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Constr. Approx. (1986) 2: 349–368

CONSTRUCTIVE
APPROXIMATION

On the Minimum Moduli of Normalized Polynomials with Two Prescribed Values

Stephan Ruscheweyh and Richard S. Varga

Abstract. With $\mathbf{P}_n$ denoting the set of complex polynomials of degree at most n $(n \geq 1)$, define, for any complex number μ, the subset

$$\mathbf{P}_n(\mu) := \{p_n(z) \in \mathbf{P}_n \colon p_n(0) = 1 \text{ and } p_n(1) = \mu\}.$$

In this paper, we determine exactly the nonnegative quantity

$$S_n(\mu) := \sup_{p_n \in \mathbf{P}_n(\mu)} \{\min_{|z| \leq 1} |p_n(z)|\},$$

as a function of n and μ. For fixed $n \geq 2$, the three-dimensional surface, generated by the points (Re μ, Im μ, $S_n(\mu)$) for all complex numbers μ, has the interesting shape of a volcano.

1. Introduction

Consider the set of all complex polynomials $p_n(z)$ of degree at most n $(n \geq 1)$, taking on two prescribed values (not both zero) in two distinct points in the complex plane:

$$p_n(z_1) = \alpha, \qquad p_n(z_2) = \beta \qquad (z_1 \neq z_2;\ \alpha \neq 0). \tag{1.1}$$

Then, what can be said about the supremum of

$$\min\{|p_n(z)| \colon |z - z_1| \leq |z_2 - z_1|\}, \tag{1.2}$$

over all polynomials $p_n(z)$ satisfying (1.1)? This problem can be normalized as follows. With $\mathbf{P}_n$ denoting the set of all complex polynomials of degree at most n $(n \geq 1)$, then for each complex number μ, consider the subset of $\mathbf{P}_n$, defined by

$$\mathbf{P}_n(\mu) := \{p_n(z) \in \mathbf{P}_n \colon p_n(0) = 1 \text{ and } p_n(1) = \mu\}. \tag{1.3}$$

Our objective in this paper is to determine precisely the nonnegative quantity

$$S_n(\mu) := \sup_{p_n \in \mathbf{P}_n(\mu)} \{\min_{|z| \leq 1} |p_n(z)|\}, \tag{1.4}$$

as a function of n and μ.

Date received: July 24, 1985. Date revised: May 27, 1986. Communicated by Peter Henrici.
AMS classification: 30A10, 30C10.
Key words and phrases: Minimum modulus of polynomials, Zero-finding algorithm for complex polynomials, Chebyshev polynomials, Volcano function.

Our interest in this question arose from the following related problem which is crucial in the study of certain *global descent methods* for finding a zero of a given complex polynomial (cf. Henrici [1], [2], Ruscheweyh [8], and references contained therein): For each $n \geq 1$, set

$$\Gamma_n := \max\left\{\min_{|z|\leq 1} |p_n(z)|: p_n(z) = 1 + \sum_{j=1}^{n} a_j z^j \text{ with } \sum_{j=1}^{n} |a_j| = 1\right\}. \tag{1.5}$$

From the results of Ruscheweyh [8], and Ruscheweyh and Varga [9], it is known that Γ_n satisfies the inequalities

$$1 - \frac{1}{n} \leq \Gamma_n \leq \sqrt{1 - \left(\frac{1}{n}\right)} < 1 - \frac{1}{2n}, \tag{1.6}$$

for every $n \geq 1$. Moreover, with

$$\tilde{\Gamma}_n := \max\{\min_{|z|\leq 1} |p_n(z)|: p_n(z) \in \mathbf{P}_n(2) \text{ with } p_n^{(j)}(0) \geq 0 \text{ for } j = 1, 2, \ldots, n\}, \tag{1.7}$$

it is further known from [9] that

$$1 - \frac{1}{n} \leq \tilde{\Gamma}_n \leq \sqrt{1 - (3/(2n+1))} = 1 - \frac{3}{4n} + o\left(\frac{1}{n}\right) \qquad (n \to \infty), \tag{1.8}$$

for every $n \geq 1$. In [9], we first conjectured that

$$\Gamma_n = \tilde{\Gamma}_n \qquad (n = 1, 2, \ldots). \tag{1.9}$$

Now, the bounds (1.6) and (1.8) suggested that there exists a positive constant γ, independent of n, such that

$$\Gamma_n = \tilde{\Gamma}_n = 1 - \frac{\gamma}{n} + o\left(\frac{1}{n}\right) \qquad (n \to \infty). \tag{1.10}$$

Indeed, extended precision calculations described in [9] led us to further conjecture that γ in (1.10) is approximately given by

$$\gamma \doteq 0.867\,189\,051. \tag{1.11}$$

By definition, we note that $\tilde{\Gamma}_n \leq S_n(2)$ for all $n \geq 1$. Interestingly enough, one consequence of this present paper (cf. (2.12) of Corollary 1) is that

$$S_n(2) = \tilde{\Gamma}_n = 1 - \frac{1}{2n}\{\operatorname{arccosh}(2)\}^2 + o\left(\frac{1}{n}\right) \qquad (n \to \infty), \tag{1.12}$$

so that the quantity γ of (1.10) is given *exactly* by

$$\gamma = \frac{\{\operatorname{arccosh}(2)\}^2}{2} = 0.867\,189\,051\,136\,318\,1\ldots. \tag{1.13}$$

Thus, our conjecture of (1.11) (to the number of digits given in (1.11)) is *correct*! The conjecture (1.9), however, remains open.

The outline of our paper is as follows. In Section 2, we state our main results concerning the determination of $S_n(\mu)$ of (1.4). As we shall see, for fixed $n \geq 2$, the three-dimensional surface, generated by the points $(\operatorname{Re}\mu, \operatorname{Im}\mu, S_n(\mu))$ for all complex numbers μ, has the interesting shape of a *volcano*. A computer-generated picture of this for the case $n=5$ is given in Fig. 1 of Section 2. Finally, the proof of Theorem 1 of Section 2 is given in Section 3, while the proofs of Corollaries 1 and 2 of Section 2 are given in Section 4.

2. Statement of Results

For any $\mu \in \mathbf{C}$, the minimum principle, applied to any $p_n(z)$ in $\mathbf{P}_n(\mu)$ (cf. (1.3)), directly gives us that $\min_{|z|\leq 1}|p_n(z)| \leq \min\{1; |\mu|\} \leq |\mu|$, whence (cf. (1.4))

$$0 \leq S_n(\mu) \leq \min\{1; |\mu|\} \leq |\mu| \qquad (n=1,2,\ldots;\ \text{all } \mu \in \mathbf{C}). \tag{2.1}$$

Furthermore, standard arguments show that the supremum in (1.4) is, in fact, a maximum. Moreover, by an elementary variation in the elements of $\mathbf{P}_n(\mu)$, it can be seen that $S_n(\mu)$ is continuous in $\mathbf{C}$.

For our subsequent analysis, it is necessary to distinguish between the following pairwise disjoint sets of $\mathbf{C}$:

$$\Delta_n := \{\mu \in \mathbf{C}\colon S_n(\mu) = |\mu|\}, \tag{2.2}$$

$$\Omega_n := \{\mu \in \mathbf{C}\colon 0 < S_n(\mu) < |\mu|\}, \tag{2.3}$$

$$\Sigma_n := \{\mu \in \mathbf{C}\colon 0 = S_n(\mu) < |\mu|\}, \tag{2.4}$$

where we note that $\Delta_n \cup \Omega_n \cup \Sigma_n = \mathbf{C}$. With $\mathbf{D} := \{z \in \mathbf{C}\colon |z| < 1\}$ and with $\bar{\mathbf{D}}$ denoting the closure of $\mathbf{D}$, it is evident from (2.1) that $\Delta_n \subset \bar{\mathbf{D}}$. Moreover, from (1.4) and (2.2), it can be verified that Δ_n has another interesting interpretation: $\mu \in \Delta_n$ iff there is a $p_n(z) \in \mathbf{P}_n$ with $p_n(0) = 1$ such that

$$\mu \in p_n(\bar{\mathbf{D}}) \quad \text{and} \quad |\mu| = \min_{|z|\leq 1}|p_n(z)|. \tag{2.5}$$

From this observation, it is somewhat surprising, as we shall see, that substantial parts of $\bar{\mathbf{D}}$ do *not* belong to Δ_n.

For additional needed notation, for each ρ with $0<\rho<1$ and for each positive integer n, set

$$Q_{n,\rho}(z^2) := \frac{-\rho}{(n+1)} z^{2n+3} \frac{d}{dz}\left\{z^{-(n+1)} T_{n+1}\left[\rho^{-1/(n+1)}\left(\frac{1+z^2}{2z}\right)\right]\right\}, \tag{2.6}$$

where $T_{n+1}(z)$ denotes the Chebyshev polynomial of degree $n+1$ of the first kind. As can be seen (cf. (4.6)), $Q_{n,\rho}(w) \in \mathbf{P}_n$.

With the above notation, our main result is

Theorem 1. *The following are valid*:

(a) *The set* Σ_n *satisfies*

$$\Sigma_n = \mathbf{C} \backslash \{z \in \mathbf{C}\colon z = (1+w)^n \text{ where } w \in \mathbf{D} \cup \{-1\}\}; \tag{2.7}$$

(b) $\Delta_1=[0,1]$. *For* $n\geq 2$, Δ_n *consists of the Jordan curve*

$$C_n := \left\{ z = -\left(\frac{\cos(\varphi/(n+1))}{\cos(\pi/(n+1))}\right)^{n+1} \cdot e^{i\varphi} : \pi \leq |\varphi| \leq 2\pi \right\}, \tag{2.8}$$

and the bounded Jordan domain having C_n *as its boundary;*

(c) $\Omega_n = \mathbf{C}\backslash(\Sigma_n \cup \Delta_n)$. *Moreover, for any* $\mu \in \Omega_n$, *there holds*

$$S_n(\mu) = \max\{0<\rho<1 : \mu \in Q_{n,\rho}(\bar{\mathbf{D}})\}. \tag{2.9}$$

When $\mu>0$, the quantity $S_n(\mu)$ can be given in the following more explicit fashion, which directly connects with the problem arising from global descent methods mentioned in Section 1.

Corollary 1. *Let* $\mu>0$. *Then, there holds:*

$$S_n(\mu) = \begin{cases} \mu & \text{if } 0<\mu\leq 1; \\ \sigma & \text{if } 1<\mu<2^n; \\ 0 & \text{if } 2^n \leq \mu. \end{cases} \tag{2.10}$$

Here, σ *is the uniquely determined solution in* $(0,1)$ *of the equation*

$$\mu = \sigma T_{n+1}\{\sigma^{-1/(n+1)}\}. \tag{2.11}$$

For n tending to infinity, the solution σ *of* (2.11) *can be expressed as*

$$\sigma = 1 - \frac{\{\operatorname{arccosh}(\mu)\}^2}{2n} + o\left(\frac{1}{n}\right) \quad (n\to\infty). \tag{2.12}$$

Finally, for $\mu \in (1, 2^n)$ *and for* σ *defined by* (2.11), $Q_{n,\sigma}(z)$ *is the unique extremal polynomial in* $\mathbf{P}_n(\mu)$ *for the problem* (1.4), *and* $Q_{n,\sigma}(z)$, *when expanded in powers of z, has positive coefficients.*

Another consequence of Theorem 1, of a more general nature, is the following:

Corollary 2. *For any* $p_n(z) \in \mathbf{P}_n$ *with* $p_n(z) \neq 0$ *in* $\bar{\mathbf{D}}$, *there holds*

$$M \leq m T_{n+1}\left(\left(\frac{|p_n(0)|}{m}\right)^{1/(n+1)}\right), \tag{2.13}$$

where

$$M := \max_{|z|\leq 1} |p_n(z)|, \qquad m := \min_{|z|\leq 1} |p_n(z)|. \tag{2.14}$$

The upper bound (2.13) *is sharp for each* $m \in (0, |p_n(0)|)$.

To conclude this section, we illustrate the result of Theorem 1, in the case $n=5$, with the computer-generated three-dimensional surface $(\operatorname{Re}\mu, \operatorname{Im}\mu, S_5(\mu))$, given in Fig. 1. It is evident that this surface has the shape of a *volcano*. Specifically, the set Δ_5, defined in (2.2), corresponds geologically to the *caldera* (i.e., the deep caldronlike cavity found at the summit of a volcano), while the sides of the volcano correspond to the set Ω_5, defined in (2.3).

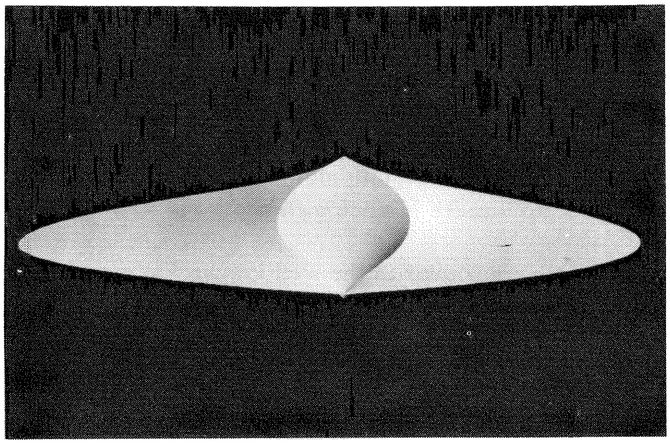

Fig. 1

3. Proof of Theorem 1

As the proof of Theorem 1 is somewhat involved, we give for convenience some intermediate results in the form of propositions. Some parts of our proof are reminiscent of the general theory of extremal problems for complex polynomials as derived, for example, in Rivlin and Shapiro [6]. However, that theory as a whole does not apply to our problem of maximizing the minimum modulus, and even similar tools need new justifications in our case. For the sake of completeness and readability, we also indicate proofs for those (small) portions which could have been transferred from the above-mentioned theory.

We start with the determination of the set Σ_n of (2.4). Clearly, $\mu \in \Sigma_n$ if and only *if* $\mu \neq 0$ and every polynomial in $\mathbf{P}_n(\mu)$ has a zero in $\bar{\mathbf{D}}$.

Proposition 1. *Let $p_n(z) \in \mathbf{P}_n$ satisfy $p_n(0) = 1$ with $p_n(z) \neq 0$ for any $z \in \mathbf{D}$. Then, $p_n(\mathbf{D}) \subset w_n(\mathbf{D})$, where $w_n(z) := (1+z)^n$.*

Proof. Simply apply Szegö's corollary (16, 1c) of Marden [3, p. 67] with $f(z) := (1+z)^n$ and with $g(z) := p_n(z)$. Another proof of this also follows from Theorems 1.1 and 1.5 of [7]. ∎

Now, consider any $\mu \in \mathbf{C} \backslash \Sigma_n$, with $\mu \neq 0$. Then, there exists a polynomial $p_n(z) \in \mathbf{P}_n(\mu)$ with $p_n(z) \neq 0$ in $\bar{\mathbf{D}}$, and for small $\varepsilon > 0$, we have $\tilde{p}_n(z) := p_n((1+\varepsilon)z) \neq 0$ in $\mathbf{D}$. Since $\tilde{p}_n(0) = 1$, we see from Proposition 1 that

$$\mu = \tilde{p}_n\left(\frac{1}{1+\varepsilon}\right) \in w_n(\mathbf{D}). \tag{3.1}$$

Conversely, if $\mu \in w_n(\mathbf{D})$, then $\mu = (1+z_0)^n$ for a certain $z_0 \in \mathbf{D}$, and $(1+z_0 z)^n$ is an element of $\mathbf{P}_n(\mu)$ with nonvanishing minimum modulus in $\bar{\mathbf{D}}$. This implies $S_n(\mu) > 0$, and thus $\mu \notin \Sigma_n$ This completes the proof of part (a) of Theorem 1.

We turn to the proof of part (c) of Theorem 1, and assume $\mu \in \Omega_n$. As we mentioned at the beginning of Section 2, there exists an extremal polynomial $E_n(z) \in \mathbf{P}_n(\mu)$ with $S_n(\mu) = \min_{|z| \leq 1} |E_n(z)|$. Since μ and n are fixed for this proof, we write $S_n(\mu) = s$. Let F denote the set of all numbers $\theta \in [0, 2\pi)$ with $|E_n(e^{i\theta})| = s$. Let $e(\theta) := |E_n(e^{i\theta})|^2$, $\theta \in \mathbf{R}$. Then, $e(\theta)$ is a trigonometric polynomial of degree at most n with s^2 as its absolute minimum. It is therefore clear that F has at most n elements. Note that $0 \notin F$ since $e(0) = |E_n(1)|^2 = |\mu|^2 > s^2$ by assumption (cf. (2.3)).

The next proposition is similar to the well-known Kolmogorov theorem from approximation theory (cf. Meinardus [4, p. 13]).

Proposition 2. *There is no polynomial $U_n(z) \in \mathbf{P}_n$ with $U_n(0) = 0$ and $U_n(1) = 0$, such that*

$$\operatorname{Re}[\overline{E_n(e^{i\theta})}\, U_n(e^{i\theta})] > 0 \qquad (\theta \in F). \tag{3.2}$$

Proof. Assume that there is such a polynomial $U_n(z)$. Then, there exists a $\gamma > 0$ such that

$$2 \operatorname{Re}[\overline{E_n(e^{i\theta})}\, U_n(e^{i\theta})] > \gamma \qquad (\theta \in I),$$

where

$$I := \{\theta \in (0, 2\pi) \text{: there exists a } \theta_j \in F \text{ with } |\theta - \theta_j| < \sigma\},$$

for a certain $\sigma > 0$. Hence, for $\varepsilon > 0$, we find

$$|E_n(e^{i\theta}) + \varepsilon U_n(e^{i\theta})|^2 \geq s^2 + \varepsilon\gamma > s^2 \qquad (\theta \in I). \tag{3.3}$$

For the compact set $I' := [0, 2\pi] \backslash I$, there exists a $\delta > 0$ such that

$$|E_n(e^{i\theta})|^2 \geq s^2 + \delta \qquad (\theta \in I').$$

Now, choose ε so small that

$$\varepsilon |2 \operatorname{Re}[\overline{E_n(e^{i\theta})}\, U_n(e^{i\theta})]| + \varepsilon^2 |U_n(e^{i\theta})|^2 < \delta \qquad (\theta \in I').$$

Then on I', we have

$$|E_n(e^{i\theta}) + \varepsilon U_n(e^{i\theta})|^2 > s^2 + \delta - \delta = s^2 \qquad (\theta \in I'). \tag{3.4}$$

For small $\varepsilon > 0$, the polynomial $E_n(z) + \varepsilon U_n(z)$ is nonvanishing in $\bar{\mathbf{D}}$, and hence, (3.3), (3.4), and the minimum principle imply $\min_{|z| \leq 1} |E_n(z) + \varepsilon U_n(z)| > s$. But since $(E_n(z) + \varepsilon U_n(z)) \in \mathbf{P}_n(\mu)$, we have a contradiction to the definition of s. ■

Assume next that F has less than n elements. Then by interpolation, we can construct a polynomial $U_n(z) \in \mathbf{P}_n$ with

$$U_n(0) = 0, \qquad U_n(1) = 0, \qquad U_n(e^{i\theta}) = E_n(e^{i\theta}) \quad \text{for} \quad \theta \in F.$$

(Note that $0 \notin F$.) But then, $U_n(z)$ contradicts Proposition 2, and we thus conclude that F has precisely n elements in $(0, 2\pi)$, say

$$0 < \theta_1 < \cdots < \theta_n < 2\pi.$$

Proposition 3. *There exists an* $\alpha \in \mathbf{C}$ *with* $|\alpha| = 1$, *such that*

$$\alpha \exp\left[-i\left(\frac{n+1}{2}\right)\theta_j\right]E_n(e^{i\theta_j}) = (-1)^j s \qquad (j = 1, \ldots, n). \tag{3.5}$$

Proof. As the assertion (3.5) is obvious for $n = 1$, assume $n > 1$. Let p be an integer with $1 \le p \le n-1$, and let $q := p+1$. Let $\varepsilon > 0$ and $\sigma = e^{i\varphi}$ with $|\varphi| < \pi/2$. By interpolation, we can find a polynomial $U_{\varepsilon,\sigma}(z)$ in $\mathbf{P}_n$ with

$$\begin{cases} U_{\varepsilon,\sigma}(0) = 0, \qquad U_{\varepsilon,\sigma}(1) = (0), \\ U_{\varepsilon,\sigma}(e^{i\theta_j}) = \varepsilon E_n(e^{i\theta_j}) \qquad \text{for } j = 1, \ldots, n; \quad j \ne p, q, \\ U_{\varepsilon,\sigma}(e^{i\theta_q}) = \sigma E_n(e^{i\theta_q}). \end{cases} \tag{3.6}$$

Using $U_{\varepsilon,\sigma}(z)$ in Proposition 2, we necessarily find that

$$\mathrm{Re}[\overline{E_n(e^{i\theta_p})}\, U_{\varepsilon,\sigma}(e^{i\theta_p})] \le 0. \tag{3.7}$$

Now, let $\varepsilon \to 0$. Then, $U_{\varepsilon,\sigma}(z) \to U_{0,\sigma}(z)$, uniformly in $\bar{\mathbf{D}}$, and the zeros of $U_{0,\sigma}(z)$ occur precisely in the points $0, 1, e^{i\theta_j}$ for $j = 1, \ldots, n; j \ne p, q$. This implies that the polynomials $U_{0,\sigma}(z)$ differ only by a factor, namely $U_{0,\sigma}(z) = \sigma U_{0,1}(z)$. Of course, (3.7) holds also for $\varepsilon = 0$, and we have

$$\mathrm{Re}[\overline{E_n(e^{i\theta_p})}\sigma U_{0,1}(e^{i\theta_p})] \le 0$$

for *every admissable* σ. Also, note that both of the numbers $E_n(e^{i\theta_p})$ and $U_{0,1}(e^{i\theta_p})$ are different from zero. Taking this into account, we obtain

$$\overline{E_n(e^{i\theta_p})}\, U_{0,1}(e^{i\theta_p}) < 0. \tag{3.8}$$

Now, $U_{0,1}(z)$ has the representation

$$U_{0,1}(z) = cz(z-1) \prod_{j=1}^{n}{}' (z - e^{i\theta_j}),$$

where c is some nonzero constant, and where the prime indicates that $j = p$ and $j = q$ are to be omitted in the above product. Writing

$$\gamma := \frac{1}{2} \sum_{j=1}^{n}{}' \theta_j,$$

we obtain for $k \in \{p; q\}$ that

$$U_{0,1}(e^{i\theta_k}) = c \exp\left[i\left(\frac{n+1}{2}\right)\theta_k\right](2i)^{n-1} e^{i\gamma} \sin\left(\frac{\theta_k}{2}\right) \prod_{j=1}^{n}{}' \sin\left(\frac{\theta_k - \theta_j}{2}\right). \tag{3.8'}$$

Setting $k = p$ and $k = q$, note that the corresponding factors from the last term in (3.8′), namely $\sin[(\theta_p - \theta_j)/2]$ and $\sin[(\theta_q - \theta_j)/2]$, are always of the same sign since θ_p and θ_q are adjacent. To identify $U_{0,1}(e^{i\theta_q})$, we use (3.8′) and the last relation of (3.6) to deduce that

$$\frac{U_{0,1}(e^{i\theta_p})}{E_n(e^{i\theta_q})} = \frac{U_{0,1}(e^{i\theta_p})}{U_{0,1}(e^{i\theta_q})} = \delta \exp\left[i\left(\frac{n+1}{2}\right)(\theta_p - \theta_q)\right],$$

for a certain $\delta>0$. Now, $E_n(e^{i\theta_k})\exp[-i((n+1)/2)\theta_k]$ has modulus s for $k\in\{p;q\}$, but using (3.8) and the above expression, we obtain

$$\frac{E_n(e^{i\theta_q})\exp\left[-i\left(\frac{n+1}{2}\right)\theta_q\right]}{E_n(e^{i\theta_p})\exp\left[-i\left(\frac{n+1}{2}\right)\theta_p\right]}=-1.$$

An inductive argument with respect to p then completes the proof of Proposition 3. ■

Proposition 4. *Let φ be such that $|E_n(e^{i\theta})|$ attains its global maximum (minimum) at φ. Then,*

$$\frac{e^{i\varphi}E_n'(e^{i\varphi})}{E_n(e^{i\varphi})}\ \begin{matrix}>\\(<)\end{matrix}\ 0.$$

In particular, $E_n'(e^{i\varphi})\neq 0$.

Proof. This is a consequence of the Julia–Wolff theorem in Pommerenke [5, p. 306]. ■

Proposition 5. *$E_n'(z)$ has all its zeros on $|z|=1$. If these zeros are $e^{i\psi_j}$ with $\psi_j\in[0,2\pi)$ in increasing order, then*

$$0<\theta_1<\psi_1<\theta_2<\psi_2<\cdots<\psi_{n-1}<\theta_n<2\pi. \tag{3.9}$$

Furthermore,

$$\alpha E_n'(z)=\bar{\alpha}z^{n-1}\overline{E_n'(1/\bar{z})}\qquad(z\in\mathbf{C}), \tag{3.10}$$

where α is as in Proposition 3.

Proof. Since $|E_n(e^{i\theta})|$ attains its global minimum in each of the points θ_j, Proposition 4 can be applied to each of these points. Using Proposition 3, this implies that

$$g(\theta):=\operatorname{Im}\left[\alpha\exp\left[-i\left(\frac{n-1}{2}\right)\theta\right]E_n'(e^{i\theta})\right]$$

vanishes in the n points $\{\theta_j\}_{j=1}^n$ in $(0,2\pi)$. Now, $G(\theta):=g(2\theta)$ is a trigonometric polynomial of degree $n-1$, which vanishes in the n points $\{\theta_j/2\}_{j=1}^n$ in $(0,\pi)$. But, as $G(\theta+\pi)=(-1)^{n-1}G(\theta)$, then $G(\theta)$ has $2n$ zeros in $(0,2\pi)$. Hence, $G(\theta)\equiv 0\equiv g(\theta)$. This shows that

$$\alpha\exp\left[-i\left(\frac{n-1}{2}\right)\theta\right]E_n'(e^{i\theta}) \tag{3.11}$$

is real-valued. Another application of Propositions 3 and 4 proves that the function of (3.11) has alternating signs in the points θ_j, and thus, it must have a zero ψ_j in each of the intervals (θ_j,θ_{j+1}), $j=1,\dots,n-1$. The points $e^{i\psi_j}$ are therefore the $n-1$ zeros of the polynomial $E_n'(z)\in \mathbf{P}_{n-1}$. Furthermore, since (3.11) is real-valued, we have

$$\alpha\, e^{-i(n-1)\theta}E_n'(e^{2i\theta})=\bar{\alpha}\, e^{i(n-1)\theta}\overline{E_n'(e^{2i\theta})},$$

or, for $|z|=1$,

$$\alpha E_n'(z^2)=\bar{\alpha} z^{2(n-1)}\overline{E_n'(1/\bar{z}^2)}.$$

This clearly extends to all $z\in \mathbf{C}$, and replacing z^2 by z, we obtain (3.10). ∎

Now, let

$$f(\theta):=\operatorname{Re}\left[\alpha \exp\left[-i\left(\frac{n+1}{2}\right)\theta\right]E_n(e^{i\theta})\right].$$

Differentiating the relation

$$2\exp\left[i\left(\frac{n+1}{2}\right)\theta\right]f(\theta)=\alpha E_n(e^{i\theta})+\bar{\alpha}\, e^{i(n+1)\theta}\overline{E_n(e^{i\theta})}$$

with respect to θ and using (3.10), we find that

$$E_n(e^{i\theta})=\bar{\alpha}\exp\left[i\left(\frac{n+1}{2}\right)\theta\right]\left[f(\theta)+\left(\frac{2i}{n+1}\right)f'(\theta)\right], \tag{3.12}$$

and, in particular, since $|\alpha|=1$,

$$e(\theta):=|E_n(e^{i\theta})|^2=f^2(\theta)+\frac{4}{(n+1)^2}(f'(\theta))^2. \tag{3.13}$$

From Proposition 3, we have $f^2(\theta_j)=s^2$, and since $e(\theta_j)=s^2$ by definition, we see from (3.13) that

$$f'(\theta_j)=0, \qquad (j=1,\ldots,n). \tag{3.14}$$

Differentiating (3.13) and (3.12) with respect to θ, we obtain

$$e'(\theta)=2f'(\theta)\cdot h(\theta), \tag{3.15}$$

and

$$E_n'(e^{i\theta})=(n+1)\bar{\alpha}\exp\left[i\left(\frac{n-1}{2}\right)\theta\right]\frac{h(\theta)}{2}, \tag{3.16}$$

where $h(\theta):=f(\theta)+[4/(n+1)^2]f''(\theta)$. Now, $e'(\theta)$ is a trigonometric polynomial of degree n. It has n zeros in the points θ_j, which correspond to the global minima of $e(\theta)$, and it has the $n-1$ zeros ψ_j from Proposition 5 and from (3.15) and (3.16). But, none of these points can be the global maximum of $e(\theta)$ since $E_n'(z)$ vanishes in these points (cf. Proposition 4). Hence, $e'(\theta)$ must have one further zero $\theta_0\in[0,2\pi)$ which corresponds to the global maximum of $e(\theta)$. But, this implies that $e'(\theta)$ has the maximal number of zeros in $[0,2\pi)$, and each zero of

$e'(\theta)$ must be simple and correspond to an extremum of $e(\theta)$. Furthermore, we observe from (3.15) that

$$f'(\theta_0)=0 \tag{3.17}$$

since $h(\theta_0)\neq 0$ from (3.16) and Proposition 4, and that all zeros of $f'(\theta)$ are simple.

Regarding θ_0, we note that it must be located in $[0, \theta_1)$ or in $(\theta_n, 2\pi)$, since the relative extrema of $e(\theta)$ in $[\theta_1, \theta_n]$ can occur only in the points $\{\theta_j\}_{j=1}^{n}$ or $\{\psi_j\}_{j=1}^{n-1}$.

We now turn to the discussion of $f(\theta)$. Our knowledge about the zeros of $f'(\theta)$ tells us that $f(\theta)$ is monotonic in each of the intervals (θ_j, θ_{j+1}), $j=1,\dots,n-1$, and since $f(\theta_j)=-f(\theta_{j+1})=\pm s$ by Proposition 3, we conclude that f oscillates between the values $\pm s$ in $[\theta_1, \theta_n]$. The nonnegative trigonometric polynomial $f^2(\theta)$ of degree $n+1$ has a relative maximum at θ_0 with $f^2(\theta_0)=|E_n(e^{i\theta_0})|^2> |E_n(0|^2=1$ by the maximum principle (cf. (3.13), (3.17), and the definition of θ_0), and relative maxima at θ_j with $f^2(\theta_j)=s^2$, $j=1,\dots,n$. The only minima of $f^2(\theta)$ are its zeros since $f'(\theta)$ has no other zeros than $\theta_0,\dots,\theta_n$. Now, θ_0 is exterior to $[\theta_1, \theta_n]$. This shows (because $s^2<1$) that there are precisely two more points $\varphi_1<\theta_1$, $\varphi_2>\theta_n$ with $0<\varphi_2-\varphi_1<2\pi$ such that $f^2(\varphi_1)=f^2(\varphi_2)=s^2$, and there are zeros of $f^2(\theta)$ in (φ_1, θ_1) and (θ_n, φ_2). In the interval $I:=[\varphi_1, \varphi_2]$ (which is not necessarily in $[0, 2\pi)$ but of length $<2\pi$), the trigonometric polynomial

$$g_1(\theta):=2\frac{f^2(\theta)}{s^2}-1$$

has the following properties:

(i) $|g_1(\theta)|\leq 1, \qquad \theta\in I$;
(ii) $g_1(\theta)$ takes on the value 1 in $n+2$ distinct points of I, including the endpoints of I;
(iii) $g_1(\theta)$ takes on the value -1 in $n+1$ distinct points of I.

Clearly, $g_1(\theta)$ is a trigonometric polynomial of degree $n+1$. Now, let $\varphi_0:=(\varphi_1+\varphi_2)/2-\pi$, and set

$$g_2(\theta):=T_{2n+2}\left(\frac{\cos\left(\frac{\theta-\varphi_0}{2}\right)}{\cos\left(\frac{\varphi_1-\varphi_0}{2}\right)}\right).$$

(We remark that $\cos[(\varphi_1-\varphi_0)/2]=\sin[(\varphi_2-\varphi_1)/4]$. Thus, since $0<\varphi_2-\varphi_1<2\pi$, this term in the denominator above is *positive.*)

Now, $g_2(\theta)$ is also a trigonometric polynomial of degree $n+1$, and also satisfies the above properties (i), (ii), (iii). A simple counting argument shows that $g_1(\theta)-g_2(\theta)$ must have at least $2n+3$ zeros in some interval of length $<2\pi$. Thus, $g_1(\theta)\equiv g_2(\theta)$ and the relation $T_{2n+2}(x)=2T_{n+1}^2(x)-1$ gives finally that

$$f(\theta)=\varepsilon s T_{n+1}\left(\sigma\cos\left(\frac{\theta-\varphi_0}{2}\right)\right), \tag{3.18}$$

with $\sigma := 1/\cos[(\varphi_1 - \varphi_0)/2]$ and $\varepsilon \in \{-1; 1\}$. Inserting this into (3.12), we obtain

$$(3.19)\qquad E_n(e^{i\theta}) = \varepsilon\bar{\alpha}s \exp\left[i\left(\frac{n+1}{2}\right)\theta\right]\left[T_{n+1}\left(\sigma\cos\left(\frac{\theta-\varphi_0}{2}\right)\right) - \frac{\sigma i}{n+1}\sin\left(\frac{\theta-\varphi_0}{2}\right)T'_{n+1}\left(\sigma\cos\left(\frac{\theta-\varphi_0}{2}\right)\right)\right].$$

Using $z = \exp[i((\theta-\varphi_0)/2)]$ and $\cos((\theta-\varphi_0)/2) = \frac{1}{2}(z+1/z)$ on $|z|=1$, the above can be written as

$$(3.20)\quad E_n(e^{i\varphi_0}z^2) = -\frac{\varepsilon\bar{\alpha}s}{(n+1)}\exp\left[i\left(\frac{n+1}{2}\right)\varphi_0\right]z^{2n+3}\frac{d}{dz}\left\{z^{-(n+1)}T_{n+1}\left[\frac{\sigma}{2}\left(z+\frac{1}{z}\right)\right]\right\},$$

which extends to all $z \in \mathbf{C}$. In particular, a short calculation with (3.20) gives, with (1.3), that

$$E_n(0) = \varepsilon\bar{\alpha}\sigma^{n+1}s\exp\left[i\left(\frac{n+1}{2}\right)\varphi_0\right] = 1,$$

which implies

$$\sigma = s^{-1/(n+1)}, \quad \text{and} \quad \varepsilon\alpha = \exp\left[i\left(\frac{n+1}{2}\right)\varphi_0\right].$$

Recalling the definition of $Q_{n,\rho}(z^2)$ of (2.6), our result of (3.20) can be written as

$$(3.21)\qquad E_n(z) = Q_{n,s}(e^{-i\varphi_0}z); \qquad \mu = Q_{n,s}(e^{-i\varphi_0}).$$

Now, let $0 < \rho \le 1$, and assume that $\mu = Q_{n,\rho}(z_\rho)$, where $z_\rho \in \bar{\mathbf{D}}$. Then, we have $Q_{n,\rho}(z_\rho z) \in \mathbf{P}_n(\mu)$ and, by the minimum principle, we see that

$$S_n(\mu) \ge \min_{|z|\le 1}|Q_{n,\rho}(z_\rho z)| \ge \min_{|z|\le 1}|Q_{n,\rho}(z)|.$$

Hence,

$$(3.22)\qquad S_n(\mu) \ge \max\{\min_{|z|\le 1}|Q_{n,\rho}(z)|: \mu \in Q_{n,\rho}(\bar{\mathbf{D}})\}.$$

On the other hand, (3.21) yields

$$(3.23)\qquad s = S_n(\mu) = \min_{|z|\le 1}|E_n(z)| = \min_{|z|\le 1}|Q_{n,s}(z)|,$$

and $\mu \in Q_{n,s}(\bar{\mathbf{D}})$. Hence, we have equality in (3.22). To complete our proof of part (c) of Theorem 1, it remains to show that

$$(3.24)\qquad \rho = \min_{|z|\le 1}|Q_{n,\rho}(z)| \qquad (0<\rho<1).$$

In view of (3.23), this is true for every ρ which equals $S_n(\mu)$ for a certain $\mu \in \Omega_n$. The interval $(1, 2^n)$ belongs to Ω_n, because of part (a) of Theorem 1 and the fact that $\Delta_n \subset \bar{\mathbf{D}}$. But, $S_n(\mu)$ is continuous on $[1, 2^n]$ with $S_n(1) = 1$, $S_n(2^n) = 0$. Thus,

$S_n(\mu)$ takes every number in $(0, 1)$ as a value in Ω_n, and the proof of part (c) of Theorem 1 is complete. ■

For further considerations, we wish to pinpoint two important facts about $Q_{n,\rho}(e^{i\theta})$ which follow from our previous deductions. We again use the notations of the last proof. We know that $|E_n(e^{i\theta})|^2$ attains its global extrema (1 maximum, n minima) precisely where f' vanishes, and from (3.18), we see that these are the points $\theta=\varphi_0$ and the solutions of

$$T'_{n+1}\left(\sigma\cos\left(\frac{\theta-\varphi_0}{2}\right)\right)=0. \tag{3.25}$$

Since $\sigma>1$, we see that φ_0 must be the global maximum and the remaining n zeros (where $T_{n+1}(x)=\pm 1$ and where x is in $(-1, +1)$) are the minima. From (3.21), we now see that $|Q_{n,s}(e^{i\theta})|$ takes its only global maximum at $\theta=0$. Also, replacing $\theta-\varphi_0$ by θ, we see from (3.21), (3.25) and the determination of σ, that $|Q_{n,\rho}(e^{i\theta})|$ takes its global minima in the points θ satisfying

$$\rho^{-1/(n+1)}\cos\left(\frac{\theta}{2}\right)=\cos\left(\frac{j\pi}{n+1}\right) \qquad (j=1,\ldots,n). \tag{3.26}$$

Here, we have made use of the explicit representation of the zeros of $T'_{n+1}(x)$.

We now turn to the remaining proof of part (b) of Theorem 1. It rests mainly on two simple ideas: to determine a simply connected closed set whose boundary consists of boundary points of Δ_n, and to show that Δ_n is itself simply connected. Although the second part seems to be very natural, it creates the main difficulty in our proof. For the case $n=1$, of course, the assertion that $\Delta_1=[0, 1]$ is easily verified.

Proposition 6. *Let $0\neq\mu\in\partial\Delta_n$. Then, there exists a $\theta\in[0, 2\pi)$ such that*

$$\mu=Q_{n,|\mu|}(e^{i\theta}). \tag{3.27}$$

Proof. Since $S_n(\mu)=|\mu|\neq 0$, we have $\mu\notin\partial\Sigma_n$, and thus $\mu\in\partial\Omega_n$. Choose a sequence of points $\mu_k\in\Omega_n$ with $\mu_k\to\mu$. From part (c) of Theorem 1, we know that there exist $\theta_k\in[0, 2\pi)$ such that

$$\mu_k=Q_{n,\rho_k}(e^{i\theta_k}), \qquad \rho_k=S_n(\mu_k)\to S_n(\mu)=|\mu|.$$

We may assume that $\theta_k\to\theta\in\mathbf{R}$ (otherwise, choose a subsequence), and since $Q_{n,\rho_k}(z)\to Q_{n,|\mu|}(z)$ uniformly in $\bar{\mathbf{D}}$, we get (3.27). ■

It follows from (3.26) that between $|\mu|$ and θ in (3.27), we have the relation

$$|\mu|=\left(\frac{\cos(\theta/2)}{\cos\left(\dfrac{j\pi}{n+1}\right)}\right)^{n+1} \qquad (j=1,\ldots,n).$$

On the other hand, a direct calculation of $Q_{n,|\mu|}(e^{i\theta})$ in these points gives

$$\mu=Q_{n,|\mu|}(e^{i\theta})=(-1)^j|\mu|\exp\left[i\left(\frac{n+1}{2}\right)\theta\right] \qquad (j=1,\ldots,n).$$

These points μ constitute the following n curves in $\bar{\mathbf{D}}$, connecting 1 and 0:

$$(3.28)\quad \begin{cases} C_{n,j} := \left\{ (-1)^j \left(\dfrac{\cos \psi}{\cos\left(\dfrac{j\pi}{n+1}\right)} e^{i\psi} \right)^{n+1} : \dfrac{j\pi}{n+1} \le \psi \le \dfrac{\pi}{2} \right\}, \\ C_{n,n+1-j} := \{\bar{\mu} : \mu \in C_{n,j}\}, \end{cases}$$

for $j = 1, \ldots, [n/2]$, while for n odd,

$$(3.29)\quad C_{n,(n+1)/2} := \{\mu : 0 \le \mu \le 1\}.$$

From (3.29), we note that $C_{1,1} = [0, 1]$. The graphs of $C_{n,j}$, for $n = 2, 3$ and $1 \le j \le n$, are shown in Figure 2(a) and (b)

The case $n = 1$ of part (b) of Theorem 1 is trivial; hence, we assume $n \ge 2$. By Proposition 6, we have

$$(3.30)\quad \partial\Delta_n \subset \bigcup_{j=1}^{n} C_{n,j} \subset \Delta_n,$$

and it is obvious from (2.2) and (2.1) that

$$(3.31)\quad \Delta_{n-1} \subset \Delta_n.$$

The set $\bigcup_{j=1}^{n} C_{n,j}$ defines a number of pairwise disjoint domains, say $\psi_n(k)$, $k = 1, \ldots, K(n)$, such that

$$(3.32)\quad \begin{cases} \bigcup_k \partial\psi_n(k) \subset \bigcup_{j=1}^{n} C_{n,j}, \\ \bigcup_k \overline{\psi_n(k)} = \bar{\mathbf{C}}. \end{cases}$$

Exactly one of them, say $\psi_n(1)$, is unbounded, and our assertion of part (b) of Theorem 1 will be shown to be equivalent to

$$(3.33)\quad \Delta_n = \mathbf{C} \backslash \psi_n(1).$$

To this end, we have to prove that

$$(3.34)\quad \psi_n(k) \subset \Delta_n \qquad (k = 2, \ldots, K(n)).$$

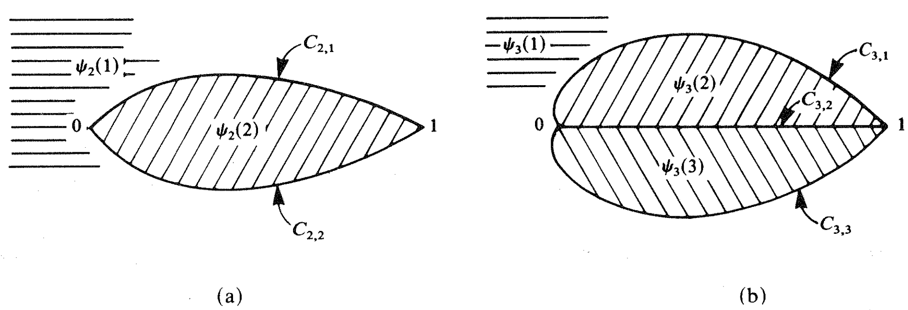

Fig. 2

Now, each point μ of $\psi_n(k)$ must, from (2.1), satisfy either

$$S_n(\mu)=|\mu| \quad (\text{whence } \mu\in\Delta_n), \quad \text{or} \quad S_n(\mu)<|\mu|.$$

But since $\psi_n(k)$ can contain no boundary point of Δ_n by definition and by (3.30), then either

$$S_n(\mu)=|\mu| \quad \text{for all } \mu\in\psi_n(k), \quad \text{or} \quad S_n(\mu)<|\mu| \quad \text{for all } \mu\in\psi_n(k).$$

To eliminate this second possibility, it suffices by (3.31) to show that

$$\psi_n(k)\cap\Delta_{n-1}\neq\varnothing \qquad (k=2,\ldots,K(n)). \tag{3.35}$$

As $\Delta_1=[0,1]$, it is evident, on superimposing Figure 2(a) and (b), that (3.35) is valid for $n=2,3$. That (3.33) and (2.8) are also equivalent in these cases is clear from Figure 2(a) and (b), and (3.28). This completes the proof for $n=2,3$, and we now may assume that $n\geq 4$. We note that $C_{n,j}$, as defined in (3.28), has the following properties ($j=1,\ldots,[n/2]$):

(3.36)
- (i) $C_{n,j}$ is a curve in $\bar{\mathbf{D}}$ connecting the points 1 and 0;
- (ii) while passing from 1 to 0, the points of $C_{n,j}$ have strictly decreasing moduli;
- (iii) similarly, their arguments are strictly increasing;
- (iv) $C_{n,j}$ crosses the negative real axis for the first time for $\psi=(j+1)\pi/(n+1)$, provided $j=1,\ldots,[n/2]-1$;
- (v) $C_{n,[n/2]}$ is a curve from 1 to 0 which lies in the closed upper-half plane.

We define the following subarcs of $C_{n,j}$:

$$\begin{cases} D_{n,j}:=\left\{(-1)^j\left(\dfrac{\cos\psi}{\cos\left(\dfrac{j\pi}{n+1}\right)}e^{i\psi}\right)^{n+1}: \dfrac{j\pi}{n+1}\leq\psi\leq\dfrac{(j+1)\pi}{n+1}\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } j=1,\ldots,[n/2]-1; \\ D_{n,j}:=\left\{(-1)^j\left(\dfrac{\cos\psi}{\cos\left(\dfrac{j\pi}{n+1}\right)}e^{i\psi}\right)^{n+1}: \dfrac{j\pi}{n+1}\leq\psi\leq\dfrac{\pi}{2}\right\} \quad \text{for } j=[n/2]; \\ D_{n,n+1-j}:=\{\bar{\mu}: \mu\in D_{n,j}\} \qquad \text{for } j=1,\ldots,[n/2], \end{cases} \tag{3.37}$$

and let $\Theta_{n,j}$ be the *bounded* Jordan domains with

$$\partial\Theta_{n,j}=D_{n,j}\cup D_{n,n+1-j} \qquad (j=1,2,\ldots,[n/2]). \tag{3.38}$$

We note that, in view of (3.36), each $\Theta_{n,j}$ is *starlike* with respect to the origin.

Proposition 7. *For $m=n$ or $m=n+1$, $1\leq j\leq[n/2]$, $1\leq k\leq[m/2]$, and either $m\neq n$ or $j\neq k$, we have*

$$\{1\}\subset\partial\Theta_{n,j}\cap\partial\Theta_{m,k}\subset\{1;0\}. \tag{3.39}$$

Proof. For reasons of symmetry, it is clear that (3.39) will follow from

$$\{1\} \subset D_{n,j} \cap D_{m,k} \subset \{1; 0\}. \tag{3.40}$$

With

$$\varphi = \frac{(j+\varepsilon)\pi}{n+1}, \qquad \psi = \frac{(k+\delta)\pi}{m+1} \qquad (0 \leq \varepsilon,\ \delta \leq 1), \tag{3.41}$$

assume that $D_{n,j}$ and $D_{m,k}$ have a common nonzero point:

$$z := (-1)^j \left(\frac{\cos\varphi}{\cos\left(\dfrac{j\pi}{n+1}\right)} e^{i\varphi} \right)^{n+1} = (-1)^k \left(\frac{\cos\psi}{\cos\left(\dfrac{k\pi}{m+1}\right)} e^{i\varphi} \right)^{m+1} =: w. \tag{3.42}$$

Then, $\arg z = \arg w$ implies $\varepsilon = \delta$. Hence, (3.42) holds if and only if

$$\left(\frac{\cos\left[\dfrac{(j+\varepsilon)\pi}{n+1}\right]}{\cos\left(\dfrac{j\pi}{n+1}\right)} \right)^{n+1} = \left(\frac{\cos\left[\dfrac{(k+\varepsilon)\pi}{m+1}\right]}{\cos\left(\dfrac{k\pi}{m+1}\right)} \right)^{m+1}, \tag{3.43}$$

or

$$\frac{\left(\cos\left[\dfrac{(j+\varepsilon)\pi}{n+1}\right]\right)^{n+1}}{\left(\cos\left(\dfrac{(k+\varepsilon)\pi}{m+1}\right)\right)^{m+1}} = \frac{\left(\cos\left(\dfrac{j\pi}{n+1}\right)\right)^{n+1}}{\left(\cos\left(\dfrac{k\pi}{m+1}\right)\right)^{m+1}}. \tag{3.44}$$

We wish to show that the left-hand side of (3.44) is strictly monotonic in ε, $0 < \varepsilon < 1$. If not, the derivative with respect to ε would vanish somewhere in $(0, 1)$, which gives

$$-\pi\left(\cos\left[\frac{(j+\varepsilon)\pi}{n+1}\right]\right)^{n} \sin\left[\frac{(j+\varepsilon)\pi}{n+1}\right]\left(\cos\left[\frac{(k+\varepsilon)\pi}{m+1}\right]\right)^{m+1}$$
$$= -\pi\left(\cos\left[\frac{(k+\varepsilon)\pi}{m+1}\right]\right)^{m} \sin\left(\frac{(k+\varepsilon)\pi}{m+1}\right)\left(\cos\left(\frac{(j+\varepsilon)\pi}{n+1}\right)\right)^{n+1}$$

or, equivalently,

$$\tan\left(\frac{(j+\varepsilon)\pi}{n+1}\right) = \tan\left(\frac{(k+\varepsilon)\pi}{m+1}\right).$$

Since both arguments are in $[0, \pi]$ where $\tan(x)$ is injective, we get the equivalent condition

$$\frac{j+\varepsilon}{n+1} = \frac{k+\varepsilon}{m+1}. \tag{3.45}$$

Now, if $n = m$, we deduce $j = k$, which violates the hypotheses of Proposition 7. However, for $m = n+1$, we get from (3.45) that $\varepsilon = k(n+1) - j(n+2)$, which is an integer and hence is not in (0, 1). This contradiction shows that the left side of (3.44) is strictly monotone in ε for $0<\varepsilon<1$, so that equality holds in (3.44) only for $\varepsilon=0$. Thus, the only nonzero point of $D_{n,j}\cup D_{m,k}$ is $\{1\}$. On the other hand, (3.36v) shows that 0 can be a point of $D_{n,j}\cap D_{m,k}$, which gives (3.39). ■

Starting from the point 1, the curves $D_{n,j}$ enter into the unit disk as follows:

$$1+i\frac{n+1}{\cos\left(\frac{j\pi}{n+1}\right)}\exp\left[i\left(\frac{j\pi}{n+1}\right)\right]\left(\psi-\frac{j\pi}{n+1}\right)+o\left(\psi-\frac{j\pi}{n+1}\right). \tag{3.46}$$

From this fact, we can deduce that *initially* $D_{n,j}\subset\Theta_{n,j-1}\cup\{1\}$, and in view of Proposition 7,

$$\Theta_{n,j}\subset\Theta_{n,j-1}\qquad (j=2,\dots,[n/2]), \tag{3.47}$$

and therefore

$$\Theta_{n,j}\subset\Theta_{n,1}\qquad (j=2,\dots,[n/2]). \tag{2.48}$$

Similarly, the initial portion of $D_{n,2}$ must be contained in $\Theta_{n-1,1}$ since (cf. (3.46))

$$\frac{2}{n+1}>\frac{1}{n}\qquad (n\geq 4).$$

Hence, again by Proposition 7 (with $m=n-1$),

$$\Theta_{n,2}\subset\Theta_{n-1,1}; \tag{3.49}$$

in fact,

$$\overline{\Theta_{n,2}}\backslash\{1;0\}\subset\Theta_{n-1,1}, \tag{3.50}$$

since the boundaries have at most the points 1 and 0 in common. As $\Theta_{n,j}$ is starlike with respect to the origin, we deduce that

$$C_{n,j}\subset\overline{\Theta_{n,j}}\subset\overline{\Theta_{n,1}}\qquad (j=1,\dots,n), \tag{3.51}$$

and, we similarly deduce from (3.47) and (3.50) that

$$C_{n,j}\subset\overline{\Theta_{n,j}}\subset\overline{\Theta_{n,2}}\subset\Theta_{n-1,1}\cup\{1;0\}\qquad (j=2,\dots,n-1). \tag{3.52}$$

From (3.38) and (3.51), it is clear that (3.33) is equivalent to

$$\Delta_n=\overline{\Theta_{n,1}}, \tag{3.53}$$

as well as to our assertion (b) in Theorem 1. We now proceed by mathematical induction. The assertion has already been established for $n=1,2,3$. Assume (3.53) holds for $n-1$. Then, by (3.52), we see that

$$C_{n,j}\backslash\{1,0\}\subset\operatorname{int}\Delta_{n-1}\qquad (j=2,\dots,n-1), \tag{3.54}$$

and (3.35) is fulfilled for every $\psi_n(k)$ which has one of these curves in its boundary. But, from (3.28), the curves $C_{n,1}\backslash D_{n,1}$, $C_{n,n}\backslash D_{n,n}$ do not intersect in $\Theta_{n,1}\backslash\Theta_{n-1,1}$

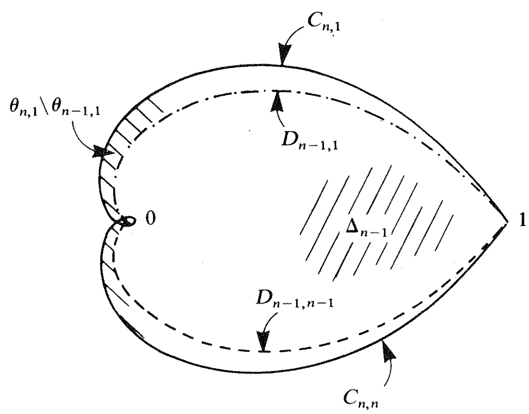

Fig. 3

(see Figure 3), so that each of the remaining domains $\psi_n(k)$, $k \neq 1$, also contains points of Δ_{n-1}. This proves (3.35), (3.33) and therefore (3.53). ■

4. Proofs of the Corollaries

We begin this section with the following

Proposition 8. *Write* $Q_{n,\rho}(w) := \sum_{k=0}^{n} a_k(\rho) w^k$, *where* $Q_{n,\rho}(z^2)$ *is defined in* (2.6). *Then,*

$$a_k(\rho) > 0 \qquad (k = 0, 1, \ldots, n; 0 < \rho < 1). \tag{4.1}$$

Proof. For any $\sigma > 1$ and any $n \geq 1$, let the coefficients $b_k(\sigma; n)$ be defined by

$$z^n T_n\left(\sigma \frac{1+z^2}{2z}\right) =: \sum_{k=0}^{2n} b_k(\sigma; n) z^k. \tag{4.2}$$

Using the known expansion of the Chebyshev polynomial $T_n(x)$ in powers of x, we immediately see that the odd coefficients $b_{2k+1}(\sigma; n)$ of (4.2) all vanish ($k = 0, \ldots, n-1$), while for the even coefficients, we have

$$b_{2k}(\sigma; n) = \frac{n\sigma^n}{2} \sum_{j=0}^{\min\{k; n-k\}} \frac{(n-j-1)!\,(-\sigma^2)^{-j}}{j!\,(k-j)!\,(n-j-k)!} \qquad (k = 0, 1, \ldots, n). \tag{4.3}$$

We next claim that $b_{2k}(\sigma; n) > 0$ for all $k = 0, 1, \ldots, n$. As $b_{2k}(\sigma; n) = b_{2(n-k)}(\sigma; n)$ from (4.3), it suffices to show that $b_{2k}(\sigma; n) > 0$ for all $k = 0, 1, \ldots, [n/2]$. Equivalently, it suffices to establish the positivity of

$$c_k(x) := \sum_{j=0}^{k} \frac{(n-j-1)!\,x^j}{j!\,(k-j)!\,(n-j-k)!} \qquad (-1 < x \leq 0; k = 0, 1, \ldots, [n/2]). \tag{4.4}$$

For $k=0$, this is obvious. For $k>0$, expanding $c_k(x)$ about $x=-1$ and applying standard combinatorial identities, we obtain

$$c_k(x)=\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(n-k-1)!\,(1+x)^{j+1}}{(n-j-k-1)!\,(j+1)!}\qquad (k=1,2,\ldots,[n/2]),$$

which *is* positive for all $-1<x\leq 0$ and for all $k=1,2,\ldots,[n/2]$. Thus, $b_{2k}(\sigma;n)>0$ for all $k=0,1,\ldots,n$ and all $\sigma>1$.

To complete the proof, from (2.6) we can write

$$(4.5)\quad Q_{n,\rho}(z^2)=-\frac{\rho}{(n+1)}z^{2n+3}\frac{d}{dz}\left\{z^{-2n-2}\left(z^{n+1}T_{n+1}\left(\rho^{-1/(n+1)}\left(\frac{1+z^2}{2z}\right)\right)\right)\right\}.$$

Replacing n by $n+1$ in (4.2) and setting $\sigma:=\rho^{-1/(n+1)}$, we have from (4.2) and (4.5) that

$$Q_{n,\rho}(z^2)=-\frac{z^{2n+3}}{(n+1)\sigma^{n+1}}\frac{d}{dz}\left\{\sum_{k=0}^{n+1}b_{2k}(\sigma;n+1)z^{2(k-n-1)}\right\}.$$

Thus, on differentiating in the above expression and then replacing z^2 by w, we have

$$(4.6)\qquad Q_{n,\rho}(w)=\frac{2}{(n+1)\sigma^{n+1}}\sum_{k=0}^{n}b_{2k}(\sigma;n+1)(n+1-k)w^k,$$

so that the positivity of the coefficients $b_{2k}(\sigma;n+1)$ implies the sought positivity of the $a_k(\rho)$ of (4.1). ■

On setting $z=1$ in (2.6), it is easily verified that

$$(4.7)\qquad Q_{n,\rho}(1)=\rho T_{n+1}(\rho^{-1/(n+1)}).$$

Proposition 9. *For each $n\geq 1$, $Q_{n,\rho}(1)$ is a strictly decreasing function of ρ for $0<\rho<1$.*

Proof. On differentiating the right side of (4.7) with respect to ρ, the strictly decreasing nature of $Q_{n,\rho}(1)$ for $0<\rho<1$ is equivalent to the property that

$$(4.8)\qquad T_{n+1}(x)<\frac{x}{(n+1)}T'_{n+1}(x)\qquad (x>1).$$

Since both sides of (4.8) are positive for $x>1$, then squaring yields

$$(4.9)\qquad T^2_{n+1}(x)<\frac{x^2}{(n+1)^2}(T'_{n+1}(x))^2=\frac{x^2}{(x^2-1)}(T^2_{n+1}(x)-1),$$

where we have used the identity that $(1-x^2)(T'_n(x))^2=n^2(1-T^2_n(x))$. Thus, (4.9) is equivalent to $T^2_{n+1}(x)>x^2$, which is obviously valid for all $x>1$ and all $n\geq 1$. ■

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Constr. Approx. (1986) 2: 349–368

CONSTRUCTIVE APPROXIMATION

On the Minimum Moduli of Normalized Polynomials with Two Prescribed Values

Stephan Ruscheweyh and Richard S. Varga

Abstract. With $\mathbf{P}_n$ denoting the set of complex polynomials of degree at most n $(n \geq 1)$, define, for any complex number μ, the subset

$$\mathbf{P}_n(\mu) := \{p_n(z) \in \mathbf{P}_n: p_n(0) = 1 \text{ and } p_n(1) = \mu\}.$$

In this paper, we determine exactly the nonnegative quantity

$$S_n(\mu) := \sup_{p_n \in \mathbf{P}_n(\mu)} \{\min_{|z| \leq 1} |p_n(z)|\},$$

as a function of n and μ. For fixed $n \geq 2$, the three-dimensional surface, generated by the points (Re μ, Im μ, $S_n(\mu)$) for all complex numbers μ, has the interesting shape of a volcano.

1. Introduction

Consider the set of all complex polynomials $p_n(z)$ of degree at most n $(n \geq 1)$, taking on two prescribed values (not both zero) in two distinct points in the complex plane:

$$p_n(z_1) = \alpha, \qquad p_n(z_2) = \beta \qquad (z_1 \neq z_2;\ \alpha \neq 0). \tag{1.1}$$

Then, what can be said about the supremum of

$$\min\{|p_n(z)|: |z - z_1| \leq |z_2 - z_1|\}, \tag{1.2}$$

over all polynomials $p_n(z)$ satisfying (1.1)? This problem can be normalized as follows. With $\mathbf{P}_n$ denoting the set of all complex polynomials of degree at most n $(n \geq 1)$, then for each complex number μ, consider the subset of $\mathbf{P}_n$, defined by

$$\mathbf{P}_n(\mu) := \{p_n(z) \in \mathbf{P}_n: p_n(0) = 1 \text{ and } p_n(1) = \mu\}. \tag{1.3}$$

Our objective in this paper is to determine precisely the nonnegative quantity

$$S_n(\mu) := \sup_{p_n \in \mathbf{P}_n(\mu)} \{\min_{|z| \leq 1} |p_n(z)|\}, \tag{1.4}$$

as a function of n and μ.

Date received: July 24, 1985. Date revised: May 27, 1986. Communicated by Peter Henrici.
AMS classification: 30A10, 30C10.
Key words and phrases: Minimum modulus of polynomials, Zero-finding algorithm for complex polynomials, Chebyshev polynomials, Volcano function.

Our interest in this question arose from the following related problem which is crucial in the study of certain *global descent methods* for finding a zero of a given complex polynomial (cf. Henrici [1], [2], Ruscheweyh [8], and references contained therein): For each $n \geq 1$, set

$$(1.5) \qquad \Gamma_n := \max\left\{ \min_{|z|\leq 1} |p_n(z)| : p_n(z) = 1 + \sum_{j=1}^{n} a_j z^j \text{ with } \sum_{j=1}^{n} |a_j| = 1 \right\}.$$

From the results of Ruscheweyh [8], and Ruscheweyh and Varga [9], it is known that Γ_n satisfies the inequalities

$$(1.6) \qquad 1 - \frac{1}{n} \leq \Gamma_n \leq \sqrt{1 - \left(\frac{1}{n}\right)} < 1 - \frac{1}{2n},$$

for every $n \geq 1$. Moreover, with

$$(1.7) \quad \tilde{\Gamma}_n := \max\{\min_{|z|\leq 1} |p_n(z)| : p_n(z) \in \mathbf{P}_n(2) \text{ with } p_n^{(j)}(0) \geq 0 \text{ for } j = 1, 2, \ldots, n\},$$

it is further known from [9] that

$$(1.8) \qquad 1 - \frac{1}{n} \leq \tilde{\Gamma}_n \leq \sqrt{1 - (3/(2n+1))} = 1 - \frac{3}{4n} + o\left(\frac{1}{n}\right) \qquad (n \to \infty),$$

for every $n \geq 1$. In [9], we first conjectured that

$$(1.9) \qquad \Gamma_n = \tilde{\Gamma}_n \qquad (n = 1, 2, \ldots).$$

Now, the bounds (1.6) and (1.8) suggested that there exists a positive constant γ, independent of n, such that

$$(1.10) \qquad \Gamma_n = \tilde{\Gamma}_n = 1 - \frac{\gamma}{n} + o\left(\frac{1}{n}\right) \qquad (n \to \infty).$$

Indeed, extended precision calculations described in [9] led us to further conjecture that γ in (1.10) is approximately given by

$$(1.11) \qquad \gamma \doteq 0.867\,189\,051.$$

By definition, we note that $\tilde{\Gamma}_n \leq S_n(2)$ for all $n \geq 1$. Interestingly enough, one consequence of this present paper (cf. (2.12) of Corollary 1) is that

$$(1.12) \qquad S_n(2) = \tilde{\Gamma}_n = 1 - \frac{1}{2n}\{\operatorname{arccosh}(2)\}^2 + o\left(\frac{1}{n}\right) \qquad (n \to \infty),$$

so that the quantity γ of (1.10) is given *exactly* by

$$(1.13) \qquad \gamma = \frac{\{\operatorname{arccosh}(2)\}^2}{2} = 0.867\,189\,051\,136\,318\,1\ldots.$$

Thus, our conjecture of (1.11) (to the number of digits given in (1.11)) is *correct*! The conjecture (1.9), however, remains open.

The outline of our paper is as follows. In Section 2, we state our main results concerning the determination of $S_n(\mu)$ of (1.4). As we shall see, for fixed $n \geq 2$, the three-dimensional surface, generated by the points $(\operatorname{Re} \mu, \operatorname{Im} \mu, S_n(\mu))$ for all complex numbers μ, has the interesting shape of a *volcano*. A computer-generated picture of this for the case $n = 5$ is given in Fig. 1 of Section 2. Finally, the proof of Theorem 1 of Section 2 is given in Section 3, while the proofs of Corollaries 1 and 2 of Section 2 are given in Section 4.

2. Statement of Results

For any $\mu \in \mathbf{C}$, the minimum principle, applied to any $p_n(z)$ in $\mathbf{P}_n(\mu)$ (cf. (1.3)), directly gives us that $\min_{|z|\leq 1}|p_n(z)| \leq \min\{1; |\mu|\} \leq |\mu|$, whence (cf. (1.4))

$$0 \leq S_n(\mu) \leq \min\{1; |\mu|\} \leq |\mu| \qquad (n = 1, 2, \ldots; \text{ all } \mu \in \mathbf{C}). \tag{2.1}$$

Furthermore, standard arguments show that the supremum in (1.4) is, in fact, a maximum. Moreover, by an elementary variation in the elements of $\mathbf{P}_n(\mu)$, it can be seen that $S_n(\mu)$ is continuous in $\mathbf{C}$.

For our subsequent analysis, it is necessary to distinguish between the following pairwise disjoint sets of $\mathbf{C}$:

$$\Delta_n := \{\mu \in \mathbf{C}: S_n(\mu) = |\mu|\}, \tag{2.2}$$

$$\Omega_n := \{\mu \in \mathbf{C}: 0 < S_n(\mu) < |\mu|\}, \tag{2.3}$$

$$\Sigma_n := \{\mu \in \mathbf{C}: 0 = S_n(\mu) < |\mu|\}, \tag{2.4}$$

where we note that $\Delta_n \cup \Omega_n \cup \Sigma_n = \mathbf{C}$. With $\mathbf{D} := \{z \in \mathbf{C}: |z| < 1\}$ and with $\bar{\mathbf{D}}$ denoting the closure of $\mathbf{D}$, it is evident from (2.1) that $\Delta_n \subset \bar{\mathbf{D}}$. Moreover, from (1.4) and (2.2), it can be verified that Δ_n has another interesting interpretation: $\mu \in \Delta_n$ iff there is a $p_n(z) \in \mathbf{P}_n$ with $p_n(0) = 1$ such that

$$\mu \in p_n(\bar{\mathbf{D}}) \quad \text{and} \quad |\mu| = \min_{|z|\leq 1}|p_n(z)|. \tag{2.5}$$

From this observation, it is somewhat surprising, as we shall see, that substantial parts of $\bar{\mathbf{D}}$ do *not* belong to Δ_n.

For additional needed notation, for each ρ with $0 < \rho < 1$ and for each positive integer n, set

$$Q_{n,\rho}(z^2) := \frac{-\rho}{(n+1)} z^{2n+3} \frac{d}{dz}\left\{z^{-(n+1)} T_{n+1}\left[\rho^{-1/(n+1)}\left(\frac{1+z^2}{2z}\right)\right]\right\}, \tag{2.6}$$

where $T_{n+1}(z)$ denotes the Chebyshev polynomial of degree $n+1$ of the first kind. As can be seen (cf. (4.6)), $Q_{n,\rho}(w) \in \mathbf{P}_n$.

With the above notation, our main result is

Theorem 1. *The following are valid*:

(a) *The set* Σ_n *satisfies*

$$\Sigma_n = \mathbf{C} \backslash \{z \in \mathbf{C}: z = (1+w)^n \text{ where } w \in \mathbf{D} \cup \{-1\}\}; \tag{2.7}$$

(b) $\Delta_1=[0,1]$. *For* $n\geq 2$, Δ_n *consists of the Jordan curve*

$$C_n:=\left\{z=-\left(\frac{\cos(\varphi/(n+1))}{\cos(\pi/(n+1))}\right)^{n+1}\cdot e^{i\varphi}\colon \pi\leq|\varphi|\leq 2\pi\right\}, \tag{2.8}$$

and the bounded Jordan domain having C_n *as its boundary;*

(c) $\Omega_n=\mathbf{C}\backslash(\Sigma_n\cup\Delta_n)$. *Moreover, for any* $\mu\in\Omega_n$, *there holds*

$$S_n(\mu)=\max\{0<\rho<1\colon \mu\in Q_{n,\rho}(\bar{\mathbf{D}})\}. \tag{2.9}$$

When $\mu>0$, the quantity $S_n(\mu)$ can be given in the following more explicit fashion, which directly connects with the problem arising from global descent methods mentioned in Section 1.

Corollary 1. *Let* $\mu>0$. *Then, there holds:*

$$S_n(\mu)=\begin{cases}\mu & \text{if } 0<\mu\leq 1;\\ \sigma & \text{if } 1<\mu<2^n;\\ 0 & \text{if } 2^n\leq\mu.\end{cases} \tag{2.10}$$

Here, σ *is the uniquely determined solution in* $(0,1)$ *of the equation*

$$\mu=\sigma T_{n+1}\{\sigma^{-1/(n+1)}\}. \tag{2.11}$$

For n tending to infinity, the solution σ *of* (2.11) *can be expressed as*

$$\sigma=1-\frac{\{\operatorname{arccosh}(\mu)\}^2}{2n}+o\left(\frac{1}{n}\right)\qquad (n\to\infty). \tag{2.12}$$

Finally, for $\mu\in(1,2^n)$ *and for* σ *defined by* (2.11), $Q_{n,\sigma}(z)$ *is the unique extremal polynomial in* $\mathbf{P}_n(\mu)$ *for the problem* (1.4), *and* $Q_{n,\sigma}(z)$, *when expanded in powers of z, has positive coefficients.*

Another consequence of Theorem 1, of a more general nature, is the following:

Corollary 2. *For any* $p_n(z)\in\mathbf{P}_n$ *with* $p_n(z)\neq 0$ *in* $\bar{\mathbf{D}}$, *there holds*

$$M\leq mT_{n+1}\left(\left(\frac{|p_n(0)|}{m}\right)^{1/(n+1)}\right), \tag{2.13}$$

where

$$M:=\max_{|z|\leq 1}|p_n(z)|,\qquad m:=\min_{|z|\leq 1}|p_n(z)|. \tag{2.14}$$

The upper bound (2.13) *is sharp for each* $m\in(0,|p_n(0)|)$.

To conclude this section, we illustrate the result of Theorem 1, in the case $n=5$, with the computer-generated three-dimensional surface $(\operatorname{Re}\mu, \operatorname{Im}\mu, S_5(\mu))$, given in Fig. 1. It is evident that this surface has the shape of a *volcano*. Specifically, the set Δ_5, defined in (2.2), corresponds geologically to the *caldera* (i.e., the deep caldronlike cavity found at the summit of a volcano), while the sides of the volcano correspond to the set Ω_5, defined in (2.3).

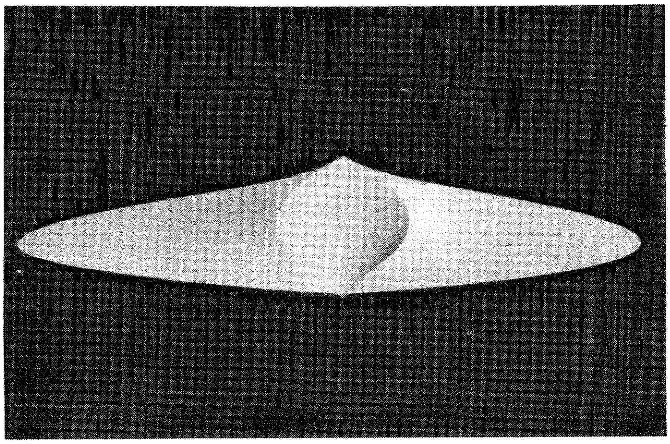

Fig. 1

3. Proof of Theorem 1

As the proof of Theorem 1 is somewhat involved, we give for convenience some intermediate results in the form of propositions. Some parts of our proof are reminiscent of the general theory of extremal problems for complex polynomials as derived, for example, in Rivlin and Shapiro [6]. However, that theory as a whole does not apply to our problem of maximizing the minimum modulus, and even similar tools need new justifications in our case. For the sake of completeness and readability, we also indicate proofs for those (small) portions which could have been transferred from the above-mentioned theory.

We start with the determination of the set Σ_n of (2.4). Clearly, $\mu \in \Sigma_n$ if and only *if* $\mu \neq 0$ and every polynomial in $\mathbf{P}_n(\mu)$ has a zero in $\bar{\mathbf{D}}$.

Proposition 1. *Let $p_n(z) \in \mathbf{P}_n$ satisfy $p_n(0) = 1$ with $p_n(z) \neq 0$ for any $z \in \mathbf{D}$. Then, $p_n(\mathbf{D}) \subset w_n(\mathbf{D})$, where $w_n(z) := (1+z)^n$.*

Proof. Simply apply Szegö's corollary (16, 1c) of Marden [3, p. 67] with $f(z) :=(1+z)^n$ and with $g(z) := p_n(z)$. Another proof of this also follows from Theorems 1.1 and 1.5 of [7]. ■

Now, consider any $\mu \in \mathbf{C}\backslash\Sigma_n$, with $\mu \neq 0$. Then, there exists a polynomial $p_n(z) \in \mathbf{P}_n(\mu)$ with $p_n(z) \neq 0$ in $\bar{\mathbf{D}}$, and for small $\varepsilon > 0$, we have $\tilde{p}_n(z) := p_n((1+\varepsilon)z) \neq 0$ in $\mathbf{D}$. Since $\tilde{p}_n(0) = 1$, we see from Proposition 1 that

$$\mu = \tilde{p}_n\left(\frac{1}{1+\varepsilon}\right) \in w_n(\mathbf{D}). \tag{3.1}$$

Conversely, if $\mu \in w_n(\mathbf{D})$, then $\mu = (1+z_0)^n$ for a certain $z_0 \in \mathbf{D}$, and $(1+z_0 z)^n$ is an element of $\mathbf{P}_n(\mu)$ with nonvanishing minimum modulus in $\bar{\mathbf{D}}$. This implies $S_n(\mu) > 0$, and thus $\mu \notin \Sigma_n$ This completes the proof of part (a) of Theorem 1.

We turn to the proof of part (c) of Theorem 1, and assume $\mu \in \Omega_n$. As we mentioned at the beginning of Section 2, there exists an extremal polynomial $E_n(z) \in \mathbf{P}_n(\mu)$ with $S_n(\mu) = \min_{|z|\le 1}|E_n(z)|$. Since μ and n are fixed for this proof, we write $S_n(\mu) = s$. Let F denote the set of all numbers $\theta \in [0, 2\pi)$ with $|E_n(e^{i\theta})| = s$. Let $e(\theta) := |E_n(e^{i\theta})|^2$, $\theta \in \mathbf{R}$. Then, $e(\theta)$ is a trigonometric polynomial of degree at most n with s^2 as its absolute minimum. It is therefore clear that F has at most n elements. Note that $0 \notin F$ since $e(0) = |E_n(1)|^2 = |\mu|^2 > s^2$ by assumption (cf. (2.3)).

The next proposition is similar to the well-known Kolmogorov theorem from approximation theory (cf. Meinardus [4, p. 13]).

Proposition 2. *There is no polynomial $U_n(z) \in \mathbf{P}_n$ with $U_n(0) = 0$ and $U_n(1) = 0$, such that*

$$\operatorname{Re}[\overline{E_n(e^{i\theta})}\,U_n(e^{i\theta})] > 0 \qquad (\theta \in F). \tag{3.2}$$

Proof. Assume that there is such a polynomial $U_n(z)$. Then, there exists a $\gamma > 0$ such that

$$2\operatorname{Re}[\overline{E_n(e^{i\theta})}\,U_n(e^{i\theta})] > \gamma \qquad (\theta \in I),$$

where

$$I := \{\theta \in (0, 2\pi): \text{there exists a } \theta_j \in F \text{ with } |\theta - \theta_j| < \sigma\},$$

for a certain $\sigma > 0$. Hence, for $\varepsilon > 0$, we find

$$|E_n(e^{i\theta}) + \varepsilon U_n(e^{i\theta})|^2 \ge s^2 + \varepsilon\gamma > s^2 \qquad (\theta \in I). \tag{3.3}$$

For the compact set $I' := [0, 2\pi]\backslash I$, there exists a $\delta > 0$ such that

$$|E_n(e^{i\theta})|^2 \ge s^2 + \delta \qquad (\theta \in I').$$

Now, choose ε so small that

$$\varepsilon|2\operatorname{Re}[\overline{E_n(e^{i\theta})}\,U_n(e^{i\theta})]| + \varepsilon^2|U_n(e^{i\theta})|^2 < \delta \qquad (\theta \in I').$$

Then on I', we have

$$|E_n(e^{i\theta}) + \varepsilon U_n(e^{i\theta})|^2 > s^2 + \delta - \delta = s^2 \qquad (\theta \in I'). \tag{3.4}$$

For small $\varepsilon > 0$, the polynomial $E_n(z) + \varepsilon U_n(z)$ is nonvanishing in $\bar{\mathbf{D}}$, and hence, (3.3), (3.4), and the minimum principle imply $\min_{|z|\le 1}|E_n(z) + \varepsilon U_n(z)| > s$. But since $(E_n(z) + \varepsilon U_n(z)) \in \mathbf{P}_n(\mu)$, we have a contradiction to the definition of s. ∎

Assume next that F has less than n elements. Then by interpolation, we can construct a polynomial $U_n(z) \in \mathbf{P}_n$ with

$$U_n(0) = 0, \qquad U_n(1) = 0, \qquad U_n(e^{i\theta}) = E_n(e^{i\theta}) \qquad \text{for} \quad \theta \in F.$$

(Note that $0 \notin F$.) But then, $U_n(z)$ contradicts Proposition 2, and we thus conclude that F has precisely n elements in $(0, 2\pi)$, say

$$0 < \theta_1 < \cdots < \theta_n < 2\pi.$$

Proposition 3. *There exists an $\alpha \in \mathbf{C}$ with $|\alpha|=1$, such that*

$$\alpha \exp\left[-i\left(\frac{n+1}{2}\right)\theta_j\right]E_n(e^{i\theta_j})=(-1)^j s \qquad (j=1,\ldots,n). \tag{3.5}$$

Proof. As the assertion (3.5) is obvious for $n=1$, assume $n>1$. Let p be an integer with $1\le p\le n-1$, and let $q:=p+1$. Let $\varepsilon>0$ and $\sigma=e^{i\varphi}$ with $|\varphi|<\pi/2$. By interpolation, we can find a polynomial $U_{\varepsilon,\sigma}(z)$ in $\mathbf{P}_n$ with

$$\begin{cases} U_{\varepsilon,\sigma}(0)=0, \qquad U_{\varepsilon,\sigma}(1)=(0), \\ U_{\varepsilon,\sigma}(e^{i\theta_j})=\varepsilon E_n(e^{i\theta_j}) \qquad \text{for } j=1,\ldots,n; \quad j\neq p, q, \\ U_{\varepsilon,\sigma}(e^{i\theta_q})=\sigma E_n(e^{i\theta_q}). \end{cases} \tag{3.6}$$

Using $U_{\varepsilon,\sigma}(z)$ in Proposition 2, we necessarily find that

$$\operatorname{Re}[\overline{E_n(e^{i\theta_p})}\,U_{\varepsilon,\sigma}(e^{i\theta_p})]\le 0. \tag{3.7}$$

Now, let $\varepsilon\to 0$. Then, $U_{\varepsilon,\sigma}(z)\to U_{0,\sigma}(z)$, uniformly in $\bar{\mathbf{D}}$, and the zeros of $U_{0,\sigma}(z)$ occur precisely in the points $0, 1, e^{i\theta_j}$ for $j=1,\ldots,n; j\neq p, q$. This implies that the polynomials $U_{0,\sigma}(z)$ differ only by a factor, namely $U_{0,\sigma}(z)=\sigma U_{0,1}(z)$. Of course, (3.7) holds also for $\varepsilon=0$, and we have

$$\operatorname{Re}[\overline{E_n(e^{i\theta_p})}\,\sigma U_{0,1}(e^{i\theta_p})]\le 0$$

for *every admissable* σ. Also, note that both of the numbers $E_n(e^{i\theta_p})$ and $U_{0,1}(e^{i\theta_p})$ are different from zero. Taking this into account, we obtain

$$\overline{E_n(e^{i\theta_p})}\,U_{0,1}(e^{i\theta_p})<0. \tag{3.8}$$

Now, $U_{0,1}(z)$ has the representation

$$U_{0,1}(z)=cz(z-1)\prod_{j=1}^{n}{}'(z-e^{i\theta_j}),$$

where c is some nonzero constant, and where the prime indicates that $j=p$ and $j=q$ are to be omitted in the above product. Writing

$$\gamma:=\frac{1}{2}\sum_{j=1}^{n}{}'\theta_j,$$

we obtain for $k\in\{p; q\}$ that

$$U_{0,1}(e^{i\theta_k})=c\exp\left[i\left(\frac{n+1}{2}\right)\theta_k\right](2i)^{n-1}e^{i\gamma}\sin\left(\frac{\theta_k}{2}\right)\prod_{j=1}^{n}{}'\sin\left(\frac{\theta_k-\theta_j}{2}\right). \tag{3.8$'$}$$

Setting $k=p$ and $k=q$, note that the corresponding factors from the last term in (3.8′), namely $\sin[(\theta_p-\theta_j)/2]$ and $\sin[(\theta_q-\theta_j)/2]$, are always of the same sign since θ_p and θ_q are adjacent. To identify $U_{0,1}(e^{i\theta_q})$, we use (3.8′) and the last relation of (3.6) to deduce that

$$\frac{U_{0,1}(e^{i\theta_p})}{E_n(e^{i\theta_q})}=\frac{U_{0,1}(e^{i\theta_p})}{U_{0,1}(e^{i\theta_q})}=\delta\exp\left[i\left(\frac{n+1}{2}\right)(\theta_p-\theta_q)\right],$$

for a certain $\delta > 0$. Now, $E_n(e^{i\theta_k})\exp[-i((n+1)/2)\theta_k]$ has modulus s for $k \in \{p; q\}$, but using (3.8) and the above expression, we obtain

$$\frac{E_n(e^{i\theta_q})\exp\left[-i\left(\frac{n+1}{2}\right)\theta_q\right]}{E_n(e^{i\theta_p})\exp\left[-i\left(\frac{n+1}{2}\right)\theta_p\right]} = -1.$$

An inductive argument with respect to p then completes the proof of Proposition 3. ■

Proposition 4. *Let φ be such that $|E_n(e^{i\theta})|$ attains its global maximum (minimum) at φ. Then,*

$$\frac{e^{i\varphi}E_n'(e^{i\varphi})}{E_n(e^{i\varphi})} \underset{(<)}{>} 0.$$

In particular, $E_n'(e^{i\varphi}) \neq 0$.

Proof. This is a consequence of the Julia–Wolff theorem in Pommerenke [5, p. 306]. ■

Proposition 5. *$E_n'(z)$ has all its zeros on $|z| = 1$. If these zeros are $e^{i\psi_j}$ with $\psi_j \in [0, 2\pi)$ in increasing order, then*

$$0 < \theta_1 < \psi_1 < \theta_2 < \psi_2 < \cdots < \psi_{n-1} < \theta_n < 2\pi. \tag{3.9}$$

Furthermore,

$$\alpha E_n'(z) = \bar{\alpha} z^{n-1}\overline{E_n'(1/\bar{z})} \qquad (z \in \mathbb{C}), \tag{3.10}$$

where α is as in Proposition 3.

Proof. Since $|E_n(e^{i\theta})|$ attains its global minimum in each of the points θ_j, Proposition 4 can be applied to each of these points. Using Proposition 3, this implies that

$$g(\theta) := \mathrm{Im}\left[\alpha \exp\left[-i\left(\frac{n-1}{2}\right)\theta\right]E_n'(e^{i\theta})\right]$$

vanishes in the n points $\{\theta_j\}_{j=1}^n$ in $(0, 2\pi)$. Now, $G(\theta) := g(2\theta)$ is a trigonometric polynomial of degree $n-1$, which vanishes in the n points $\{\theta_j/2\}_{j=1}^n$ in $(0, \pi)$. But, as $G(\theta + \pi) = (-1)^{n-1}G(\theta)$, then $G(\theta)$ has $2n$ zeros in $(0, 2\pi)$. Hence, $G(\theta) \equiv 0 \equiv g(\theta)$. This shows that

$$\alpha \exp\left[-i\left(\frac{n-1}{2}\right)\theta\right]E_n'(e^{i\theta}) \tag{3.11}$$

is real-valued. Another application of Propositions 3 and 4 proves that the function of (3.11) has alternating signs in the points θ_j, and thus, it must have a zero ψ_j in each of the intervals (θ_j, θ_{j+1}), $j = 1, \ldots, n-1$. The points $e^{i\psi_j}$ are therefore the $n-1$ zeros of the polynomial $E_n'(z) \in \mathbf{P}_{n-1}$. Furthermore, since (3.11) is real-valued, we have

$$\alpha\, e^{-i(n-1)\theta} E_n'(e^{2i\theta}) = \bar{\alpha}\, e^{i(n-1)\theta} \overline{E_n'(e^{2i\theta})},$$

or, for $|z|=1$,

$$\alpha E_n'(z^2) = \bar{\alpha} z^{2(n-1)} \overline{E_n'(1/\bar{z}^2)}.$$

This clearly extends to all $z \in \mathbf{C}$, and replacing z^2 by z, we obtain (3.10). ■

Now, let

$$f(\theta) := \operatorname{Re}\left[\alpha \exp\left[-i\left(\frac{n+1}{2}\right)\theta\right] E_n(e^{i\theta})\right].$$

Differentiating the relation

$$2 \exp\left[i\left(\frac{n+1}{2}\right)\theta\right] f(\theta) = \alpha E_n(e^{i\theta}) + \bar{\alpha}\, e^{i(n+1)\theta} \overline{E_n(e^{i\theta})}$$

with respect to θ and using (3.10), we find that

$$E_n(e^{i\theta}) = \bar{\alpha} \exp\left[i\left(\frac{n+1}{2}\right)\theta\right]\left[f(\theta) + \left(\frac{2i}{n+1}\right) f'(\theta)\right], \tag{3.12}$$

and, in particular, since $|\alpha|=1$,

$$e(\theta) := |E_n(e^{i\theta})|^2 = f^2(\theta) + \frac{4}{(n+1)^2}(f'(\theta))^2. \tag{3.13}$$

From Proposition 3, we have $f^2(\theta_j) = s^2$, and since $e(\theta_j) = s^2$ by definition, we see from (3.13) that

$$f'(\theta_j) = 0, \qquad (j = 1, \ldots, n). \tag{3.14}$$

Differentiating (3.13) and (3.12) with respect to θ, we obtain

$$e'(\theta) = 2f'(\theta) \cdot h(\theta), \tag{3.15}$$

and

$$E_n'(e^{i\theta}) = (n+1)\bar{\alpha} \exp\left[i\left(\frac{n-1}{2}\right)\theta\right] \frac{h(\theta)}{2}, \tag{3.16}$$

where $h(\theta) := f(\theta) + [4/(n+1)^2] f''(\theta)$. Now, $e'(\theta)$ is a trigonometric polynomial of degree n. It has n zeros in the points θ_j, which correspond to the global minima of $e(\theta)$, and it has the $n-1$ zeros ψ_j from Proposition 5 and from (3.15) and (3.16). But, none of these points can be the global maximum of $e(\theta)$ since $E_n'(z)$ vanishes in these points (cf. Proposition 4). Hence, $e'(\theta)$ must have one further zero $\theta_0 \in [0, 2\pi)$ which corresponds to the global maximum of $e(\theta)$. But, this implies that $e'(\theta)$ has the maximal number of zeros in $[0, 2\pi)$, and each zero of

$e'(\theta)$ must be simple and correspond to an extremum of $e(\theta)$. Furthermore, we observe from (3.15) that

$$f'(\theta_0)=0 \tag{3.17}$$

since $h(\theta_0)\neq 0$ from (3.16) and Proposition 4, and that all zeros of $f'(\theta)$ are simple.

Regarding θ_0, we note that it must be located in $[0, \theta_1)$ or in $(\theta_n, 2\pi)$, since the relative extrema of $e(\theta)$ in $[\theta_1, \theta_n]$ can occur only in the points $\{\theta_j\}_{j=1}^n$ or $\{\psi_j\}_{j=1}^{n-1}$.

We now turn to the discussion of $f(\theta)$. Our knowledge about the zeros of $f'(\theta)$ tells us that $f(\theta)$ is monotonic in each of the intervals (θ_j, θ_{j+1}), $j=1,\ldots,n-1$, and since $f(\theta_j)=-f(\theta_{j+1})=\pm s$ by Proposition 3, we conclude that f oscillates between the values $\pm s$ in $[\theta_1, \theta_n]$. The nonnegative trigonometric polynomial $f^2(\theta)$ of degree $n+1$ has a relative maximum at θ_0 with $f^2(\theta_0)=|E_n(e^{i\theta_0})|^2>|E_n(0|^2=1$ by the maximum principle (cf. (3.13), (3.17), and the definition of θ_0), and relative maxima at θ_j with $f^2(\theta_j)=s^2$, $j=1,\ldots,n$. The only minima of $f^2(\theta)$ are its zeros since $f'(\theta)$ has no other zeros than $\theta_0,\ldots,\theta_n$. Now, θ_0 is exterior to $[\theta_1, \theta_n]$. This shows (because $s^2<1$) that there are precisely two more points $\varphi_1<\theta_1$, $\varphi_2>\theta_n$ with $0<\varphi_2-\varphi_1<2\pi$ such that $f^2(\varphi_1)=f^2(\varphi_2)=s^2$, and there are zeros of $f^2(\theta)$ in (φ_1, θ_1) and (θ_n, φ_2). In the interval $I:=[\varphi_1, \varphi_2]$ (which is not necessarily in $[0, 2\pi)$ but of length $<2\pi$), the trigonometric polynomial

$$g_1(\theta):=2\frac{f^2(\theta)}{s^2}-1$$

has the following properties:

(i) $|g_1(\theta)|\leq 1, \qquad \theta\in I$;
(ii) $g_1(\theta)$ takes on the value 1 in $n+2$ distinct points of I, including the endpoints of I;
(iii) $g_1(\theta)$ takes on the value -1 in $n+1$ distinct points of I.

Clearly, $g_1(\theta)$ is a trigonometric polynomial of degree $n+1$. Now, let $\varphi_0:=(\varphi_1+\varphi_2)/2-\pi$, and set

$$g_2(\theta):=T_{2n+2}\left(\frac{\cos\left(\frac{\theta-\varphi_0}{2}\right)}{\cos\left(\frac{\varphi_1-\varphi_0}{2}\right)}\right).$$

(We remark that $\cos[(\varphi_1-\varphi_0)/2]=\sin[(\varphi_2-\varphi_1)/4]$. Thus, since $0<\varphi_2-\varphi_1<2\pi$, this term in the denominator above is *positive.*)

Now, $g_2(\theta)$ is also a trigonometric polynomial of degree $n+1$, and also satisfies the above properties (i), (ii), (iii). A simple counting argument shows that $g_1(\theta)-g_2(\theta)$ must have at least $2n+3$ zeros in some interval of length $<2\pi$. Thus, $g_1(\theta)\equiv g_2(\theta)$ and the relation $T_{2n+2}(x)=2T_{n+1}^2(x)-1$ gives finally that

$$f(\theta)=\varepsilon s T_{n+1}\left(\sigma\cos\left(\frac{\theta-\varphi_0}{2}\right)\right), \tag{3.18}$$

with $\sigma := 1/\cos[(\varphi_1-\varphi_0)/2]$ and $\varepsilon \in \{-1; 1\}$. Inserting this into (3.12), we obtain

$$(3.19)\qquad E_n(e^{i\theta}) = \varepsilon\bar{\alpha}s\exp\left[i\left(\frac{n+1}{2}\right)\theta\right]\left[T_{n+1}\left(\sigma\cos\left(\frac{\theta-\varphi_0}{2}\right)\right)\right.$$
$$\left.-\frac{\sigma i}{n+1}\sin\left(\frac{\theta-\varphi_0}{2}\right)T'_{n+1}\left(\sigma\cos\left(\frac{\theta-\varphi_0}{2}\right)\right)\right].$$

Using $z=\exp[i((\theta-\varphi_0)/2)]$ and $\cos((\theta-\varphi_0)/2)=\frac{1}{2}(z+1/z)$ on $|z|=1$, the above can be written as

$$(3.20)\quad E_n(e^{i\varphi_0}z^2) = -\frac{\varepsilon\bar{\alpha}s}{(n+1)}\exp\left[i\left(\frac{n+1}{2}\right)\varphi_0\right]z^{2n+3}\frac{d}{dz}\left\{z^{-(n+1)}T_{n+1}\left[\frac{\sigma}{2}\left(z+\frac{1}{z}\right)\right]\right\},$$

which extends to all $z\in\mathbf{C}$. In particular, a short calculation with (3.20) gives, with (1.3), that

$$E_n(0)=\varepsilon\bar{\alpha}\sigma^{n+1}s\exp\left[i\left(\frac{n+1}{2}\right)\varphi_0\right]=1,$$

which implies

$$\sigma = s^{-1/(n+1)},\quad \text{and}\quad \varepsilon\alpha=\exp\left[i\left(\frac{n+1}{2}\right)\varphi_0\right].$$

Recalling the definition of $Q_{n,\rho}(z^2)$ of (2.6), our result of (3.20) can be written as

$$(3.21)\qquad E_n(z)=Q_{n,s}(e^{-i\varphi_0}z);\qquad \mu=Q_{n,s}(e^{-i\varphi_0}).$$

Now, let $0<\rho\le 1$, and assume that $\mu=Q_{n,\rho}(z_\rho)$, where $z_\rho\in\bar{\mathbf{D}}$. Then, we have $Q_{n,\rho}(z_\rho z)\in\mathbf{P}_n(\mu)$ and, by the minimum principle, we see that

$$S_n(\mu)\ge\min_{|z|\le 1}|Q_{n,\rho}(z_\rho z)|\ge\min_{|z|\le 1}|Q_{n,\rho}(z)|.$$

Hence,

$$(3.22)\qquad S_n(\mu)\ge\max\{\min_{|z|\le 1}|Q_{n,\rho}(z)|\colon \mu\in Q_{n,\rho}(\bar{\mathbf{D}})\}.$$

On the other hand, (3.21) yields

$$(3.23)\qquad s=S_n(\mu)=\min_{|z|\le 1}|E_n(z)|=\min_{|z|\le 1}|Q_{n,s}(z)|,$$

and $\mu\in Q_{n,s}(\bar{\mathbf{D}})$. Hence, we have equality in (3.22). To complete our proof of part (c) of Theorem 1, it remains to show that

$$(3.24)\qquad \rho=\min_{|z|\le 1}|Q_{n,\rho}(z)|\qquad (0<\rho<1).$$

In view of (3.23), this is true for every ρ which equals $S_n(\mu)$ for a certain $\mu\in\Omega_n$. The interval $(1,2^n)$ belongs to Ω_n, because of part (a) of Theorem 1 and the fact that $\Delta_n\subset\bar{\mathbf{D}}$. But, $S_n(\mu)$ is continuous on $[1,2^n]$ with $S_n(1)=1$, $S_n(2^n)=0$. Thus,

$S_n(\mu)$ takes every number in $(0, 1)$ as a value in Ω_n, and the proof of part (c) of Theorem 1 is complete. ■

For further considerations, we wish to pinpoint two important facts about $Q_{n,\rho}(e^{i\theta})$ which follow from our previous deductions. We again use the notations of the last proof. We know that $|E_n(e^{i\theta})|^2$ attains its global extrema (1 maximum, n minima) precisely where f' vanishes, and from (3.18), we see that these are the points $\theta = \varphi_0$ and the solutions of

$$T'_{n+1}\left(\sigma \cos\left(\frac{\theta - \varphi_0}{2}\right)\right) = 0. \tag{3.25}$$

Since $\sigma > 1$, we see that φ_0 must be the global maximum and the remaining n zeros (where $T_{n+1}(x) = \pm 1$ and where x is in $(-1, +1)$) are the minima. From (3.21), we now see that $|Q_{n,s}(e^{i\theta})|$ takes its only global maximum at $\theta = 0$. Also, replacing $\theta - \varphi_0$ by θ, we see from (3.21), (3.25) and the determination of σ, that $|Q_{n,\rho}(e^{i\theta})|$ takes its global minima in the points θ satisfying

$$\rho^{-1/(n+1)} \cos\left(\frac{\theta}{2}\right) = \cos\left(\frac{j\pi}{n+1}\right) \qquad (j = 1, \ldots, n). \tag{3.26}$$

Here, we have made use of the explicit representation of the zeros of $T'_{n+1}(x)$.

We now turn to the remaining proof of part (b) of Theorem 1. It rests mainly on two simple ideas: to determine a simply connected closed set whose boundary consists of boundary points of Δ_n, and to show that Δ_n is itself simply connected. Although the second part seems to be very natural, it creates the main difficulty in our proof. For the case $n = 1$, of course, the assertion that $\Delta_1 = [0, 1]$ is easily verified.

Proposition 6. *Let $0 \neq \mu \in \partial\Delta_n$. Then, there exists a $\theta \in [0, 2\pi)$ such that*

$$\mu = Q_{n,|\mu|}(e^{i\theta}). \tag{3.27}$$

Proof. Since $S_n(\mu) = |\mu| \neq 0$, we have $\mu \notin \partial\Sigma_n$, and thus $\mu \in \partial\Omega_n$. Choose a sequence of points $\mu_k \in \Omega_n$ with $\mu_k \to \mu$. From part (c) of Theorem 1, we know that there exist $\theta_k \in [0, 2\pi)$ such that

$$\mu_k = Q_{n,\rho_k}(e^{i\theta_k}), \qquad \rho_k = S_n(\mu_k) \to S_n(\mu) = |\mu|.$$

We may assume that $\theta_k \to \theta \in \mathbf{R}$ (otherwise, choose a subsequence), and since $Q_{n,\rho_k}(z) \to Q_{n,|\mu|}(z)$ uniformly in $\bar{\mathbf{D}}$, we get (3.27). ■

It follows from (3.26) that between $|\mu|$ and θ in (3.27), we have the relation

$$|\mu| = \left(\frac{\cos(\theta/2)}{\cos\left(\frac{j\pi}{n+1}\right)}\right)^{n+1} \qquad (j = 1, \ldots, n).$$

On the other hand, a direct calculation of $Q_{n,|\mu|}(e^{i\theta})$ in these points gives

$$\mu = Q_{n,|\mu|}(e^{i\theta}) = (-1)^j |\mu| \exp\left[i\left(\frac{n+1}{2}\right)\theta\right] \qquad (j = 1, \ldots, n).$$

These points μ constitute the following n curves in $\bar{\mathbf{D}}$, connecting 1 and 0:

$$\text{(3.28)}\qquad \begin{cases} C_{n,j} := \left\{ (-1)^j \left(\dfrac{\cos\psi}{\cos\left(\dfrac{j\pi}{n+1}\right)} e^{i\psi} \right)^{n+1} : \dfrac{j\pi}{n+1} \le \psi \le \dfrac{\pi}{2} \right\}, \\ C_{n,n+1-j} := \{\bar{\mu} : \mu \in C_{n,j}\}, \end{cases}$$

for $j = 1, \ldots, [n/2]$, while for n odd,

$$\text{(3.29)}\qquad C_{n,(n+1)/2} := \{\mu : 0 \le \mu \le 1\}.$$

From (3.29), we note that $C_{1,1} = [0, 1]$. The graphs of $C_{n,j}$, for $n = 2, 3$ and $1 \le j \le n$, are shown in Figure 2(a) and (b)

The case $n = 1$ of part (b) of Theorem 1 is trivial; hence, we assume $n \ge 2$. By Proposition 6, we have

$$\text{(3.30)}\qquad \partial\Delta_n \subset \bigcup_{j=1}^{n} C_{n,j} \subset \Delta_n,$$

and it is obvious from (2.2) and (2.1) that

$$\text{(3.31)}\qquad \Delta_{n-1} \subset \Delta_n.$$

The set $\bigcup_{j=1}^{n} C_{n,j}$ defines a number of pairwise disjoint domains, say $\psi_n(k)$, $k = 1, \ldots, K(n)$, such that

$$\text{(3.32)}\qquad \begin{cases} \bigcup_k \partial\psi_n(k) \subset \bigcup_{j=1}^{n} C_{n,j}, \\ \bigcup_k \overline{\psi_n(k)} = \bar{\mathbf{C}}. \end{cases}$$

Exactly one of them, say $\psi_n(1)$, is unbounded, and our assertion of part (b) of Theorem 1 will be shown to be equivalent to

$$\text{(3.33)}\qquad \Delta_n = \mathbf{C} \backslash \psi_n(1).$$

To this end, we have to prove that

$$\text{(3.34)}\qquad \psi_n(k) \subset \Delta_n \qquad (k = 2, \ldots, K(n)).$$

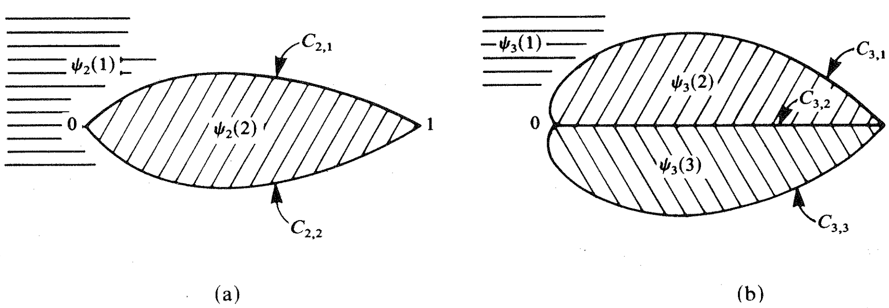

Fig. 2

Now, each point μ of $\psi_n(k)$ must, from (2.1), satisfy either

$$S_n(\mu)=|\mu| \quad (\text{whence } \mu\in\Delta_n), \quad \text{or} \quad S_n(\mu)<|\mu|.$$

But since $\psi_n(k)$ can contain no boundary point of Δ_n by definition and by (3.30), then either

$$S_n(\mu)=|\mu| \quad \text{for all } \mu\in\psi_n(k), \quad \text{or} \quad S_n(\mu)<|\mu| \quad \text{for all } \mu\in\psi_n(k).$$

To eliminate this second possibility, it suffices by (3.31) to show that

$$\psi_n(k)\cap\Delta_{n-1}\neq\varnothing \qquad (k=2,\ldots,K(n)). \tag{3.35}$$

As $\Delta_1=[0,1]$, it is evident, on superimposing Figure 2(a) and (b), that (3.35) is valid for $n=2,3$. That (3.33) and (2.8) are also equivalent in these cases is clear from Figure 2(a) and (b), and (3.28). This completes the proof for $n=2,3$, and we now may assume that $n\geq 4$. We note that $C_{n,j}$, as defined in (3.28), has the following properties $(j=1,\ldots,[n/2])$:

(3.36)
- (i) $C_{n,j}$ is a curve in $\bar{\mathbf{D}}$ connecting the points 1 and 0;
- (ii) while passing from 1 to 0, the points of $C_{n,j}$ have strictly decreasing moduli;
- (iii) similarly, their arguments are strictly increasing;
- (iv) $C_{n,j}$ crosses the negative real axis for the first time for $\psi=(j+1)\pi/(n+1)$, provided $j=1,\ldots,[n/2]-1$;
- (v) $C_{n,[n/2]}$ is a curve from 1 to 0 which lies in the closed upper-half plane.

We define the following subarcs of $C_{n,j}$:

$$\begin{cases} D_{n,j}:=\left\{(-1)^j\left(\dfrac{\cos\psi}{\cos\left(\dfrac{j\pi}{n+1}\right)}e^{i\psi}\right)^{n+1}: \dfrac{j\pi}{n+1}\leq\psi\leq\dfrac{(j+1)\pi}{n+1}\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } j=1,\ldots,[n/2]-1; \\ D_{n,j}:=\left\{(-1)^j\left(\dfrac{\cos\psi}{\cos\left(\dfrac{j\pi}{n+1}\right)}e^{i\psi}\right)^{n+1}: \dfrac{j\pi}{n+1}\leq\psi\leq\dfrac{\pi}{2}\right\} \quad \text{for } j=[n/2]; \\ D_{n,n+1-j}:=\{\bar{\mu}: \mu\in D_{n,j}\} \qquad \text{for } j=1,\ldots,[n/2], \end{cases} \tag{3.37}$$

and let $\Theta_{n,j}$ be the *bounded* Jordan domains with

$$\partial\Theta_{n,j}=D_{n,j}\cup D_{n,n+1-j} \qquad (j=1,2,\ldots,[n/2]). \tag{3.38}$$

We note that, in view of (3.36), each $\Theta_{n,j}$ is *starlike* with respect to the origin.

Proposition 7. *For $m=n$ or $m=n+1$, $1\leq j\leq[n/2]$, $1\leq k\leq[m/2]$, and either $m\neq n$ or $j\neq k$, we have*

$$\{1\}\subset\partial\Theta_{n,j}\cap\partial\Theta_{m,k}\subset\{1;0\}. \tag{3.39}$$

Proof. For reasons of symmetry, it is clear that (3.39) will follow from

$$\{1\} \subset D_{n,j} \cap D_{m,k} \subset \{1; 0\}. \tag{3.40}$$

With

$$\varphi = \frac{(j+\varepsilon)\pi}{n+1}, \qquad \psi = \frac{(k+\delta)\pi}{m+1} \qquad (0 \le \varepsilon, \delta \le 1), \tag{3.41}$$

assume that $D_{n,j}$ and $D_{m,k}$ have a common nonzero point:

$$z := (-1)^j \left(\frac{\cos\varphi}{\cos\left(\frac{j\pi}{n+1}\right)} e^{i\varphi} \right)^{n+1} = (-1)^k \left(\frac{\cos\psi}{\cos\left(\frac{k\pi}{m+1}\right)} e^{i\varphi} \right)^{m+1} =: w. \tag{3.42}$$

Then, $\arg z = \arg w$ implies $\varepsilon = \delta$. Hence, (3.42) holds if and only if

$$\left(\frac{\cos\left[\frac{(j+\varepsilon)\pi}{n+1}\right]}{\cos\left(\frac{j\pi}{n+1}\right)} \right)^{n+1} = \left(\frac{\cos\left[\frac{(k+\varepsilon)\pi}{m+1}\right]}{\cos\left(\frac{k\pi}{m+1}\right)} \right)^{m+1}, \tag{3.43}$$

or

$$\frac{\left(\cos\left[\frac{(j+\varepsilon)\pi}{n+1}\right]\right)^{n+1}}{\left(\cos\left(\frac{(k+\varepsilon)\pi}{m+1}\right)\right)^{m+1}} = \frac{\left(\cos\left(\frac{j\pi}{n+1}\right)\right)^{n+1}}{\left(\cos\left(\frac{k\pi}{m+1}\right)\right)^{m+1}}. \tag{3.44}$$

We wish to show that the left-hand side of (3.44) is strictly monotonic in ε, $0 < \varepsilon < 1$. If not, the derivative with respect to ε would vanish somewhere in $(0, 1)$, which gives

$$\begin{aligned} -\pi\left(\cos\left[\frac{(j+\varepsilon)\pi}{n+1}\right]\right)^{n} \sin\left[\frac{(j+\varepsilon)\pi}{n+1}\right]\left(\cos\left[\frac{(k+\varepsilon)\pi}{m+1}\right]\right)^{m+1} \\ = -\pi\left(\cos\left[\frac{(k+\varepsilon)\pi}{m+1}\right]\right)^{m} \sin\left(\frac{(k+\varepsilon)\pi}{m+1}\right)\left(\cos\left(\frac{(j+\varepsilon)\pi}{n+1}\right)\right)^{n+1} \end{aligned}$$

or, equivalently,

$$\tan\left(\frac{(j+\varepsilon)\pi}{n+1}\right) = \tan\left(\frac{(k+\varepsilon)\pi}{m+1}\right).$$

Since both arguments are in $[0, \pi]$ where $\tan(x)$ is injective, we get the equivalent condition

$$\frac{j+\varepsilon}{n+1} = \frac{k+\varepsilon}{m+1}. \tag{3.45}$$

Now, if $n = m$, we deduce $j = k$, which violates the hypotheses of Proposition 7. However, for $m = n+1$, we get from (3.45) that $\varepsilon = k(n+1) - j(n+2)$, which is an integer and hence is not in (0, 1). This contradiction shows that the left side of (3.44) is strictly monotone in ε for $0<\varepsilon<1$, so that equality holds in (3.44) only for $\varepsilon=0$. Thus, the only nonzero point of $D_{n,j}\cup D_{m,k}$ is $\{1\}$. On the other hand, (3.36v) shows that 0 can be a point of $D_{n,j}\cap D_{m,k}$, which gives (3.39). ∎

Starting from the point 1, the curves $D_{n,j}$ enter into the unit disk as follows:

$$1+i\frac{n+1}{\cos\left(\frac{j\pi}{n+1}\right)}\exp\left[i\left(\frac{j\pi}{n+1}\right)\right]\left(\psi-\frac{j\pi}{n+1}\right)+o\left(\psi-\frac{j\pi}{n+1}\right). \tag{3.46}$$

From this fact, we can deduce that *initially* $D_{n,j}\subset\Theta_{n,j-1}\cup\{1\}$, and in view of Proposition 7,

$$\Theta_{n,j}\subset\Theta_{n,j-1}\qquad(j=2,\ldots,[n/2]), \tag{3.47}$$

and therefore

$$\Theta_{n,j}\subset\Theta_{n,1}\qquad(j=2,\ldots,[n/2]). \tag{2.48}$$

Similarly, the initial portion of $D_{n,2}$ must be contained in $\Theta_{n-1,1}$ since (cf. (3.46))

$$\frac{2}{n+1}>\frac{1}{n}\qquad(n\geq 4).$$

Hence, again by Proposition 7 (with $m=n-1$),

$$\Theta_{n,2}\subset\Theta_{n-1,1}; \tag{3.49}$$

in fact,

$$\overline{\Theta_{n,2}}\backslash\{1;0\}\subset\Theta_{n-1,1}, \tag{3.50}$$

since the boundaries have at most the points 1 and 0 in common. As $\Theta_{n,j}$ is starlike with respect to the origin, we deduce that

$$C_{n,j}\subset\overline{\Theta_{n,j}}\subset\overline{\Theta_{n,1}}\qquad(j=1,\ldots,n), \tag{3.51}$$

and, we similarly deduce from (3.47) and (3.50) that

$$C_{n,j}\subset\overline{\Theta_{n,j}}\subset\overline{\Theta_{n,2}}\subset\Theta_{n-1,1}\cup\{1;0\}\qquad(j=2,\ldots,n-1). \tag{3.52}$$

From (3.38) and (3.51), it is clear that (3.33) is equivalent to

$$\Delta_n=\overline{\Theta_{n,1}}, \tag{3.53}$$

as well as to our assertion (b) in Theorem 1. We now proceed by mathematical induction. The assertion has already been established for $n=1,2,3$. Assume (3.53) holds for $n-1$. Then, by (3.52), we see that

$$C_{n,j}\backslash\{1,0\}\subset\operatorname{int}\Delta_{n-1}\qquad(j=2,\ldots,n-1), \tag{3.54}$$

and (3.35) is fulfilled for every $\psi_n(k)$ which has one of these curves in its boundary. But, from (3.28), the curves $C_{n,1}\backslash D_{n,1}$, $C_{n,n}\backslash D_{n,n}$ do not intersect in $\Theta_{n,1}\backslash\Theta_{n-1,1}$

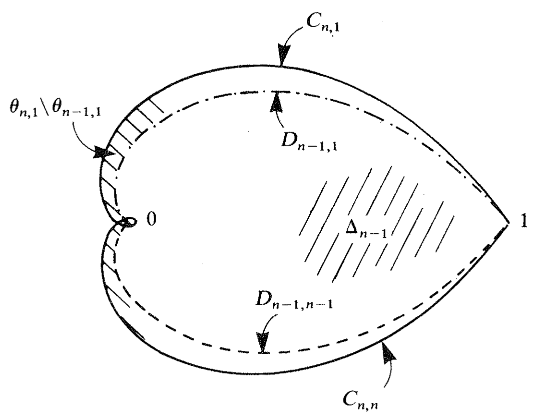

Fig. 3

(see Figure 3), so that each of the remaining domains $\psi_n(k)$, $k \neq 1$, also contains points of Δ_{n-1}. This proves (3.35), (3.33) and therefore (3.53). ∎

4. Proofs of the Corollaries

We begin this section with the following

Proposition 8. *Write* $Q_{n,\rho}(w) := \sum_{k=0}^{n} a_k(\rho) w^k$, *where* $Q_{n,\rho}(z^2)$ *is defined in* (2.6). *Then,*

$$a_k(\rho) > 0 \qquad (k = 0, 1, \ldots, n;\ 0 < \rho < 1). \tag{4.1}$$

Proof. For any $\sigma > 1$ and any $n \geq 1$, let the coefficients $b_k(\sigma; n)$ be defined by

$$z^n T_n\left(\sigma \frac{1+z^2}{2z}\right) =: \sum_{k=0}^{2n} b_k(\sigma; n) z^k. \tag{4.2}$$

Using the known expansion of the Chebyshev polynomial $T_n(x)$ in powers of x, we immediately see that the odd coefficients $b_{2k+1}(\sigma; n)$ of (4.2) all vanish $(k = 0, \ldots, n-1)$, while for the even coefficients, we have

$$b_{2k}(\sigma; n) = \frac{n\sigma^n}{2} \sum_{j=0}^{\min\{k; n-k\}} \frac{(n-j-1)!\,(-\sigma^2)^{-j}}{j!\,(k-j)!\,(n-j-k)!} \qquad (k = 0, 1, \ldots, n). \tag{4.3}$$

We next claim that $b_{2k}(\sigma; n) > 0$ for all $k = 0, 1, \ldots, n$. As $b_{2k}(\sigma; n) = b_{2(n-k)}(\sigma; n)$ from (4.3), it suffices to show that $b_{2k}(\sigma; n) > 0$ for all $k = 0, 1, \ldots, [n/2]$. Equivalently, it suffices to establish the positivity of

$$c_k(x) := \sum_{j=0}^{k} \frac{(n-j-1)!\, x^j}{j!\,(k-j)!\,(n-j-k)!} \qquad (-1 < x \leq 0;\ k = 0, 1, \ldots, [n/2]). \tag{4.4}$$

For $k=0$, this is obvious. For $k>0$, expanding $c_k(x)$ about $x=-1$ and applying standard combinatorial identities, we obtain

$$c_k(x)=\sum_{j=0}^{k-1}\binom{k-1}{j}\frac{(n-k-1)!\,(1+x)^{j+1}}{(n-j-k-1)!\,(j+1)!}\qquad (k=1,2,\ldots,[n/2]),$$

which *is* positive for all $-1<x\leq 0$ and for all $k=1,2,\ldots,[n/2]$. Thus, $b_{2k}(\sigma;n)>0$ for all $k=0,1,\ldots,n$ and all $\sigma>1$.

To complete the proof, from (2.6) we can write

$$(4.5)\quad Q_{n,\rho}(z^2)=-\frac{\rho}{(n+1)}z^{2n+3}\frac{d}{dz}\left\{z^{-2n-2}\left(z^{n+1}T_{n+1}\left(\rho^{-1/(n+1)}\left(\frac{1+z^2}{2z}\right)\right)\right)\right\}.$$

Replacing n by $n+1$ in (4.2) and setting $\sigma:=\rho^{-1/(n+1)}$, we have from (4.2) and (4.5) that

$$Q_{n,\rho}(z^2)=-\frac{z^{2n+3}}{(n+1)\sigma^{n+1}}\frac{d}{dz}\left\{\sum_{k=0}^{n+1}b_{2k}(\sigma;n+1)z^{2(k-n-1)}\right\}.$$

Thus, on differentiating in the above expression and then replacing z^2 by w, we have

$$(4.6)\qquad Q_{n,\rho}(w)=\frac{2}{(n+1)\sigma^{n+1}}\sum_{k=0}^{n}b_{2k}(\sigma;n+1)(n+1-k)w^k,$$

so that the positivity of the coefficients $b_{2k}(\sigma;n+1)$ implies the sought positivity of the $a_k(\rho)$ of (4.1). ■

On setting $z=1$ in (2.6), it is easily verified that

$$(4.7)\qquad Q_{n,\rho}(1)=\rho T_{n+1}(\rho^{-1/(n+1)}).$$

Proposition 9. *For each $n\geq 1$, $Q_{n,\rho}(1)$ is a strictly decreasing function of ρ for $0<\rho<1$.*

Proof. On differentiating the right side of (4.7) with respect to ρ, the strictly decreasing nature of $Q_{n,\rho}(1)$ for $0<\rho<1$ is equivalent to the property that

$$(4.8)\qquad T_{n+1}(x)<\frac{x}{(n+1)}T'_{n+1}(x)\qquad (x>1).$$

Since both sides of (4.8) are positive for $x>1$, then squaring yields

$$(4.9)\qquad T^2_{n+1}(x)<\frac{x^2}{(n+1)^2}(T'_{n+1}(x))^2=\frac{x^2}{(x^2-1)}(T^2_{n+1}(x)-1),$$

where we have used the identity that $(1-x^2)(T'_n(x))^2=n^2(1-T^2_n(x))$. Thus, (4.9) is equivalent to $T^2_{n+1}(x)>x^2$, which is obviously valid for all $x>1$ and all $n\geq 1$. ■

Proof of Corollary 1. From Theorem 1, we know that $[0, 1] \in \Delta_n$, while $[2^n, +\infty) \in \Sigma_n$. Thus, with the definitions of Δ_n and Σ_n in (2.2) and (2.4), the first and third assertions in (2.10) of Corollary 1 are clearly valid. It remains to consider then the remaining interval $(1, 2^n)$, which is in Ω_n.

First, we note from (4.7) that

$$\lim_{\rho\to 1-} Q_{n,\rho}(1) = 1, \quad \text{and} \quad \lim_{\rho\to 0+} Q_{n,\rho}(1) = 2^n.$$

Then, with Proposition 9, it follows that for each μ with $1 < \mu < 2^n$, there is a unique $\sigma \in (0, 1)$ for which $\mu = \sigma T_{n+1}(\sigma^{-1/(n+1)})$, so that (cf. (4.7))

$$\mu = Q_{n,\sigma}(1) \in Q_{n,\sigma}(\bar{\mathbf{D}}). \tag{4.10}$$

Next, as a consequence of Proposition 8, for $0 < \rho < 1$, $|Q_{n,\rho}(z)|$ evidently takes its global maximum in $\bar{\mathbf{D}}$ only in the point $z = 1$. Thus, using Proposition 9, if $1 > \rho > \sigma$, then for all $z \in \bar{\mathbf{D}}$,

$$|Q_{n,\rho}(z)| \le Q_{n,\rho}(1) < Q_{n,\sigma}(1) = \mu,$$

whence $\mu \notin Q_{n,\rho}(\bar{\mathbf{D}})$ for all $1 > \rho > \sigma$. This fact, combined with (4.10) and part (c) of Theorem 1, establishes $S_n(\mu) = \sigma$. Consequently, $Q_{n,\sigma}(z)$ is the unique extremal polynomial in $\mathbf{P}_n(\mu)$ for the problem in (1.2), and from Proposition 8, $Q_{n,\sigma}(z)$ has positive coefficients.

To conclude the proof of Corollary 1, it remains to establish (2.12) of Corollary 1. Fixing μ in $(1, 2^n)$, let σ_n be the unique solution of (2.11) in $(0, 1)$. Making use of the following well-known representation for Chebyshev polynomials,

$$2T_n(y) = (y + \sqrt{y^2 - 1})^n + (y - \sqrt{y^2 - 1})^n \qquad (y > 1),$$

we have from (2.11) that

$$2\mu = (1 + \sqrt{1 - \sigma_n^{2/(n+1)}})^{n+1} + (1 - \sqrt{1 - \sigma_n^{2/(n+1)}})^{n+1}. \tag{4.11}$$

Writing

$$\sqrt{1 - \sigma_n^{2/(n+1)}} = \frac{K}{n+1} + o\left(\frac{1}{n+1}\right) \qquad (n \to \infty), \tag{4.12}$$

it follows from (4.11) that $K = \operatorname{arccosh}(\mu)$. With this value for K, solving for σ_n in (4.12) directly gives the desired asymptotic expression (2.12). ■

Proof of Corollary 2. Let $p_n(z)$ in $\mathbf{P}_n$ be such that $p_n(z) \neq 0$ in $\bar{\mathbf{D}}$. If $p_n(z) \equiv K (K \neq 0)$, then the inequality (2.13) of Corollary 2 is trivially satisfied. Hence, we may assume that $p_n(0) = 1$, with $p_n(z) \not\equiv 1$. By the maximum principle, there is a real θ such that

$$|p_n(e^{i\theta})| = M := \max_{|z|\le 1} |p_n(z)|, \qquad \text{where} \quad M > 1.$$

Now, with $\mu := p_n(e^{i\theta})$ so that $|\mu| = M > 1$, set $\tilde{p}_n(z) := p_n(e^{i\theta} z)$. Clearly, $\tilde{p}_n(z) \in \mathbf{P}_n(\mu)$. With

$$m := \min_{|z|\le 1} |p_n(z)| = \min_{|z|\le 1} |\tilde{p}_n(z)|,$$

the hypothesis $p_n(z) \neq 0$ in $\bar{\mathbf{D}}$ gives that $0 < m < 1$. As $\tilde{p}_n(z) \in \boldsymbol{P}_n(\mu)$, it is also evident (cf. (1.4)) that $m \leq S_n(\mu) < 1$, so that

$$0 < m \leq S_n(\mu) < 1 < |\mu|. \tag{4.13}$$

Thus from (2.3), $\mu \in \Omega_n$. With definition (2.6), assume $|\mu| > Q_{n,m}(1)$. Because $Q_{n,m}(w)$ has positive coefficients from Proposition 8, then $\mu \notin \dot{Q}_{n,m}(\bar{\mathbf{D}})$, and, from Proposition 9, it further follows that $\mu \notin Q_{n,\rho}(\bar{\mathbf{D}})$ for any $m \leq \rho < 1$. But as $\mu \in \Omega_n$, (2.9) of Theorem 1, coupled with the previous statement, gives us that $S_n(\mu) = \rho$ for some ρ with $0 < \rho < m$, whence $S_n(\mu) < m$. As this contradicts (4.13), then $|\mu| \leq Q_{n,m}(1)$, which (cf. (4.7)) yields the desired result of (2.13) of Corollary 2. It is further evident that all polynomials $cQ_{n,m/|c|}(z)$, $c \in \mathbf{C}\backslash\{0\}$, gives the case of equality in (2.13). ∎

Acknowledgments. We sincerely wish to thank Mr. Geoffrey Morgan of the Department of Mathematical Sciences of Kent State University for his dedication and tireless efforts in developing the color-graphics capability for the computer-generated surface given in Fig. 1. (Unfortunately, it was not possible to reproduce his results in color here.) The work of R.S.V. was supported in part by the Department of Energy and by the Alexander von Humboldt-Stiftung.

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S. Ruscheweyh
Mathematisches Institut
Universität Würzburg
Würzburg
F.R.G.

R. S. Varga
Institute for Computational Mathematics
Kent State University
Kent
Ohio 44242
U.S.A.