximal extension*. Then, the image of the unit circle ∂D_1 is a simple closed curve which separates $(1/p)(\bar{D}_1)$ and its complement, i.e., the image of ∂D_1 is the boundary of $S(p)$. For the same reasons, it follows that for $1 < \eta < \hat{\eta}(p)$, the image of the circle with radius η under the mapping $1/p$ is the boundary of $S_\eta(p)$. Examples will follow in Sect. 7. As a consequence, there holds the

Corollary. *If p is holomorphic and univalent in some neighborhood of the closed unit disk, then for $1 < \eta \leqq \hat{\eta}(p)$, equality holds in (2.13) if and only if there is at least one eigenvalue of T on the boundary of $S_\eta(p)$.*

§ 3. Local Convergence of Stationary k-Step Methods

In view of the linear theory cited in § 2, the following local convergence result for iteration (1.2) is now but a simple application of Ostrowski's theorem and a related result on the linear rate of convergence:

Theorem 2. *Let p be of the form (2.2) and let $\sigma(\Phi'(x)) \subseteq S_\eta(p)$ at some fixed point x of Φ. Then the nonlinear k-step stationary iterative method (1.2) converges locally near x, and the linear root-convergence factor κ is equal to $\rho(\mathbf{T}(\Phi'(x), p))$ and satisfies*

$$\kappa \leqq 1/\eta. \tag{3.1}$$

If p is univalent in some neighborhood of the closed unit disk, then equality holds in (3.1) if and only if there is an eigenvalue of $\Phi'(x)$ on the boundary of $S_\eta(p)$.

Proof. In analogy with the equivalence between (2.1) and (2.4), we write (1.2) as a nonlinear first-order difference equation,

$$\mathbf{y}_m := \Psi(\mathbf{y}_{m-1}), \qquad m = k, k+1, \ldots, \tag{3.2a}$$

where

$$\Psi(y_{m-1}) := \begin{bmatrix} \mu_0 \Phi(y_{m-1}) + \mu_1 y_{m-1} + \mu_2 y_{m-2} + \ldots + \mu_k y_{m-k} \\ y_{m-1} \\ \vdots \\ y_{m-k+1} \end{bmatrix}. \tag{3.2b}$$

Since

$$\Psi'(\mathbf{x}) = T(\Phi'(x), p), \tag{3.3}$$

the assertion follows from Ostrowski's Theorem [15, Thm. 10.1.3; 16, Thm. 22.1], the associated linear convergence theorem [15, Thm. 10.1.4], Lemma 1, Theorem 1, and its Corollary. □

§ 4. Local Convergence of Asymptotically Stationary k-Step Methods

In order to prove a local convergence theorem for the asymptotically stationary method (1.5) satisfying (1.6), we have to replace Ostrowski's Theorem by the following more powerful theorem due to Perron. For notation, $D_\eta^n := D_\eta \times D_\eta \times \ldots \times D_\eta$, where D_η is defined in (2.10).

Perron's Theorem [17, Thms. 5 & 7]. *Let A be a complex $n\times n$ matrix, and let (for some $\eta>0$) the functions χ_m: $D_\eta^n\to\mathbb{C}^n$, $m=0,1,\ldots$, satisfy the conditions*

(a) *there exists a constant $K>0$ such that $\|\chi_m(z)\|\le K\|z\|$ for all $z\in D^n$ and all $m=0,1,\ldots$;*

(b) $\chi_m(z)/\|z\|\to 0$ *as* $\|z\|\to 0$ *and* $m\to\infty$.

Then, the (trivial) null solution of the recursion

$$z_m:=Az_{m-1}+\chi_m(z_{m-1})$$

is stable (i.e., *every solution $\{z_m\}$ generated from a sufficiently small starting vector z_0 tends to 0) if $\rho(A)<1$, and it is unstable if $\rho(A)>1$.*

Remark. Perron used the l_1-norm in D_η^n and the l_∞-norm in $\mathbb{C}^n$, but this is not essential, of course.

In view of assumptions (1.1) and (1.6) and by introducing the errors $e_m:=y_m-x$, iteration (1.5) may be written in the form

$$e_m=\mu_0^{(m)}[\Phi(x+e_{m-1})-\Phi(x)]+\mu_1^{(m)}e_{m-1}+\ldots+\mu_k^{(m)}e_{m-k}. \tag{4.1}$$

In order to make use of the Fréchet differentiability of Φ and of (1.6b), we write this as

$$e_m=[\mu_0\Phi'(x)+\mu_1]\,e_{m-1}+\mu_2e_{m-2}+\ldots+\mu_ke_{m-k}+\chi_m(\mathbf{e}_{m-1}), \tag{4.2}$$

with $\mathbf{e}_{m-1}:=(e_{m-1}^T,\ldots,e_{m-k}^T)^T$ (in analogy to (2.6)) and

$$\begin{aligned}\chi_m(\mathbf{h}):=&[(\mu_0^{(m)}-\mu_0)\,\Phi'(x)+(\mu_1^{(m)}-\mu_1)]\,h_1\\&+(\mu_2^{(m)}-\mu_2)\,h_2+\ldots+(\mu_k^{(m)}-\mu_k)\,h_k\\&+\mu_0^{(m)}[\Phi(x+h_1)-\Phi(x)-\Phi'(x)\,h_1],\end{aligned} \tag{4.3}$$

where $\mathbf{h}:=(h_1^T,\ldots,h_k^T)^T$ with $h_j\in\mathbb{C}^n$ arbitrary ($j=1,\ldots,k$). By the Fréchet differentiability of Φ, the last bracket in (4.3) is $o(\|h_1\|)$ as $h_1\to 0$, so obviously

$$\|\chi_m(\mathbf{h})\|=O(\|\mathbf{h}\|)\quad\text{as }\mathbf{h}\to\mathbf{0}, \tag{4.4}$$

uniformly in m, and

$$\frac{\|\chi_m(\mathbf{h})\|}{\|\mathbf{h}\|}\to 0\quad\text{as}\quad m\to\infty,\quad \mathbf{h}\to\mathbf{0}. \tag{4.5}$$

Writing (4.2) as a system (similar to (3.2), thereby introducing $\boldsymbol{\chi}_m(\mathbf{h}):=(\chi_m(\mathbf{h})^T,O^T,\ldots,O^T)^T$), makes clear that Perron's Theorem can be applied, showing that $\mathbf{e}_m=\mathbf{y}_m-\mathbf{x}=\mathbf{0}$ ($\forall m$) is a stable solution of (4.2). We thus conclude:

Theorem 3. *Let p satisfy the assumptions of Theorem 2. Then an associated iterative method (1.5) satisfying (1.6) converges locally near x.*

One can show by a detailed analysis that also inequality (3.1) for the root-convergence factor still holds [6].

§ 5. Invariance of the Contraction Property

In this section, we assume that Φ is a contraction, but we drop the Fréchet differentiability of Φ. More precisely, we assume:

(i) There is $X \subseteq \mathbb{C}^n$ and an $L \in (0, 1)$ such that $\Phi\colon X \to X$ is continuous and

$$\|\Phi(y) - \Phi(\tilde{y})\| \leqq L \|y - \tilde{y}\| \qquad (\forall\, y, \tilde{y} \in X); \tag{5.1}$$

(ii) the weights $\mu_l^{(m)}$ in iteration (1.5) satisfy (1.6a), (1.6b), and are non-negative real numbers;

(iii) $X \subseteq \mathbb{C}^n \cong \mathbb{R}^{2n}$ is convex.

Assumption (i) is well known to imply that there is exactly one fixed point x of Φ in X and that the sequence $\{x_m\}$, generated by $x_m := \Phi(x_{m-1})$ with an arbitrary $x_0 \in X$, converges to x, at least with the linear rate L.

One can show that iteration (1.5) and, a fortiori, iteration (1.2) can be viewed as a contraction also. In particular, *global convergence* is guaranteed.

Theorem 4. *Under assumptions* (i), (ii), *and* (iii), *the sequence* $\{y_m\}$ *generated by* (1.5) *with arbitrary starting values* $y_0, \ldots, y_{k-1}$ *in* X, *converges linearly to the unique fixed point* $x \in X$ *of* Φ.

Proof. First, if $y_0, \ldots, y_{k-1} \in X$, then $\Phi(y_{k-1}) \in X$, and $y_k \in X$ also, since y_k is a weighted mean of $k+1$ points in X, which is convex. By induction, $y_m \in X$ for every m.

Let $\varepsilon_m := \|y_m - x\|$. Using (5.1), we conclude from (4.1) that

$$0 \leqq \varepsilon_m \leqq \mu_0^{(m)} L \varepsilon_{m-1} + \mu_1^{(m)} \varepsilon_{m-1} + \ldots + \mu_k^{(m)} \varepsilon_{m-k}. \tag{5.2}$$

If the inequality sign for vectors is understood componentwise, we may write

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \leqq \begin{bmatrix} \varepsilon_m \\ \vdots \\ \varepsilon_{m-k+1} \end{bmatrix} \leqq M^{(m)} \cdot \begin{bmatrix} \varepsilon_{m-1} \\ \vdots \\ \varepsilon_{m-k} \end{bmatrix}, \tag{5.3}$$

where

$$M^{(m)} := \begin{bmatrix} \mu_0^{(m)} L + \mu_1^{(m)} & \mu_2^{(m)} & \ldots & \mu_{k-1}^{(m)} & \mu_k^{(m)} \\ 1 & 0 & \ldots & & 0 \\ & 1 & \ddots & & \vdots \\ 0 & & \ddots & 1 & 0 \end{bmatrix}$$

is a real $k \times k$ matrix.

Obviously, $M^{(m)}$ is a companion matrix of the type (2.5), but with $n = 1$. Hence, its eigenvalues λ_1 are related to the eigenvalue L of the 1×1 matrix L by $q_m(L, \lambda_1) = 0$, where q_m is defined by

$$q_m(\tau, \lambda) := \mu_0^{(m)} \lambda^{k-1} \left[\tau - \frac{1}{p_m(1/\lambda)} \right],$$

$$p_m(z) := \frac{\mu_0^{(m)} z}{1 - \mu_1^{(m)} z - \ldots - \mu_k^{(m)} z^k}$$

(cf. (2.9) and (2.2)). Now, for $|\lambda| \geqq 1$,

$$\frac{1}{|p_m(1/\lambda)|} = \left| \frac{1 + \mu_0^{(m)} - [\mu_0^{(m)} + \mu_1^{(m)} \lambda^{-1} + \mu_2^{(m)} \lambda^{-2} + \ldots + \mu_k^{(m)} \lambda^{-k}]}{\mu_0^{(m)} \lambda^{-1}} \right| \geqq |\lambda| \geqq 1,$$

since the bracket has modulus at most 1. Therefore, $|q_m(L, \lambda)| \geqq \mu_0^{(m)}[1-L] > 0$ if $|\lambda| \geqq 1$, i.e., all eigenvalues of $M^{(m)}$ and of $M := \lim M^{(m)}$ lie in D_1 (and cannot accumulate at a boundary point of D_1). By a well known result on matrix norms, there exists a norm $\|\cdot\|^\wedge$ in $\mathbb{R}^k$ such that in the associated operator norm, $\|M\|^\wedge = 1 - \delta < 1$, which, by the continuity of the norm, implies that $\|M^{(m)}\|^\wedge \leqq 1 - \delta/2 < 1$ for sufficiently large m. From (5.3), it is then clear that $\varepsilon_m \to 0$. (Defining $\varepsilon_m := \|y_m - \tilde{y}_m\|$ instead, one likewise shows that, for sufficiently large m, the asymptotically stationary version of iteration (3.2) defines a contraction in a suitable norm.) □

§ 6. Cyclic Systems

We call the system of equations (1.1) *weakly cyclic of index* ν, if it is of the form

$$\begin{bmatrix} x^1 \\ x^2 \\ \vdots \\ x^\nu \end{bmatrix} = \begin{bmatrix} \varphi_1(x^\nu) \\ \varphi_2(x^1) \\ \vdots \\ \varphi_\nu(x^{\nu-1}) \end{bmatrix}, \tag{6.1}$$

where $x^j \in \mathbb{C}^{n_j}$, and where $n = n_1 + \ldots + n_\nu$. Systems of the form (6.1) were treated by Ullrich [19]. The case $\nu = 2$ came up in Gutknecht [4], and the linear case with $\nu = 3$ is covered by Niethammer, de Pillis, and Varga [13]. The Fréchet derivative of Φ is then an $n \times n$ weakly cyclic of index ν matrix [20, Def. 2.3, p. 39]

$$\Phi'(x) = \begin{bmatrix} 0 & 0 & \cdots & 0 & \varphi_1'(x^\nu) \\ \varphi_2'(x^1) & 0 & \ddots & & \\ 0 & \varphi_3'(x^2) & & 0 & \\ \vdots & & \ddots & & \\ 0 & & & \varphi_\nu'(x^{\nu-1}) & 0 \end{bmatrix}. \tag{6.2}$$

By noting that $[\Phi'(x)]^\nu$ is block-diagonal, i.e.,

$$[\Phi'(x)]^\nu = \operatorname{diag}(B_1, B_2, \ldots, B_\nu), \tag{6.3a}$$

with entries

$$B_j := \varphi_j'(x^{j-1})\, \varphi_{j-1}'(x^{j-2}) \ldots \varphi_1'(x^\nu)\, \varphi_\nu'(x^{\nu-1}) \ldots \varphi_{j+1}'(x^j) \tag{6.3b}$$

($j = 1, \ldots, \nu$), which all have the same nonzero eigenvalues (since only the order of the factors in (6.3b) depends on j), one concludes that the nonzero eigenvalues of $\Phi'(x)$ are ν-th roots of eigenvalues of B_1 (or any other B_j). In fact, Romanovsky's theorem [20, Thm. 2.4, Exercise 12, p. 45] says that the spectrum of $\Phi'(x)$ is ν-fold (i.e., with respect to the angle $2\pi/\nu$) rotationally symmetric about 0. Hence, if $\Phi'(x)$ is regular,

$$\sigma(\Phi'(x))=\{\lambda\in\mathbb{C}:\ \lambda^{\nu}\in\sigma(B_1)\}. \tag{6.4}$$

(Otherwise, 0 is an additional eigenvalue of $\Phi'(x)$ and of at least one B_j.)

Our k-step methods for solving (1.1) can be modified in two ways to take account of the cyclic of index ν structure of (6.1). First, with the usual notation $(f_1\circ f_2)(y):=f_1(f_2(y))$, (6.1) can be reduced to the equation

$$x^1=(\varphi_1\circ\varphi_\nu\circ\varphi_{\nu-1}\circ\ldots\circ\varphi_2)(x^1), \tag{6.5}$$

which contains only x^1. (In view of a later application, we prefer this equation to $x^\nu=(\varphi_\nu\circ\varphi_{\nu-1}\circ\ldots\circ\varphi_1)(x^\nu)$.) Once x^1 is computed, the other subvectors $x^2,\ldots,x^\nu$ can be directly computed from (6.1). Clearly, (6.5) is itself a fixed point equation, and the Fréchet derivative of the operator on the right-hand side of (6.5) is exactly B_1, whose nonzero eigenvalues are ν-th powers of eigenvalues of $\Phi'(x)$. Hence, we may apply a k-step method of the form (1.2) or (1.6) to (6.5). The relevant condition on the related function p is now

$$\{\lambda^\nu:\ \lambda\in\sigma(\Phi'(x))\}\subseteq S_\eta(p)\qquad\text{for some } \eta\in(1,\hat{\eta}(p)]. \tag{6.6}$$

Clearly, if $\rho(\Phi'(x))>1$, $\rho(B_1)$ is even larger, but this does not imply that the convergence factor of the k-step method associated with (6.5) is necessarily worse than the convergence factor of the method associated with (6.1). In fact, in some important examples (cf. § 7), this convergence factor is the ν-th power of the one for the latter method, and hence better, even though the computational effort in one step costs about the same for either method.

Our second proposal is based on applying a k-step method of the form (1.2) or (1.6) to the whole system (6.1). However, since we know that the spectrum of Φ' is ν-fold rotationally symmetric, we restrict our attention to functions p in (2.2) which (for some $\eta>1$) map D_η onto a region with this symmetry: these functions are of the form

$$p(z)=\frac{\mu_0 z}{1-\mu_\nu z^\nu-\mu_{2\nu}z^{2\nu}-\ldots-\mu_k z^k} \tag{6.7}$$

with $k=\nu l$. Application of the associated k-step method to (6.1) yields the iteration

$$\begin{aligned} y_m^1&:=\mu_0\varphi_1(y_{m-1}^\nu)+\mu_\nu y_{m-\nu}^1+\ldots+\mu_{l\nu}y_{m-l\nu}^1\\ y_m^2&:=\mu_0\varphi_2(y_{m-1}^1)+\mu_\nu y_{m-\nu}^2+\ldots+\mu_{l\nu}y_{m-l\nu}^2,\\ &\ \ \vdots\\ y_m^\nu&:=\mu_0\varphi_\nu(y_{m-1}^{\nu-1})+\mu_\nu y_{m-\nu}^\nu+\ldots+\mu_{l\nu}y_{m-l\nu}^\nu \end{aligned} \tag{6.8}$$

$(m=l\nu, l\nu+1,\ldots)$. Let us consider ν steps of iteration (6.8) with iteration index $m, m+1,\ldots,m+\nu-1$. From each of the corresponding iteration vectors $y_m, y_{m+1},\ldots,y_{m+\nu-1}$, we extract one subvector in the following way:

$$\begin{aligned} y^1_m &:= \mu_0 \varphi_1(y^\nu_{m-1}) + \mu_\nu y^1_{m-\nu} + \dots + \mu_{l\nu} y^1_{m-l\nu}, \\ y^2_{m+1} &:= \mu_0 \varphi_2(y^1_m) + \mu_\nu y^2_{m+1-\nu} + \dots + \mu_{l\nu} y^2_{m+1-l\nu}, \\ &\;\vdots \\ y^\nu_{m+\nu-1} &:= \mu_0 \varphi_\nu(y^{\nu-1}_{m+\nu-2}) + \mu_\nu y^\nu_{m-1} + \dots + \mu_{l\nu} y^\nu_{m+\nu-1-l\nu} \end{aligned} \tag{6.9}$$

$(m = l\nu, l\nu+\nu, \dots)$. It can be seen directly from (6.9) that this sequence of subvectors can be computed without using any other subvector of the iterates $y_m, y_{m+1}, \dots, y_{m+\nu-1}$. Thus, it becomes clear that once the vectors

$$y^1_{m-j\nu}, y^2_{m-j\nu+1}, \dots, y^\nu_{m-j\nu+\nu-1} \qquad (j = 1, \dots, l)$$

are known, one sweep through (6.9) produces

$$y^1_m, y^2_{m+1}, \dots, y^\nu_{m+\nu-1}. \tag{6.10}$$

Now, let us assume that, under some assumptions on the function p of (6.7) and on $\sigma(\Phi'(x))$, Theorem 2 predicts a certain linear root-convergence factor $\kappa \leqq 1/\eta$ for iteration (6.8). Then, one iteration of (6.9) has a asymptotically the *same* effect as ν iterations of (6.8), i.e., under the same assumptions on p and $\sigma(\Phi'(x))$ as above, iteration (6.9) converges locally with a root-convergence factor $\kappa^\nu \leqq 1/\eta^\nu$. Remember here that one iteration of (6.9) needs about the same computational effort as one iteration of (6.8). We have thus established

Theorem 5. *Under the assumptions of Theorem* 2, *if p is of the special form* (6.7), *the iteration* (6.9) *for the cyclic system of index* ν (6.1) *converges locally and with the linear root-convergence factor* $\kappa^\nu \leqq 1/\eta^\nu$.

In analogy with Theorem 3 the result remains valid for any nonstationary variant of (6.9) satisfying (1.6), and it is also clear that the global convergence result of Theorem 4 still holds.

The iteration (6.9) has the important advantage that the function φ_j is evaluated at y^{j-1}_{m+j-2} (which is supposed to converge to x^{j-1}). In contrast, in the k-step method for (6.5), φ_j is evaluated at

$$\hat{y}^{j-1}_m := (\varphi_{j-1} \circ \varphi_{j-2} \circ \dots \circ \varphi_2)(y^1_m),$$

and though this point is also supposed to converge to x^{j-1}, it may be far from x^{j-1}, since $\hat{y}^{j-1}_m$ can be thought of as a result of $j-2$ applications of the map Φ, which may be expansive.

§ 7. Examples

Since we do not know the fixed point x of Φ, we know neither $\Phi'(x)$ nor the eigenvalues of $\Phi'(x)$. But let us assume that two compact sets $\Lambda \subset \mathbb{C}$ and $\Omega \subset \mathbb{C}$ are known with $x \in \Lambda$ and with $\sigma(\Phi'(\tilde{x})) \subseteq \Omega$ for all $\tilde{x} \in \Lambda$. Then, Theorem 2 says that if there is a p of the form (2.2) such that $\Omega \subset S(p)$, our iteration (1.2) converges locally to x, and if $\Omega \subseteq S_\eta(p)$ for some $\eta > 1$, then the convergence factor κ is at most $1/\eta$. Since all further considerations are made with respect to the set Ω, it makes no difference whether the nonlinear iteration (1.2), together with the eigenvalues of $\Phi'(x)$, or the linear iteration (2.1) and the eigenvalues of T, are considered. Thus, we can use the results in [12] where (2.1) was extensively examined.

For deciding whether $\Omega \subset S(p)$ for some p, it is useful to describe $S(p)$ and $S_\eta(p)$ for some specific functions p of the form (2.2). In the case $k=1$, we have $p(z)=\mu_0/(1-(1-\mu_0)z)$, i.e., p is a linear fractional mapping which is univalent for $1<\eta<\infty$. It can be directly seen that $S(p)$ is the disk with center

$$\delta := 1 - 1/\mu_0, \tag{7.1}$$

and radius $1/|\mu_0|$ (which has 1 on its boundary); $S_\eta(p)$ is the concentric closed disk with radius $1/(\eta|\mu_0|)$. Conversely, if $D(\delta, \rho)$ is a closed disk with center $\delta \in \mathbb{C}$ and radius $\rho > 0$ such that $\Omega \subseteq D(\delta, \rho)$ and $1 \notin D(\delta, \rho)$, then we can look for a p of the form (2.2) with $k=1$ such that $S_\eta(p)=D(\delta, \rho)$. From (7.1), we get $\mu_0 := 1/(1-\delta)$; i.e., method (1.2) with $k=1$ and this value of μ_0 *converges locally*. The convergence factor is at most $\rho/|1-\delta|$ (namely, the quotient of the radii ρ of $D(\delta, \rho)$ and $1/|\mu_0|$ of $S(p)$). Of course, as there are in general an infinite number of such disks $D(\delta, \rho) \supseteq \Omega$, then one wishes to minimize the convergence factor $\rho/|1-\delta|$. In a different context and with a slightly different notation, this was done in Opfer and Schober [14] for the case where Ω is a line segment or an ellipse.

If $k=2$,

$$(1/p)(z) = \frac{1-\mu_1 z - \mu_2 z^2}{\mu_0 z} = \frac{1}{\mu_0}\left(\frac{1}{z} - \mu_1 - \mu_2 z\right) \tag{7.2}$$

is a mapping of the Joukowski type. The following results have been derived in [12]: p is univalent in a neighborhood of the unit disk iff $|\mu_2|<1$; furthermore $\hat{\eta}(p)=1/|\mu_2|$ and $S(p)$ is the interior of an ellipse E with $1 \in E$ and foci α, β with

$$\alpha, \beta = (-\mu_1 \pm 2\sqrt{-\mu_2})/\mu_0. \tag{7.3}$$

For $1<\eta<\hat{\eta}$, the region $S_\eta(p)$ is the closed interior of a confocal ellipse E_η within E, and $S_{\hat{\eta}}(p)$ is the interval between the foci α and β. If the set E_η is given, the value of η has to be determined as follows:

If z is an arbitrary point on E_η, solve the quadratic equation $\zeta = 1/p(z)$ for z. There is a solution with $1<|z| \leq \hat{\eta}$ which yields $\eta = |z|$.

Conversely, let us assume that there is an ellipse E' such that the closed interior $\bar{E}'$ contains Ω and $1 \notin \bar{E}'$. Then, we wish to find parameters μ_0, μ_1, μ_2 such that for the corresponding p, there holds $\bar{E}' = S_\eta(p)$ for some η with $1<\eta<\hat{\eta}(p)$. Let α and β be the foci of E', and set

$$\theta^{\pm} := \frac{(\sqrt{1-\alpha} \pm \sqrt{1-\beta})^2}{\beta - \alpha}, \tag{7.4}$$

so that $\theta^{+} = 1/\theta^{-}$. Let θ be that value with $|\theta|>1$. Then with $\gamma := (\alpha-\beta)/2$, $\delta := (\alpha+\beta)/2$, we get (see [12], formula (7.3))

$$\mu_0 = \frac{2}{\gamma\theta}, \quad \mu_1 = \frac{-2\delta}{\gamma\theta}, \quad \mu_2 = \frac{-1}{\theta^2}. \tag{7.5}$$

The value of η can be determined as mentioned above. It should be noted that, for all confocal ellipses E' with $1 \notin \bar{E}'$, we get by (7.5) the same iteration parameters μ_0, μ_1, μ_2; only the value of η will be different.

The set Ω represents our information on the spectrum of T or $\Phi'(x)$, respectively. If $\Omega = S_\eta(p)$ for some p of the form (2.2), then from [12] if follows that with respect to this information, the resulting convergence factor is minimal, i.e., we get an *optimal asymptotic rate of convergence*. In the linear positive definite case, i.e., $\Phi(y) = Ty + c$ with a symmetric matrix T such that $I - T$ is positive definite, Ω is an interval on the real line not containing 1. Here, Golub and Varga [3] have shown that the weights $\mu_l^{(m)}$, $0 \leqq l \leqq 2 = k$, derived from the recursion formula for appropriately shifted Chebyshev polynomials yield a method with optimal *average* rate of convergence [20]. For the related complex linear case where Ω is a complex interval, Wrigley [22] and Manteuffel [9] have considered the corresponding method resulting from suitably shifted Chebyshev polynomials. There results a method of type (1.5) with $k = 2$, which however has not an optimal average rate, but only an optimal asymptotic rate of convergence (as the stationary method described in [12] and outlined above).

Numerical results on the application of the 2-step method associated to a degenerate ellipse $\hat{E}$ with foci $\pm i\alpha$ and of the related iteration (6.9) (which happens to be a nonlinear SOR iteration) to a highly nonlinear system of between 16 and 128 equations are reported in [5, Examples 2.1 and 3.1]. In this application it turned out that both methods not only behaved as predicted, but that the effect of the nonlinearity on the convergence speed is small, except that the iteration may not converge at all if the initial approximation is too bad.

References

1. Bittner, L.: Über ein mehrstufiges Verfahren zur Lösung von linearen Gleichungen. Numer. Math. **6**, 161–180 (1964)
2. Gekeler, E.: Über mehrstufige Iterationsverfahren und die Lösung der Hammersteinschen Gleichung. Numer. Math. **19**, 351–360 (1972)
3. Golub, G.H., Varga, R.S.: Chebyshev Semiiterative Methods, Successive Overrelaxation Iterative Methods, and Second Order Richardson Iterative Methods. Numer. Math. **3**, 147–168 (1961)
4. Gutknecht, M.H.: Solving Theodorsen's Integral Equation for Conformal Maps with the Fast Fourier Transform and Various Nonlinear Iterative Methods. Numer. Math. **36**, 405–429 (1981)
5. Gutknecht, M.H.: Numerical Experiments on Solving Theodorsen's Integral Equation for Conformal Maps with the Fast Fourier Transform and Various Nonlinear Iterative Methods. SIAM J. Sci. Stat. Comput. **4**, 1–30 (1983)
6. Gutknecht, M.H., Kaiser, A.: Iterative *k*-Step Methods for Computing Possibly Repulsive Fixed Points in Banach Spaces. J. Math. Anal. Appl. (To appear)
7. Kublanovskaja, V.N.: Application of Analytic Continuation in Numerical Analysis by Means of Change of Variables. Trudy Mat. Inst. Steklov **53**, 145–185 (1959)
8. Mann, R.W.: Averaging to Improve Convergence of Iterative Processes. Functional Analysis Methods in Numerical Analysis. Lect. Notes in Math. 701, pp. 169–179. Berlin, Heidelberg, New York: Springer 1979
9. Manteuffel, T.A.: The Tchebychev Iteration for Nonsymmetric Linear Systems. Numer. Math. **28**, 307–327 (1977)
10. Niethammer, W.: Iterationsverfahren und allgemeine Euler-Verfahren. Math. Z. **102**, 288–317 (1967)

11. Niethammer, W.: Konvergenzbeschleunigung bei einstufigen Iterationsverfahren durch Summierungsmethoden. Iterationsverfahren, Numerische Mathematik, Approximationstheorie, pp. 235-243. Basel: Birkhäuser 1970
12. Niethammer, W., Varga, R.S.: The Analysis of k-Step Iterative Methods for Linear Systems from Summability Theory. Numer. Math. **41**, 177-206 (1983)
13. Niethammer, W., de Pillis, J., Varga, R.S.: Convergence of Block Iterative Methods Applied to Sparse Least-Squares Problems. Linear Algebra Appl. **58**, 327-341 (1984)
14. Opfer, G., Schober, G.: Richardson's Iteration for Nonsymmetric Matrices. Linear Algebra Appl. **58**, 343-361 (1984)
15. Ortega, J.M., Rheinboldt, W.C.: Iterative Solution of Nonlinear Equations in Several Variables. New York: Academic Press 1970
16. Ostrowski, A.M.: Solution of Equations and Systems of Equations. New York: Academic Press 1966
17. Perron, O.: Über Stabilität und asymptotisches Verhalten der Lösungen eines Systems endlicher Differenzengleichungen. J. Reine Angew. Math. **161**, 41-64 (1929)
18. Rjabenki, V.S., Filippow, A.F.: Über die Stabilität von Differenzengleichungen. Berlin: VEB Deutscher Verlag der Wissenschaften 1960
19. Ullrich, Chr.: Über schwach zyklische Abbildungen in nichtlinearen Produkträumen und einige Monotonieaussagen. Apl. Mat. **24**, 209-234 (1979)
20. Varga, R.S.: Matrix Iterative Analysis. Englewood Cliffs: Prentice Hall 1962
21. Voigt, R.G.: Rates of Convergence for a Class of Iterative Procedures. SIAM J. Number Anal. **8**, 127-134 (1971)
22. Wrigley, H.E.: Accelerating the Jacobi Method for Solving Simultaneous Equations by Chebyshev Extrapolation when the Eigenvalues of the Iteration Matrix are Complex. Comput. J. **6**, 159-176 (1963)

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