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CONSTRUCTIVE APPROXIMATION

Moment Inequalities and the Riemann Hypothesis

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Dedicated to the memory of Peter Henrici, 1923–1987

Abstract. It is known that the Riemann hypothesis is equivalent to the statement that all zeros of the Riemann ξ-function are real. On writing $\xi(x/2) = 8\int_0^\infty \Phi(t)\cos(xt)\,dt$, it is known that a necessary condition that the Riemann hypothesis be valid is that the moments $\hat{b}_m(\lambda) := \int_0^\infty t^{2m} e^{\lambda t^2}\Phi(t)\,dt$ satisfy the Turán inequalities

$$(*)\qquad (\hat{b}_m(\lambda))^2 > \left(\frac{2m-1}{2m+1}\right)\hat{b}_{m-1}(\lambda)\hat{b}_{m+1}(\lambda) \qquad (m \geq 1, \lambda \geq 0).$$

We give here a constructive proof that $\log \Phi(\sqrt{t})$ is strictly concave for $0 < t < \infty$, and with this we deduce in Theorem 2.4 a general class of moment inequalities which, as a special case, establishes that the inequalities $(*)$ are in fact valid for all real λ. As the case $\lambda = 0$ of $(*)$ corresponds to the Pólya conjecture of 1927, this gives a new proof of the Pólya conjecture.

1. Introduction

Motivated by our recent solution of a 58-year-old problem of Pólya (see [13, p. 16], [4], and also [6]), we will establish in this paper a general class of moment inequalities related to the Riemann hypothesis. Moreover, we will also establish here a new property of the kernel function $\Phi(t)$ (cf. (1.2) below) figuring in the definition of the Riemann ξ-function, $\xi(x)$, where

$$(1.1)\qquad \xi\left(\frac{x}{2}\right) := 8\int_0^\infty \Phi(t)\cos(xt)\,dt,$$

and where

$$(1.2)\qquad \Phi(t) := \sum_{n=1}^{\infty} (2n^4\pi^2 e^{9t} - 3n^2\pi e^{5t})\exp(-n^2\pi e^{4t}).$$

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(We have deleted here the usual factor of 4 in the definition of $\Phi(t)$.) Now $\xi(x)$ is an entire function of order one (see p. 16 of Titchmarsh [17]), whose Taylor series about the origin can be written in the form

$$\tfrac{1}{8}\xi\left(\frac{x}{2}\right)=\sum_{m=0}^{\infty}\frac{(-1)^m\hat{b}_m}{(2m)!}x^{2m}, \tag{1.3}$$

where

$$\hat{b}_m:=\int_0^{\infty}t^{2m}\Phi(t)\,dt \qquad (m=0,1,2,\ldots). \tag{1.4}$$

On setting $z=-x^2$ in (1.3), the function $F(z)$, defined by

$$F(z):=\sum_{m=0}^{\infty}\frac{\hat{b}_m}{(2m)!}z^m, \tag{1.5}$$

is an entire function of order $\frac{1}{2}$. If x_0 is a real zero of $\xi(x/2)$, then $z_0:=-x_0^2$ is a negative real zero of $F(z)$, and the Riemann hypothesis is equivalent to the statement that all the zeros of $F(z)$ are real and negative (see, for example, [13] and [17]). It is known (see p. 24 of Boas [2], or Pólya and Schur [15]) that a *necessary condition* that $F(z)$ have only real zeros is that

$$(\hat{b}_m)^2>\left(\frac{2m-1}{2m+1}\right)\hat{b}_{m-1}\hat{b}_{m+1} \qquad (m=1,2,\ldots). \tag{1.6}$$

While the inequalities (1.6) are today commonly referred to as *Turán inequalities* (associated with the function $F(z)$ of (1.5)), they may be more precisely called the Euler-Laguerre-Pólya-Schur-Turán inequalities.

In 1927 Pólya [13, p. 16] raised the question of whether or not the Turán inequalities (1.6) are all valid. In [4] we have proved the inequalities (1.6), for $m\geq 2$, by showing that the function

$$K(t):=\int_t^{\infty}\Phi(\sqrt{u})\,du \qquad (t\geq 0) \tag{1.7}$$

is strictly logarithmically concave on the interval $(0,\infty)$, i.e. $\log K(t)$ is strictly concave on $(0,\infty)$. The case $m=1$ of (1.6) was settled numerically in [4].

The interest in the more general moment inequalities, which we shall establish in Section 2 (cf. Theorem 2.4), stems in part from the fact that they also provide necessary conditions for $\xi(x)$ (cf. (1.1)) to have only real zeros. Indeed, as a by-product of the present investigations we will obtain here a new proof of the Turán inequalities (1.6). More importantly, our proof of the general moment inequalities is based on the key result (Theorem 2.1) which asserts that the function $\log(\Phi(\sqrt{t}))$, $t\geqslant 0$, is strictly concave on $(0,\infty)$, where $\Phi(t)$ is defined by (1.2). Our verification of the logarithmic concavity of $\Phi(\sqrt{t})$ on $(0,\infty)$ entails a series of involved, but elementary, estimates. These results (Lemma 3.1–3.10) have been gathered separately in Section 3. Also, in Section 2 we will provide some additional background information pertaining to the work of de Bruijn [3], Newman [8], and Pólya [9]–[13].

For the readers convenience we mention that all original papers of G. Pólya ([9]-[13] and [15]) can be found in [14].

2. The Problem and the Main Results

The ideas developed in [4] can also be applied to problems involving some more general trigonometric integrals such as those studied by de Bruijn [3], Newman [8], Pólya [11]-[13], and Prather [16]. In order to facilitate the description of one such problem, assume that $\varphi(t)$: $\mathbf{R} \to \mathbf{R}$ and assume that

$$\text{(2.1)} \qquad \begin{array}{rl} \text{(i)} & \varphi \text{ is integrable over } \mathbf{R}, \\ \text{(ii)} & \varphi(t) = \varphi(-t),\ t \in \mathbf{R}, \text{ and} \\ \text{(iii)} & \varphi(t) = O(e^{-|t|^\alpha}),\ \alpha > 2, \text{ as } t \to \pm\infty. \end{array}$$

Now let $f(t)$ be a real entire function of genus 0 or 1 with only real zeros, and let λ satisfy $\lambda \geqslant 0$. If the function $\varphi(t)$ satisfies the conditions of (2.1) and if $\varphi(t)$ is such that all the zeros of the Fourier transform of $\varphi(t)$, namely

$$\text{(2.2)} \qquad H(x) := \int_{-\infty}^{\infty} \varphi(t)\, e^{ixt}\, dt,$$

are real, then, by a classical result of Pólya [12], the entire function

$$\text{(2.3)} \qquad \int_{-\infty}^{\infty} e^{\lambda t^2} f(it) \varphi(t)\, e^{ixt}\, dt$$

also has only real zeros. That is, in Pólya's terminology (see p. 7 of [12]), the functions of the form $e^{\lambda t^2} f(it)$, $\lambda \geq 0$, are *universal factors* which preserve the reality of the zeros of any entire function of the form (2.2), where $H(x)$ has only real zeros and where $\varphi(t)$ satisfies the conditions of (2.1). In fact, Pólya [12] has shown that the functions $e^{\lambda t^2} f(it)$, $\lambda \geq 0$, are the *only* analytic functions which enjoy the aforementioned property.

Now choose $\varphi(t) = \Phi(t)$, where $\Phi(t)$ is defined by (1.2). Then it is known that $\Phi(t)$ satisfies the conditions of (2.1) (see, for example, Theorem A of [4]). Thus, if the Riemann hypothesis is true, then for *any* universal factor $e^{\lambda t^2} f(it)$, with $\lambda \geq 0$, the entire function

$$\text{(2.4)} \qquad \int_{-\infty}^{\infty} e^{\lambda t^2} f(it) \Phi(t)\, e^{ixt}\, dt$$

must have only real zeros. In particular, with $f(it) \equiv 1$ the entire function given by (2.4) reduces to the function

$$\text{(2.5)} \qquad \int_{-\infty}^{\infty} e^{\lambda t^2} \Phi(t)\, e^{ixt}\, dt = 2 \int_{0}^{\infty} e^{\lambda t^2}\, \Phi(t) \cos(xt)\, dt,$$

and the Riemann hypothesis implies that the above function of x has only real zeros for any $\lambda \geq 0$.

Relevant to the foregoing considerations is a recent interesting result of Newman [8, Theorem 3] which, using the notations adopted here, may be expressed as follows. There exists a real number λ_0, with $-\infty<\lambda_0\le\frac{1}{2}$, such that the function $G_\lambda(x)$ of (2.5) has only real zeros when $\lambda\ge\lambda_0$ and has nonreal zeros when $\lambda<\lambda_0$. Thus, the Riemann hypothesis is the statement that $\lambda_0\le 0$. In [8] Newman makes the complementary conjecture that $\lambda_0\ge 0$ and remarks that "This new conjecture is a quantitative version of the dictum that the Riemann Hypothesis, if true, is only barely so."

Now, for an *arbitrary* real number λ, consider the function

$$H_\lambda(x):=\int_0^\infty e^{\lambda t^2}\Phi(t)\cos(xt)\,dt, \tag{2.6}$$

and the corresponding moments

$$\hat{b}_m(\lambda):=\int_0^\infty t^{2m}e^{\lambda t^2}\Phi(t)\,dt \qquad (\lambda\in\mathbf{R},\ m=0,1,\ldots). \tag{2.7}$$

Since $H_0(x)$ has all its zeros in the horizontal strip $|\mathrm{Im}\ z|\le 1$ (see p. 221 of [3] or p. 246 of [8]), it follows from a result of de Bruijn [3, Theorem 13] that $H_\lambda(x)$ has only real zeros for all $\lambda\ge\frac{1}{2}$. Moreover, if $H_\lambda(x)$ has only real zeros for some λ, then $H_{\lambda'}(x)$ also has only real zeros for all $\lambda'\ge\lambda$ (see Theorem 13 of [3]). Thus, in particular, the Turán inequalities associated with the general moments (2.7), namely

$$(\hat{b}_m(\lambda))^2>\left(\frac{2m-1}{2m+1}\right)\hat{b}_{m-1}(\lambda)\hat{b}_{m+1}(\lambda) \qquad (m=1,2,\ldots), \tag{2.8}$$

hold for all $\lambda\ge\frac{1}{2}$. But if the Riemann hypothesis is true, then, by the above cited facts, $H_\lambda(x)$ must have only real zeros for all $\lambda\ge 0$, and in particular for $0\le\lambda\le\frac{1}{2}$. Therefore, a *necessary condition* for the Riemann hypothesis to be true is that the inequalities (2.8) should hold for $m=1,2,\ldots$ and for $0<\lambda<\frac{1}{2}$.

Preliminaries aside we will now proceed to establish the inequalities (2.8) for all $m\ge 1$ and for all real λ, as a special case of a more general result (cf. Theorem 2.4). Set

$$g(t):=t[(\Phi'(t))^2-\Phi(t)\Phi''(t)]+\Phi(t)\Phi'(t) \qquad (t\ge 0). \tag{2.9}$$

Then the proof of our first main result here (cf. Theorem 2.1 below) is based on the inequality

$$g(t)>0 \qquad (0<t<\infty). \tag{2.10}$$

Since the lengthy calculations leading to the proof of (2.10) (cf. Theorem 3.11) might detract from the basic ideas of this section, they have been gathered separately in Section 3.

Theorem 2.1. *The function* $\log(\Phi(\sqrt{t}))$ *is strictly concave on the interval* $(0,\infty)$, *where* $\Phi(t)$ *is defined in* (1.2).

Proof. Observing that $\Phi(t)\in C^\infty(\mathbf{R})$ and $\Phi(t)>0$ for all $t\ge 0$ (see Theorem A of [4]), an elementary calculation shows that $(d^2/dt^2)\log(\Phi(\sqrt{t}))<0$ on $(0,\infty)$

if and only if $g(t)>0$ on $(0, \infty)$, where $g(t)$ is defined in (2.9). Thus, Theorem 3.11 yields the desired result. ∎

Consider now any *even* real entire function $f(z) \not\equiv 0$ of genus 0 or 1 with only real zeros, so that $f(z)$ can be expressed as

$$f(z)=Cz^{2n}\prod_{j=1}^{\omega}\left(1-\frac{z^2}{z_j^2}\right) \qquad (\omega \leq \infty), \tag{2.11}$$

where C is a real nonzero constant, n is a nonnegative integer, and z_j satisfies $z_j>0$ $(1 \leq j \leq \omega)$ with $\sum_{j=1}^{\omega} 1/z_j^2<\infty$. Note that $f(z)\equiv 1$ is of the form (2.11). With our earlier discussion we further note that $e^{\lambda t^2} f(it)$ is a *universal factor* for any $f(z)$ of the form (2.11) and for any $\lambda \geq 0$. In addition we have from (2.11) that

$$f(it)=\tilde{C}t^{2n}\prod_{j=1}^{\omega}\left(1+\frac{t^2}{z_j^2}\right) \qquad (\tilde{C}:=(-1)^n C), \tag{2.12}$$

so that $f(it)$ is also a real *even* entire function. Without loss of generality we may assume that $\tilde{C}>0$. As $\Phi(t)$ of (1.2) is also an even function, then, for any $\lambda \geq 0$,

$$G(x; f, \lambda):=\tfrac{1}{2}\int_{-\infty}^{+\infty} e^{\lambda t^2} f(it)\Phi(t)\, e^{ixt}\, dt=\int_0^{\infty} e^{\lambda t^2} f(it)\Phi(t)\cos(xt)\, dt, \tag{2.13}$$

and $G(x{:}\, f, \lambda)$ is an entire function (see Theorem A of [4]). As previously noted, if the Riemann hypothesis true, then $G(x; f, \lambda)$ must have only real zeros. Now the Taylor series of $G(x; f, \lambda)$ about the origin can be expressed as

$$G(x; f, \lambda)=\sum_{m=0}^{\infty}\frac{(-1)^m \hat{b}_m(f; \lambda)x^{2m}}{(2m)!} \tag{2.14}$$

where

$$\hat{b}_m(f; \lambda):=\int_0^{\infty} t^{2m} e^{\lambda t^2} f(it)\Phi(t)\, dt \qquad (m=0, 1, \ldots). \tag{2.15}$$

On setting $z=-x^2$ in (2.14), the function

$$F(z; f, \lambda):=\sum_{m=0}^{\infty}\frac{\hat{b}_m(f; \lambda)z^m}{(2m)!}$$

has only real negative zeros if the Riemann ξ-function has only real zeros. As in (1.6), a *necessary condition* that $F(z; f, \lambda)$ have only real zeros is that

$$D_m(f; \lambda):=(\hat{b}_m(f; \lambda))^2-\left(\frac{2m-1}{2m+1}\right)\hat{b}_{m-1}(f; \lambda)\hat{b}_{m+1}(f; \lambda)>0 \qquad (m=1, 2, \ldots) \tag{2.16}$$

for *all* $\lambda \geq 0$ and *all* functions $f(z)$ of the form (2.11).

We will deduce from Theorem 2.1 the inequalities of (2.16) for all real λ and all $f(z)$ of the form (2.11). This can be done using two distinct approaches. The first approach is patterned after Matiyasevich's triple integral representation [6] of the *Turán differences* of (2.16).

Proposition 2.2. *Let λ be a fixed but arbitrary real number, let $f(z)$ be of the form* (2.11), *and set*

$$\Psi(t)=\Psi(t;f,\lambda):=e^{\lambda t^2}f(it)\Phi(t), \tag{2.17}$$

where $\Phi(t)$ is defined in (1.2). *Then, for $m\geq 1$,*

$$2(2m+1)D_m(f;\lambda)=\int_0^\infty\int_0^\infty u^{2m}v^{2m}(v^2-u^2)I(u,v;f,\lambda)\,du\,dv, \tag{2.18}$$

where

$$I(u,v;f,\lambda):=\Psi(u)\Psi(v)\int_u^v -\frac{d}{dt}\left(\frac{\Psi'(t)}{t\Psi(t)}\right)dt,$$

and where $D_m(f;\lambda)$ is defined in (2.16).

Proof. From (2.15) and (2.17) $\hat{b}_m(f;\lambda)=\int_0^\infty t^{2m}\Psi(t)\,dt$, and an integration by parts, applied to this integral, yields

$$\hat{b}_m(f;\lambda)=-\frac{1}{(2m+1)}\int_0^\infty t^{2m+1}\Psi'(t)\,dt \qquad (\lambda\in\mathbf{R},\ m\geq 0). \tag{2.19}$$

Hence, using (2.19) we have, for $\lambda\in\mathbf{R}$ and $m\geq 1$, that

$$\begin{aligned}
&\int_0^\infty\int_0^\infty u^{2m}v^{2m}\Psi(u)\Psi(v)(v^2-u^2)\left(\int_u^v -\frac{d}{dt}\left(\frac{\Psi'(t)}{t\Psi(t)}\right)dt\right)du\,dv\\
&=-(2m-1)\hat{b}_{m-1}(f;\lambda)\int_0^\infty v^{2m+2}\Psi(v)\,dv\\
&\quad+(2m+1)\hat{b}_m(f;\lambda)\int_0^\infty v^{2m}\Psi(v)\,dv\\
&\quad+(2m+1)\hat{b}_m(f;\lambda)\int_0^\infty u^{2m}\Psi(u)\,du\\
&\quad-(2m-1)\hat{b}_{m-1}(f;\lambda)\int_0^\infty u^{2m+2}\Psi(u)\,du\\
&=2(2m+1)D_m(f,\lambda).
\end{aligned}$$

■

The second proof of the inequality (2.16) will be based on the following slight extension of the known result (see, for example, Marshall and Olkin [7, Proposition E.4 part (2)] of Barlow, Marschall, and Proschan [1]).

Proposition 2.3. *Let λ be a fixed but arbitrary real number, let $f(z)$ be of the form* (2.11), *and set*

$$\mu_x:=\frac{1}{\Gamma(x+1)}\int_0^\infty t^x\Psi(\sqrt{t})\,dt \qquad (x>-1,\ \lambda\in\mathbf{R}), \tag{2.20}$$

where $\Gamma(x)$ denotes the gamma function and where $\Psi(t)$ is defined by (2.17). *If $\log(\Psi(\sqrt{t}))$ is strictly concave for $0<t<\infty$, then $\log\mu_x$ is strictly concave for $-1<x<\infty$.*

Proof. The Barlow, Marshall, and Proschan result asserts that if $\log(\Psi\sqrt{t}))$ is concave for $0 \leq t < \infty$, then $\log \mu_x$ is concave for $x \geq 0$. The extension of this result from the concave case to the strictly concave case is *mutatis mutandis* the same as the proof of Proposition 2.4 of [4]. ■

Theorem 2.4. *For any* $f(z)$ *of the form* (2.11), *set*

$$\hat{b}_m(f;\lambda) := \int_0^\infty t^{2m} e^{\lambda t^2} f(it)\Phi(t)\,dt \qquad (\lambda \in \mathbf{R},\ m = 0, 1, 2, \ldots).$$

Then, the following Turán inequalities

$$(\hat{b}_m(f;\lambda))^2 > \left(\frac{2m-1}{2m+1}\right)\hat{b}_{m-1}(f;\lambda)\hat{b}_{m+1}(f;\lambda) \tag{2.21}$$

hold for all $m = 1, 2, \ldots$ *and for all real* λ.

First Proof of Theorem 2.4. To begin we note, from (2.17) and (2.12), that $\Psi(t) > 0$ for all $t \neq 0$ and all real λ (see Theorem A of [4]). Thus, by Proposition 2.2, it suffices to show that

$$E(t) := \frac{d}{dt}\left(-\frac{\Psi'(t)}{t\Psi(t)}\right) > 0 \qquad (t > 0). \tag{2.22}$$

Now an elementary calculation shows that

$$E(t) = -4t\left\{\frac{d^2}{du^2}\log \Psi(\sqrt{u})\big|_{u=t^2}\right\} \qquad (t > 0). \tag{2.23}$$

Thus, to establish (2.22), it suffices to show that

$$\frac{d^2}{du^2}\log \Psi(\sqrt{u}) < 0 \qquad (u > 0). \tag{2.24}$$

Now, from (2.17), $\Psi(\sqrt{u}) = e^{\lambda u} f(i\sqrt{u})\Phi(\sqrt{u})$ for all $u > 0$ so that

$$\log \Psi(\sqrt{u}) = \lambda u + \log f(i\sqrt{u}) + \log \Phi(\sqrt{u}) \qquad (u > 0)$$

and hence

$$\frac{d^2}{du^2}\log \Psi(\sqrt{u}) = \frac{d^2}{du^2}\log f(i\sqrt{u}) + \frac{d^2 \log \Phi(\sqrt{u})}{du^2} \qquad (u > 0). \tag{2.25}$$

From (2.12), $\log f(i\sqrt{u}) = \log \tilde{C} + n \log u + \sum_{j=1}^{\omega} \log(1 + u/z_j^2)$, so that

$$\frac{d^2}{du^2}\log f(i\sqrt{u}) = -\left\{\frac{n}{u^2} + \sum_{j=1}^{\omega}\frac{1}{(z_j^2+u)^2}\right\} < 0 \qquad (u > 0).$$

Consequently, the first term on the right in (2.25) is negative for each $u > 0$ and, from Theorem 2.1, the second term on the right in (2.25) is also negative for each $u > 0$. Thus, (2.24) is valid. ■

Second Proof of Theorem 2.4. For a fixed but arbitrary real λ, consider

$$\frac{1}{2}\mu_x := \frac{1}{2\Gamma(x+1)}\int_0^\infty t^x \Psi(\sqrt{t})\,dt \qquad (x>-1,\ \lambda\in\mathbf{R}) \tag{2.26}$$

$$= \frac{1}{\Gamma(x+1)}\int_0^\infty s^{2x+1}\Psi(s)\,ds \qquad (t := s^2)$$

$$=: T(x) = T(x; f, \lambda),$$

where $\Gamma(x)$ denotes the gamma function and $\Psi(t)$ is defined in (2.17). From the first proof of Theorem 2.4, $\log \Psi(\sqrt{t})$ is strictly concave for $0<t<\infty$. Hence, by Proposition 2.3, $\log T(x)$ is strictly concave for $-1<x<\infty$ and for any real λ. Therefore,

$$T^2(x) > T(x-\delta)T(x+\delta) \qquad (\delta>0)$$

for any x with $x-\delta>-1$. In particular, choosing $\delta=1$ and $x=n-\frac{1}{2}$ for $n=1,2,\ldots$ gives

$$T^2(n-\tfrac{1}{2}) > T(n-\tfrac{3}{2})T(n+\tfrac{1}{2}) \qquad (n=1,2,\ldots). \tag{2.27}$$

But, by (2.26), $T(n-\frac{1}{2})$ can be expressed (cf. (2.15)) as $\hat{b}_n(f;\lambda)/\Gamma(n+\frac{1}{2})$, and it follows that inequality (2.27) reduces to the desired result of (2.21). ■

Remark. Our results are also applicable in fairly general situations. Indeed, consider any entire function of the form

$$\hat{H}(x) := \int_0^\infty \Psi(t)\cos(xt)\,dt, \tag{2.28}$$

where $\Psi(t)$ is any $C^2(\mathbf{R})$ function which satisfies (2.1). Let

$$\hat{c}_m := \int_0^\infty t^{2m}\Psi(t)\,dt \qquad (m=0,1,2,\ldots) \tag{2.29}$$

denote the moments corresponding to the function $\Psi(t)$. Then, a necessary condition for the entire function $\hat{H}(x)$ to have only real zeros is that

$$\hat{c}_m^2 > \left(\frac{2m-1}{2m+1}\right)\hat{c}_{m-1}\hat{c}_{m+1} \qquad (m=1,2,\ldots). \tag{2.30}$$

By Theorem 2.4 a sufficient condition for (2.30) to hold is that

$$\frac{d^2}{dt^2}\log(\Psi(\sqrt{t})) < 0 \qquad (t>0). \tag{2.31}$$

As an example of how (2.31) can be applied, consider first the function $\hat{\Psi}(t) := \exp(-2\cosh(t))$. Then it is known (see [11]) that the cosine transform of $\hat{\Psi}(t)$, namely

$$\int_0^\infty \exp(-2\cosh(t))\cos(xt)\,dt,$$

is a real entire function having only real zeros. Since

$$\log \hat{\Psi}(\sqrt{t}) = -2\cosh(\sqrt{t}) \qquad (t \geq 0),$$

then

$$\frac{d^2}{dt^2}\log \hat{\Psi}(\sqrt{t}) = \frac{1}{2}\left\{\frac{\sinh(\sqrt{t})}{t^{3/2}} - \frac{\cosh(\sqrt{t})}{t}\right\} \qquad (t > 0).$$

But, as the Taylor expansion (in the variable $u := \sqrt{t}$) of the quantity in braces has all *negative* coefficients, we see that the sufficient condition (2.31) for the inequalities of (2.30) to hold is satisfied. It is interesting to remark that $\hat{\Psi}(t) = \exp(-2\cosh(t))$ *cannot* be expressed as $f(it)$ where $f(z)$ is of the form (2.11), so that this example does not involve universal factors.

As another application of the previous results we have the following

Corollary 2.5. *Let*

$$\Psi_\lambda(t) := \Phi(t)\cosh(\lambda t) \qquad (\lambda \in \mathbf{R}), \tag{2.32}$$

where $\Phi(t)$ *is defined in* (1.2), *and let*

$$\hat{c}_m(\lambda) := \int_0^\infty t^{2m}\Psi_\lambda(t)\,dt \qquad (m = 0, 1, 2, \ldots). \tag{2.33}$$

Then,

$$(\hat{c}_m(\lambda))^2 > \left(\frac{2m-1}{2m+1}\right)\hat{c}_{m-1}(\lambda)\hat{c}_{m+1}(\lambda) \qquad (\lambda \in \mathbf{R},\ m = 1, 2, \ldots). \tag{2.34}$$

Remark. The inequalities (2.34) are known in the special cases $\lambda \geq 1$ (see p. 32 of [13]) and $\lambda = 0$ (see [4]). For $\lambda = 1$, the kernel $\Psi_1(t)$ of (2.32) is of particular interest since, in [13], Pólya has shown that the Fourier cosine transform of $\Psi_1(t)$, i.e.,

$$F_1(x) := \int_0^\infty \Psi_1(t)\cos(xt)\,dt, \tag{2.35}$$

has only real zeros. Pólya's method also shows that the entire function

$$F_\lambda(x) := \int_0^\infty \Psi_\lambda(t)\cos(xt)\,dt$$

has only real zeros if $\lambda \geq 1$, and, consequently, (2.34) holds for all $\lambda \geq 1$.

Proof. By Proposition 2.3 it suffices to show that the function

$$h_\lambda(t) := \log(\Psi_\lambda(\sqrt{t})) \qquad (\lambda \in \mathbf{R},\ t \geq 0) \tag{2.36}$$

is strictly concave for $t > 0$. By the above Remark we may assume that $\lambda \neq 0$. Thus, from (2.32),

$$\frac{d^2}{dt^2}h_\lambda(t) = \frac{d^2}{dt^2}\log(\Phi(\sqrt{t}) + \frac{d^2}{dt^2}\log(\cosh(\lambda\sqrt{t})) \qquad (t > 0). \tag{2.37}$$

From Theorem 2.1, the first term on the right above is evidently negative for all $t>0$. With the identity $\cosh^2(\lambda u)-\sinh^2(\lambda u)=1$, a calculation shows that the final term of (2.37) is negative for all $t>0$ iff

$$\sigma_\lambda(u):=-\lambda^2 u+\lambda \sinh(2\lambda u)/2>0 \qquad (0<u<\infty). \tag{2.38}$$

But, as $\sigma_\lambda(0)=0$ and $\sigma'_\lambda(u)=\lambda^2\{-1+\cosh(2\lambda u)\}>0$ for all $\lambda\neq 0$ and all $u>0$, it follows that the inequality in (2.38) holds, and hence the function $\log(\Psi_\lambda(\sqrt{t})$ of (2.36) is strictly concave for all $t>0$. ∎

3. Background Analysis

With

$$g(t):=t[(\Phi'(t))^2-\Phi(t)\Phi''(t)]+\Phi(t)\Phi'(t) \qquad (t\geq 0), \tag{3.1}$$

which can also be expressed as

$$g(t)=(\Phi'(t))^2\frac{d}{dt}\left\{\frac{t\Phi(t)}{\Phi'(t)}\right\} \qquad (t\geq 0),$$

we see that $g(0)=0$, since it is known (see Theorem A of [4]) that $\Phi'(0)=0$. The object of the section is to show that

$$g(t)>0 \qquad (t>0), \tag{3.2}$$

and this will be shown by separately establishing that

$$g(t)>0 \qquad (0<t\leq 0.03), \tag{3.3}$$

$$g(t)>0 \qquad (0.03\leq t\leq 0.06), \tag{3.4}$$

$$g(t)>0 \qquad (0.056\leq t<\infty). \tag{3.5}$$

Lemmas 3.1-3.4 give the result of (3.3), while (3.4) is established from Lemmas 3.5 and 3.6, and (3.5) is established in Lemmas 3.7-3.10.

Throughout this section we will use the following notations, notations which are consistent with those used in [4]:

$$\begin{cases} a_n(t):=\pi n^2(2\pi n^2 e^{4t}-3)\exp(5t-\pi n^2 e^{4t}) & (n=1,2,\ldots). \\ \Phi(t):=\sum\limits_{n=1}^{\infty} a_n(t), & \\ \Phi_j(t):=\sum\limits_{n=j+1}^{\infty} a_n(t) & (j=1,2,\ldots). \end{cases} \tag{3.6}$$

On differentiating $a_n(t)$ of (3.6), we can write

$$a_n^{(j)}(t)=\pi n^2 p_{j+1}(\pi n^2 e^{4t})\exp(5t-\pi n^2 e^{4t}) \qquad (j=0,1,\ldots), \tag{3.7}$$

where the polynomials $\{p_k(y)\}_{k=1}^{\infty}$ are defined recursively from

$$p_{k+1}(y):=4yp'_k(y)+(5-4y)p_k(y) \qquad (k=1,2,\ldots), \tag{3.8}$$

where (cf. (3.6)) $p_1(y):=2y-3$. Now (3.8) gives

$$e^{-4y}y^4(p_k(y))^3 p_{k+1}(y)=\frac{d}{dy}[e^{-4y}y^5(p_k(y))^4] \qquad (k=1,2,\ldots),$$

and, with Rolle's theorem, the above expression can be used to show, inductively, that $p_k(y)$ has k distinct positive zeros $\{r_k^{(j)}\}_{j=1}^k$, ordered as $0<r_k^{(1)}<r_k^{(2)}<\cdots<r_k^{(k)}$, for each $k\geq 1$, and that the zeros of $p_k(y)$ and $p_{k+1}(y)$ *interlace*, i.e.,

$$0<r_{k+1}^{(1)}<r_k^{(1)}<r_{k+1}^{(2)}<r_k^{(2)}<\cdots<r_k^{(k)}<r_{k+1}^{(k+1)} \qquad (k=1,2,\ldots).$$

Because of their use in results to follow in this section, we list below the following particular polynomials $p_k(x)$:

$$(3.9)\quad \begin{cases} p_2(y):=-8y^2+30y-15, \\ p_3(y):=32y^3-224y^2+330y-75, \\ p_4(y):=-128y^4+1{,}440y^3-4{,}232y^2+3{,}270y-375, \\ p_5(y):=512y^5-8{,}448y^4+41{,}408y^3-68{,}096y^2+30{,}930y-1{,}875, \\ p_7(y):=8{,}192y^7-245{,}760y^6+2{,}536{,}960y^5-11{,}109{,}120y^4+20{,}633{,}312y^3 \\ \qquad -14{,}260{,}064y^2+2{,}610{,}330y-46{,}875. \end{cases}$$

In this section, numerical values for the sums of the rapidly convergent series of (3.6), as well as numerical values of the zeros of the polynomials $p_j(y)$ of (3.9), appear. These have been determined with high precision using the VAXIMA package on a VAX-11/780. The reader may find it useful to have a hand calculator available while reading portions of what follows.

Lemma 3.1. *We have*

$$(3.10)\qquad |\Phi_1''(t)|<(1.031)2^{13}\pi^4\exp(17t-4\pi e^{4t}) \qquad (t\geq 0).$$

Proof. From (3.6) and (3.7) we have

$$(3.11)\qquad \Phi_1''(t)=\sum_{n=2}^{\infty}\pi n^2 p_3(\pi n^2 e^{4t})\exp(5t-\pi n^2 e^{4t}).$$

From (3.9) it can be verified that $p_3(y)$ has three distinct positive zeros, namely $r_3^{(1)}:=0.277\,455\ldots$, $r_3^{(2)}:=1.672\,823\ldots$, and $r_3^{(3)}:=5.049\,720\ldots$, so that $p_3(y)>0$ for all $y>r_3^{(3)}$. Next, $p_3(y)<32y^3$ iff (cf. (3.9))

$$224y^2-330y+75=224(y-0.280\,790\ldots)(y-1.192\,423\ldots)>0.$$

As this last inequality holds for all $y>1.192\,423\ldots$, then

$$(3.12)\qquad 0<p_3(y)<32y^3 \qquad (y>r_3^{(3)}:=5.049\,720\ldots).$$

But as $\pi n^2 e^{4t}\geq 4\pi>r_3^{(3)}$ for all $n\geq 2$ and $t\geq 0$, then applying (3.12) to (3.11) gives

$$(3.13)\qquad |\Phi_1''(t)|<32\pi^4 e^{17t}\sum_{n=2}^{\infty}\frac{n^8}{\exp(\pi n^2 e^{4t})} \qquad (t\geq 0).$$

To bound the sum in (3.13), set $u:=\pi e^{4t}$. We then seek to find a number $K=K(u)>1$ for which

$$(3.14)\qquad \sum_{n=2}^{\infty}\frac{n^8}{e^{n^2u}}=\sum_{n=2}^{\infty}\frac{1}{e^{n^2u-8\log n}}\leq\sum_{n=2}^{\infty}\frac{1}{K^n}=\sum_{n=2}^{\infty}\frac{1}{e^{n\log K}}.$$

Indeed, it is sufficient from (3.14) that such a K should satisfy

$$n^2u - 8\log n \geq n \log K \qquad (n = 2, 3, \ldots)$$

or, equivalently,

$$nu - (8 \log n)/n \geq \log K \qquad (n = 2, 3, \ldots). \tag{3.15}$$

Set $h(s) := us - (8 \log s)/s$ for all $s \geq 2$, so that $h'(s) = u + 8(\log s - 1)/s^2$. As $\log s/s^2 > 0$ for all $s \geq 2$, then $h'(s) > u - 8/s^2 \geq \pi - 2 > 0$ for all $s \geq 2$, since $u \geq \pi$. Thus, $h(s)$ is strictly increasing for $s \geq 2$, and $h(s) \geq h(2)$ for all $s \geq 2$. This implies that

$$nu - (8 \log n)/n \geq 2u - 4 \log 2 =: \log K \qquad (n = 2, 3, \ldots).$$

Thus, $K = K(u) = e^{2u}/2^4$ satisfies (3.15). But then, the last sum in (3.14) is just

$$\frac{1}{K^2(1 - 1/K)} = \frac{2^8}{e^{4u}(1 - 16/e^{2u})} \leq \frac{2^8}{e^{4u}(1 - 16/e^{2\pi})}, \tag{3.16}$$

where the last inequality follows from $u \geq \pi$. Since $(1 - 16/e^{2\pi})^{-1} < 1.031$, inserting the above upper bound in (3.13) gives the desired result of (3.10). ∎

Lemma 3.2. *We have*

(3.17) $\Phi''(t)$ *is strictly increasing on the interval* $I := [0, 0.06]$,

and

$$\Phi''(t) < 0 \qquad \textit{for all} \quad t \in I. \tag{3.18}$$

Proof. To prove (3.17), it suffices to show that

$$\Phi'''(t) > 0 \qquad (0 < t \leq 0.06). \tag{3.19}$$

We first note that $\Phi'''(0) = 0$ (see Theorem A of [4]). Moreover, it is known (see inequality (3.36) in [4]) that

$$\Phi^{(4)}(t) > a_1^{(4)}(t) \qquad (t \geq 0), \tag{3.20}$$

where

$$a_1^{(4)}(t) = \pi p_5(\pi e^{4t}) \exp(5t - \pi e^{4t}), \tag{3.21}$$

and where $p_5(x)$ is the polynomial defined in (3.9). Now one readily verifies that $p_5(y)$ has the five distinct positive zeros $r_5^{(1)} := 0.071\,349\ldots$, $r_5^{(2)} := 0.604\,398\ldots$, $r_5^{(3)} := 1.996\,885\ldots$, $r_5^{(4)} := 4.617\,597\ldots$, and $r_5^{(5)} := 9.209\,769\ldots$. Thus, $p_5(y) > 0$ on the interval $(r_5^{(3)}, r_5^{(4)})$. But, as πe^{4t} falls in this interval for all $0 \leq t < \log(r_5^{(4)}/\pi)/4 = 0.096\,286\ldots$, it follows that

$$a_1^{(4)}(t) > 0 \qquad (t \in I). \tag{3.22}$$

Therefore, as $\Phi'''(t) = \Phi'''(0) + t\Phi^{(4)}(\xi(t)) = t\Phi^{(4)}(\xi(t))$, where $0 \leq \xi(t) \leq t$, then, from (3.20) and (3.22), $\Phi'''(t) > ta_1^{(4)}(\xi(t)) > 0$ for all $0 < t \leq 0.06$, and $\Phi''(t)$ is thus strictly increasing on I, which establishes (3.17).

To establish (3.18) we use (3.6) to obtain

$$\Phi''(t) = a_1''(t) + \Phi_1''(t) \leq a_1''(t) + |\Phi_1''(t)| \qquad (t \geq 0). \tag{3.23}$$

From (3.10) of Lemma 3.1

$$|\Phi_1''(t)| < (1.031) \cdot 2^{13}\pi^4 \exp(17t - 4\pi e^{4t}) \qquad (t \geq 0).$$

As this upper bound for $|\Phi_1''(t)|$ is strictly decreasing for $t \geq 0$, then evaluating this upper bound for $t = 0$ yields

$$|\Phi_1''(t)| < 2.869\,080\ldots \qquad (t \geq 0). \tag{3.24}$$

Next, with (3.6) and (3.7), we can write

$$a_1''(t) = \pi p_3(\pi e^{4t}) \exp(5t - \pi e^{4t}),$$

where $p_3(y)$, defined in (3.9), has the three distinct positive zeros $r_3^{(1)} := 0.277\,455\ldots$, $r_3^{(2)} := 1.672\,823\ldots$, and $r_3^{(3)} := 5.049\,720\ldots$, so that $p_3(y) < 0$ on the interval $(r_3^{(2)}, r_3^{(3)})$. Now πe^{4t} is contained in the interval $(r_3^{(2)}, r_3^{(3)})$ for $0 \leq t < \log(r_3^{(3)}/\pi)/4 = 0.118\,650\ldots$, so that

$$a_1''(t) < 0 \qquad (0 \leq t < 0.118\,650\ldots). \tag{3.25}$$

On the other hand (cf. (3.7)),

$$a_1^{(3)}(t) = \pi p_4(\pi e^{4t}) \exp(5t - \pi e^{4t}),$$

where $p_4(y)$ is given in (3.9). Now $p_4(y)$ has the four distinct positive zeros $r_4^{(1)} := 0.138\,273\ldots$, $r_4^{(2)} := 0.981\,795\ldots$, $r_4^{(3)} := 3.046\,710\ldots$, and $r_4^{(4)} := 7.083\,220\ldots$. As $p_4(y) > 0$ for all y in the interval $(r_4^{(3)}, r_4^{(4)})$, then

$$p_4(\pi e^{4t}) > 0 \qquad (0 \leq t < \log(r_4^{(4)}/\pi)/4 = 0.203\,249\ldots),$$

so that

$$a_1^{(3)}(t) > 0 \qquad (0 \leq t < 0.203\,249\ldots). \tag{3.26}$$

In particular, (3.26) gives that $a_1''(t)$ is strictly increasing on the interval $0 \leq t < 0.118\,650$, whence a calculation based on (3.7) yields

$$a_1''(t) \leq a_1''(0.06) = -22.779\,786\ldots \qquad (t \in I). \tag{3.27}$$

Thus, from (3.23), (3.24), and (3.27),

$$\begin{aligned}\Phi''(t) &\leq a_1''(t) + |\Phi_1''(t)| \\ &< -22.779\,786\ldots + 2.869\,080\ldots < 0\end{aligned}$$

for all $t \in I$, which gives (3.18). ∎

Lemma 3.3. *$\Phi^{(4)}(t)$ is strictly decreasing on $I := [0, 0.06]$. Moreover, $\Phi^{(4)}(t) > 0$ on I.*

Proof. To show that $\Phi^{(4)}(t)$ is strictly decreasing on I, it suffices to show that $\Phi^{(5)}(t) < 0$ for $0 < t \le 0.06$. Since $\Phi(t)$ is an even function (see Theorem A of [4]), $\Phi^{(5)}(0) = 0$. Thus, by Taylor's theorem, for each $t \in I$ there is a $\xi(t)$, where $0 \le \xi(t) \le t$, such that

$$\Phi^{(5)}(t) = t\Phi^{(6)}(\xi(t)). \tag{3.28}$$

Consider first (cf. (3.6))

$$\Phi_2^{(6)}(t) = \sum_{n=3}^{\infty} a_n^{(6)}(t) = \sum_{n=3}^{\infty} \pi n^2 p_7(\pi n^2 e^{4t}) \exp(5t - \pi n^2 e^{4t}), \tag{3.29}$$

where $p_7(y)$ is given in (3.9). Then, $p_7(y) < 8192 y^7 = 2^{13} y^7$ iff

$$q_6(y) := 245{,}760y^6 - 2{,}536{,}960y^5 + 11{,}109{,}120y^4 - 20{,}633{,}312y^3 + 14{,}260{,}064y^2 - 2{,}610{,}330y + 46{,}875 > 0.$$

Now, $q_6(y)$ has the four distinct positive zeros $s_6^{(1)} := 0.020\,101\ldots$, $s_6^{(2)} := 0.246\,796\ldots$, $s_6^{(3)} := 0.954\,957\ldots$, $s_6^{(4)} = 2.112\,148\ldots$, and two nonreal zeros $(3.494\,456\ldots \pm i2.617\,300\ldots)$, so that

$$p_7(y) < 2^{13} y^7 \qquad (y > s_6^{(4)} := 2.112\,148\ldots). \tag{3.30}$$

But as $\pi n^2 e^{4t} \ge 9\pi > s_6^{(4)}$ for all $n \ge 3$ and $t \ge 0$, then, by (3.30) and (3.29),

$$\Phi_2^{(6)}(t) < \pi^8 2^{13} e^{33t} \sum_{n=3}^{\infty} \frac{n^{16}}{\exp(\pi n^2 e^{4t})} \qquad (t \ge 0). \tag{3.31}$$

The above sum can be bounded above, as in the proof of Lemma 3.1, by

$$\sum_{n=3}^{\infty} \frac{1}{K^n} = \frac{1}{K^3(1 - 1/K)},$$

with $K := \exp(3\pi e^{4t})/3^{16/3}$. As $(1 - 1/K)^{-1} \le (1 - 3^{16/3}/e^{3\pi}) < 1.030$, then substituting these upper bounds for the sum in (3.31) gives

$$\Phi_2^{(6)}(t) < (1.030)\pi^8 2^{13} 3^{16} \exp(33t - 9\pi e^{4t}) \qquad (t \ge 0). \tag{3.32}$$

Thus, from (3.6), (3.7), and (3.32), we have

$$\Phi^{(6)}(t) < \pi e^{5t - \pi e^{4t}} \{ p_7(\pi e^{4t}) + 4p_7(4\pi e^{4t})\, e^{-3\pi e^{4t}} + (1.030)\pi^7 2^{13} 3^{16} e^{28t - 8\pi e^{4t}} \} \tag{3.33}$$

for all $t \ge 0$.

We next bound above the three terms in the braces in (3.33) for $t \in I := [0, 0.06]$. Now $p_7(y)$ has seven distinct positive zeros given by $r_7^{(1)} := 0.020\,101\ldots$, $r_7^{(2)} := 0.246\,797\ldots$, $r_7^{(3)} := 0.952\,621\ldots$, $r_7^{(4)} := 2.347\,817\ldots$, $r_7^{(5)} := 4.638\,584\ldots$, $r_7^{(6)} := 8.145\,988\ldots$, and $r_7^{(7)} := 13.648\,090\ldots$. Since $\pi \le \pi e^{4t} \le \pi e^{0.24} = 3.993\,746\ldots$ falls in the interval $(r_7^{(4)}, r_7^{(5)})$, for all $t \in I$, the first term, $p_7(\pi e^{4t})$, in the braces of (3.33) is clearly negative for all $t \in I$. Moreover, as $p_7'(y)$ has a unique zero $\tau_1 := 3.827\,186\ldots$ in $(r_7^{(4)}, r_7^{(5)})$, then $p_7(y)$ is strictly decreasing in $(r_7^{(4)}, \tau_1)$ and strictly increasing in $(\tau_1, r_7^{(5)})$. Thus,

$$B_1(t) := p_7(\pi e^{4t}) \le \max\{p_7(\pi);\, p_7(\pi e^{0.24})\} < -10{,}123{,}638 \qquad (t \in I).$$

Next, the second term in the braces of (3.33) can be written as $4p_7(4y)\,e^{-3y}$ (where $y=\pi\, e^{4t}$), and its derivative with respect to y is

$$4\,e^{-3y}[4p_7'(4y)-3p_7(4y)].$$

The polynomial of degree seven in brackets above has the seven distinct positive zeros 0.028 954..., 0.149 240..., 0.412 752..., 0.863 681..., 1.556 575..., 2.589 347..., and 4.232 781.... As this polynomial is positive at $y=0$, this polynomial is then positive on $[2.59, 4.23]$. As $y=\pi\, e^{4t}$ falls in this interval for all $t\in I$, then $B_2(t):=4p_7(4\pi\, e^{4t})\,e^{-3\pi\, e^{4t}}$ is strictly increasing on I, hence

$$B_2(t)\le B_2(0.06)<2{,}176{,}400 \qquad (t\in I).$$

Finally, the last term in the braces of (3.33) is clearly positive and strictly decreasing for all $t\ge 0$, so that

$$B_3(t):=(1.030)\pi^7 2^{13}3^{16}\,e^{28t-8\pi\, e^{4t}}\le B_3(0)<13{,}342 \qquad (t\in I).$$

On adding, the above bounds give

$$B_1(t)+B_2(t)+B_3(t)<0 \qquad (t\in I),$$

so that, from (3.33), $\Phi^{(6)}(t)<0$ for all $t\in I$. Consequently (cf. (3.28)), $\Phi^{(5)}(t)<0$ for $0<t\le 0.06$, and $\Phi^{(4)}(t)$ is strictly decreasing on I. Finally, from (3.20) and (3.22), we have that $\Phi^{(4)}(t)>0$ on I. ■

Lemma 3.4. *With the definition of* (3.1),

(3.34) $$g(t)>0 \qquad (0<t\le 0.03).$$

Proof. Since $\Phi'(0)=\Phi'''(0)=0$ (see Theorem A of [4]), it follows from (3.1) that $g(0)=g'(0)=g''(0)=0$. Thus, to prove (3.34) it suffices to show that $g''(t)>0$ for $0<t\le 0.03$. Now, by Taylor's theorem, for each fixed t, $0<t\le 0.03$, there are numbers $\eta_j(t)$ $(j=0,1,2,3)$ satisfying $0\le\eta_j(t)\le t$, such that

(3.35) $$\Phi(t)=\Phi(0)+\frac{t^2}{2}\Phi''(\eta_0(t)),$$

(3.36) $$\Phi'(t)=\Phi''(0)t+\frac{t^3}{6}\Phi^{(4)}(\eta_1(t)).$$

(3.37) $$\Phi''(t)=\Phi''(0)+\frac{t^2}{2}\Phi^{(4)}(\eta_2(t)),$$

(3.38) $$\Phi'''(t)=t\Phi^{(4)}(\eta_3(t)).$$

Since

(3.39) $$g''(t)=5\Phi'(t)\Phi''(t)+t[(\Phi''(t))^2-\Phi(t)\Phi^{(4)}(t)]-\Phi(t)\Phi'''(t),$$

we obtain, using (3.35)-(3.38),

(3.40) $$g''(t)=t[S_1(t)+S_2(t)+S_3(t)],$$

where

$$S_1(t) := 6(\Phi''(0))^2 - \Phi(0)[\Phi^{(4)}(t) + \Phi^{(4)}(\eta_3(t))], \tag{3.41}$$

$$S_2(t) := t^2\{\Phi''(0)[\tfrac{7}{2}\Phi^{(4)}(\eta_2(t)) + \tfrac{5}{6}\Phi^{(4)}(\eta_1(t))] - \tfrac{1}{2}\Phi''(\eta_0(t))[\Phi^{(4)}(\eta_3(t)) + \Phi^{(4)}(t)]\}, \tag{3.42}$$

and

$$S_3(t) := t^4\{\Phi^{(4)}(\eta_2(t))[\tfrac{5}{12}\Phi^{(4)}(\eta_1(t)) + \tfrac{1}{4}\Phi^{(4)}(\eta_2(t))]\}. \tag{3.43}$$

By Lemma 3.2, $\Phi''(t) < 0$ on $I := [0, 0.06]$ and $\Phi''(t)$ is strictly increasing on I, while by Lemma 3.3, $\Phi^{(4)}(t) > 0$ on I and $\Phi^{(4)}(t)$ is strictly decreasing on I. Also, it is known that $\Phi(t) > 0$ for all $t \geq 0$ (see Theorem A of [4]). Consequently, for $0 < t \leq 0.03$, we obtain the estimates

$$S_1(t) > 6(\Phi''(0))^2 - 2\Phi(0)\Phi^{(4)}(0) > 913 \tag{3.44}$$

and

$$S_2(t) > (0.03)^2\{\tfrac{13}{3}\Phi''(0)\Phi^{(4)}(0) + |\Phi''(0.03)|\Phi^{(4)}(0.03)\} > -689. \tag{3.45}$$

Now, by Lemma 3.3, $\Phi^{(4)}(t) > 0$ for $0 < t \leq 0.06$ and *a fortiori* $S_3(t) > 0$ for $0 < t \leq 0.03$. Therefore, by (3.40), (3.44), and (3.45), we conclude that $g(t) > 0$ for $0 < t \leq 0.03$. ∎

Lemma 3.5. *Let*

$$R(t) := [(\Phi'(t))^2/\Phi(t)] - \Phi''(t) \qquad (t \geq 0). \tag{3.46}$$

Then $R(t)$ is strictly decreasing for $0.03 \leq t \leq 0.06$.

Proof. Set

$$L(t) := (\Phi(t))^2 R'(t) \qquad (t \geq 0).$$

Then, since $\Phi(t) > 0$ for $t \geq 0$ (see Theorem A of [4]), it suffices to show that $L(t) < 0$ for $0.03 \leq t \leq 0.06$. Using (3.46) it follows that

$$L(t) = (-\Phi'(t))^3 + \Phi(t)\{2(-\Phi'(t))(-\Phi''(t)) - \Phi(t)\Phi'''(t)\}. \tag{3.47}$$

Since $\Phi'(t) < 0$ for $t > 0$ (see Theorem A of [4], $\Phi''(t) < 0$ on $[0, 0.06]$ (cf. (3.18)), and $\Phi'''(t) > 0$ on $(0, 0.06]$ (cf. 3.19)), the quantity in braces in (3.47) can be written as

$$V(t) := 2|\Phi'(t)|\,|\Phi''(t)| - |\Phi(t)|\,|\Phi'''(t)| \qquad (0 \leq t \leq 0.06).$$

Consequently, for any closed subinterval $[a, b]$ of $[0.03, 0.06]$,

$$\max_{[a,b]} V(t) \leq 2 \max_{[a,b]} |\Phi'(t)| \max_{[a,b]} |\Phi''(t)| - \min_{[a,b]} |\Phi(t)| \min_{[a,b]} |\Phi'''(t)|.$$

In addition, because $\Phi''(t) < 0$ on $[0, 0.06]$ (cf. (3.18)) and because $\Phi^{(4)}(t) > 0$ for $0 \leq t \leq 0.06$ (cf. Lemma 3.3), then

$$\max_{[a,b]} V(t) \leq 2|\Phi'(b)|\,|\Phi''(a)| - |\Phi(b)|\,|\Phi'''(a)|. \tag{3.48}$$

On setting $J_1 := [0.03, 0.035]$, $J_2 := [0.035, 0.04]$, $J_3 := [0.04, 0.05]$, and $J_4 := [0.05, 0.06]$, and on determining $\Phi^{(k)}(t)$ $(0 \le k \le 3)$ at the endpoints of these intervals, we find, using (3.48), that

$$\begin{cases} \max_{J_1} V(t) \le -11.200 =: \mu_1, \\ \max_{J_2} V(t) \le -15.645 =: \mu_2, \\ \max_{J_3} V(t) \le -11.011 =: \mu_3, \\ \max_{J_4} V(t) \le -21.817 =: \mu_4. \end{cases} \tag{3.49}$$

Thus, from (3.49), $V(t) < 0$ on $[0.03, 0.06]$, and we can write

$$L(t) = |\Phi'(t)|^3 + |\Phi(t)|\,V(t) \qquad (0.03 \le t \le 0.06).$$

On setting $J_i := [\alpha_i, \beta_i]$, then

$$L(t) \le \max_{J_j} (|\Phi'(t)|^3) + (\min_{J_j} |\Phi(t)|)(\max_{J_j} V(t)) =: M_j \tag{3.50}$$

for all $t \in J_j$ $(1 \le j \le 4)$, and, by the same reasoning used in determining (3.48),

$$M_j \le |\Phi'(\beta_j)|^3 + |\Phi(\beta_j)|\mu_j \qquad (1 \le j \le 4).$$

With the numbers of (3.49), we determine that $M_1 \le -3.352\ldots$, $M_2 \le -4.529\ldots$, $M_3 \le -0.810\ldots$ and $M_4 \le -2.826\ldots$ Thus, from (3.50), $L(t) < 0$ for $0.03 \le t \le 0.06$. ■

Lemma 3.6. *With the definition of* (3.1)

$$g(t) > 0 \qquad (0.03 \le t \le 0.06). \tag{3.51}$$

Proof. Let

$$\begin{aligned} &h := 0.005, \\ &t_j := 0.03 + jh && (0 \le j \le 5), \\ &J_j := [0.03 + jh, 0.03 + (j+1)h] && (0 \le j \le 5). \end{aligned} \tag{3.52}$$

Now, by Taylor's theorem, for $t \in J_j$, there is a number $\xi_j(t)$ satisfying $0.03 + jh \le \xi_j(t) \le t$, such that

$$\Phi'(t) = \Phi'(t_j) + (t - t_j)\Phi''(\xi_j(t)). \tag{3.53}$$

Since $\Phi(t) > 0$ $(0 \le t < \infty)$, it suffices to prove that $g(t)/\Phi(t) > 0$ for $t \in J := \bigcup_{j=0}^{5} J_j$. Now, from (3.1) and (3.46), $g(t)/\Phi(t) = tR(t) + \Phi'(t)$, so that, for $t \in J_j$, we have, from (3.53), that

$$\frac{g(t)}{\Phi(t)} = (t - t_j)[R(t) + \Phi''(\xi_j(t))] + \Phi'(t_j) + t_j R(t). \tag{3.54}$$

Next, using Lemma 3.2 and Lemma 3.5, we find that

$$\min_{t \in J_j} [R(t) + \Phi''(\xi_j(t))] \ge R(t_{j+1}) + \Phi''(t_j) \qquad (0 \le j \le 5). \tag{3.55}$$

By evaluating the right-hand side of (3.55), for $0 \le j \le 5$, we obtain

$$\min_{0 \le j \le 5} [R(t_{j+1}) + \Phi''(t_j)] > 1.959\,461 \ldots. \tag{3.56}$$

Consequently, using (3.54) and (3.56), we obtain the lower bound

$$\frac{g(t)}{\Phi(t)} > \Phi'(t_j) + t_j R(t) =: Q(t) \qquad (t \in J_j). \tag{3.57}$$

Thus, to prove (3.51), it suffices to establish that $Q(t) > 0$ for $t \in J = \bigcup_{j=0}^{5} J_j$. But, by Lemma 3.5, $R(t)$ is strictly decreasing for $0.03 \le t \le 0.06$. Hence,

$$Q(t) \ge \Phi'(t_j) + t_j R(t_{j+1}) \qquad (t \in J_j). \tag{3.58}$$

Finally, by evaluating the right-hand side of (3.58) for $0 \le j \le 5$, we obtain

$$Q(t) \ge \min_{0 \le j \le 5} [\Phi'(t_j) + t_j R(t_{j+1})]$$

$$> 0.001\,616 \ldots$$

for all $0.03 \le t \le 0.06$. Therefore, from (3.57), the assertion (3.51) is valid. ■

We next turn to the proof of inequality (3.5). From (3.1),

$$g(t) = t[(\Phi'(t))^2 - \Phi(t)\Phi''(t)] + \Phi(t)\Phi'(t) \qquad (t \ge 0), \tag{3.59}$$

and write $\Phi(t) = a_1(t) + \Phi_1(t)$ (cf. (3.6)). Then a computation shows that $g(t)$ can be expressed in the form

$$g(t) = T_1(t) + T_2(t) + T_3(t) + T_4(t) + T_5(t) + t(\Phi_1'(t))^2, \tag{3.60}$$

where, for $t \ge 0$,

$$T_1(t) := t[(a_1'(t))^2 - a_1(t)a_1''(t)] + a_1(t)a_1'(t), \tag{3.61}$$

$$T_2(t) := 2ta_1'(t)\Phi_1'(t), \tag{3.62}$$

$$T_3(t) := -t\Phi(t)\Phi_1''(t), \tag{3.63}$$

$$T_4(t) := \Phi(t)\Phi_1'(t), \tag{3.64}$$

$$T_5(t) := (a_1'(t) - ta_1''(t))\Phi_1(t). \tag{3.65}$$

Since $t(\Phi_1'(t))^2 \ge 0$ for $t \ge 0$, we have the lower bound

$$g(t) \ge E(t) \qquad (t \ge 0), \tag{3.66}$$

where

$$E(t) := \sum_{j=1}^{5} T_j(t). \tag{3.67}$$

With the aid of the following lemmas we will prove that the function $E(t)$, defined in (3.67), is positive on the interval $I_2 := [0.056, \infty)$.

From the definitions in (3.61) and (3.6) it can be verified that

$$T_1(t) = 16\pi^5 \exp(22t - 2\pi e^{4t})\theta_1(t) \qquad (t \ge 0), \tag{3.68}$$

where

$$\theta_1(t) := t\left[4 - \frac{12}{\pi} e^{-4t} + \frac{15}{\pi^2} e^{-8t}\right] + \left[-1 + \frac{21}{4\pi} e^{-4t} - \frac{15}{2\pi^2} e^{-8t} + \frac{45}{16\pi^3} e^{-12t}\right]. \tag{3.69}$$

Lemma 3.7. *We have*

$$\theta_1(t) > 0 \qquad (t \geq 0.04623) \tag{3.70}$$

and

$$\theta_1'(t) > 0 \qquad (t \geq 0.04623). \tag{3.71}$$

Proof. First, a calculation using (3.69) shows that

$$\theta_1(0.04623) = 0.002\,784\ldots. \tag{3.72}$$

Thus, to prove (3.70) and (3.71), it is enough to show that $\theta_1'(t) > 0$ for $t \geq 0.04623$. To this end we express $\theta_1'(t)$ as

$$\theta_1'(t) = \frac{48}{\pi} e^{-4t}\{\theta_{11}(t) + \theta_{12}(t)\}, \tag{3.73}$$

where

$$\theta_{11}(t) := \frac{\pi}{12} e^{4t} - \frac{11}{16} + \frac{25}{16\pi} e^{-4t} - \frac{45}{64\pi^2} e^{-8t} \tag{3.74}$$

and

$$\theta_{12}(t) := t\left(1 - \frac{5}{2\pi} e^{-4t}\right). \tag{3.75}$$

We now proceed to verify that $\theta_{11}(t) + \theta_{12}(t) > 0$ for $t \geq 0.04623$. Consider $\theta_{11}(t)$ given by (3.74). Then a computation shows that

$$\theta_{11}(0.04623) = -0.008\,349\ldots \tag{3.76}$$

and that

$$\theta_{11}'(t) = \frac{\pi}{3} e^{4t} - \frac{25}{4\pi} e^{-4t} + \frac{45}{8\pi^2} e^{-8t}. \tag{3.77}$$

For $t \geq 0$, set $x := \pi e^{4t} (\geq \pi)$ and let

$$\theta_{11}'(t) = R(x) := \frac{1}{24x^2}(8x^3 - 150x + 135) \qquad (x > 0). \tag{3.78}$$

Then the zeros of the rational function $R(x)$ are given by $x_1 := -4.724\,588\ldots$, $x_2 := 0.945\,009\ldots$, and $x_3 := 3.779\,578\ldots$. Thus, $R(x) > 0$ if $x > x_3$, and so

$$\theta_{11}'(t) > 0 \qquad (t \geq t_1), \tag{3.79}$$

where

$$t_1 := \tfrac{1}{4}\log\left(\frac{x_3}{\pi}\right) = 0.046\,220\ldots. \tag{3.80}$$

Turning to the function $\theta_{12}(t)$ given by (3.75), we find that

$$\theta_{12}(0.04623) = 0.015\,652\,336\ldots \tag{3.81}$$

and that

$$\theta_{12}'(t) = 1 - \frac{5}{2\pi} e^{-4t} + t\frac{10}{\pi} e^{-4t} \geq 1 - \frac{5}{2\pi} e^{-4t} > 0 \qquad (t \geq 0). \tag{3.82}$$

Consequently, we infer from (3.79) and (3.82) that

$$\theta_{11}'(t) + \theta_{12}'(t) > 0 \qquad (t \geq 0.04623 > t_1), \tag{3.83}$$

where t_1 is given by (3.80). But, by (3.76) and (3.81), we have

$$\theta_{11}(0.04623) + \theta_{12}(0.04623) = 0.007\,302\ldots > 0, \tag{3.84}$$

and so the desired results of (3.70) and (3.71) follow from (3.72), (3.73), (3.83), and (3.84). ■

Lemma 3.8. *We have (cf. (3.62)-(3.64))*

$$|T_2(t)| < 9{,}040\pi^6 t \exp(26t - 5\pi e^{4t}) \qquad (t \geq 0), \tag{3.85}$$

$$|T_3(t)| < (1.0362)2^{14}\pi^6 t \exp(26t - 5\pi e^{4t}) \qquad (t \geq 0), \tag{3.86}$$

$$|T_4(t)| < (1.109)2^{10}\pi^5 \exp(22t - 5\pi e^{4t}) \qquad (t \geq 0). \tag{3.87}$$

Proof. From (3.7) $a_1'(t) = \pi p_2(\pi e^{4t}) \exp(5t - \pi e^{4t})$, where (cf. (3.9)) $p_2(y) = -8y^2 + 30y - 15$. As $|p_2(y)| < 8y^2$ for all $y > \frac{1}{2}$, then $|p_2(\pi e^{4t})| < 8\pi^2 e^{8t}$ for all $t \geq 0$, so that

$$|a_1'(t)| < 8\pi^3 \exp(13t - \pi e^{4t}) \qquad (t \geq 0). \tag{3.88}$$

Next, in [4, Lemma 3.3] it was proved that

$$|\Phi_1'(t)| < 565\pi^3 \exp(13t - 4\pi e^{4t}) \qquad (t \geq 0). \tag{3.89}$$

Thus, applying the bounds of (3.88) and (3.89) to the definition of $T_2(t)$ in (3.62) gives (3.85).

In order to prove (3.86), we first recall that in [4, equation (3.41)] it was shown that

$$0 < \Phi(t) < \frac{203}{202} a_1(t) \qquad (t \geq 0).$$

Since $a_1(t) = 2\pi^2(1 - (3/2\pi)\, e^{-4t}) \exp(9t - \pi e^{4t}) < 2\pi^2 \exp(9t - \pi e^{4t})$ for all $t \geq 0$, it follows that

$$0 < \Phi(t) < 2\left(\frac{203}{202}\right)\pi^2 \exp(9t - \pi e^{4t}) \qquad (t \geq 0). \tag{3.90}$$

Hence, applying the bound of (3.90) and the bound for $|\Phi_1''(t)|$ of (3.10) of Lemma 3.1 to the definition of $T_3(t)$ in (3.63), gives the desired result of (3.86).

Finally, the estimate (3.87) follows directly from (3.89) and (3.90). ■

Our next lemma provides an upper estimate for the function $T_5(t)$ defined by (3.65). Note that since $a_n(t) > 0$ for all $t \geq 0$ and $n \geq 1$ (see Theorem A of [4]), the function $\Phi_1(t)$ (cf. (3.6)), figuring in the definition of $T_5(t)$, is positive for all $t \geq 0$.

Lemma 3.9. *We have* (*cf.* (3.65))

$$|T_5(t)| < 2^{11}\pi^6 \exp(22t - 5\pi e^{4t})\left(\frac{1}{4\pi} + t\, e^{4t}\right) \qquad (t \geq 0). \tag{3.91}$$

Proof. First, on p. 415 of [5], Haviland has shown that

$$\Phi_1(t) < 64\pi^2 \exp(9t - 4\pi e^{4t}) \qquad (t \geq 0). \tag{3.92}$$

Second, a computation based on (3.7) shows that

$$|a_1'(t) - t a_1''(t)| = 32\pi^4 \exp(13t - \pi e^{4t})\theta_2(t), \tag{3.93}$$

where

$$\theta_2(t) := \left|\frac{1}{4\pi} - \frac{15}{16\pi^2} e^{-4t} + \frac{15}{32\pi^3} e^{-8t} + t\left(e^{4t} - \frac{7}{\pi} + \frac{165}{16\pi^2} e^{-4t} - \frac{75}{32\pi^3} e^{-8t}\right)\right|. \tag{3.94}$$

Thus, by (3.92) and (3.93), we have that

$$|T_5(t)| < 2^{11}\pi^6 \exp(22t - 5\pi e^{4t})\theta_2(t), \tag{3.95}$$

where $\theta_2(t)$ was defined by (3.94). Therefore, it suffices to prove that

$$\theta_2(t) \leq \frac{1}{4\pi} + t\, e^{4t} \qquad (t \geq 0). \tag{3.96}$$

In order to prove (3.96) we let

$$A(t) := \frac{1}{16\pi^2}\left\{-15e^{-4t} + \frac{15}{2\pi} e^{-8t} - 112\pi t + 165 t\, e^{-4t} - \frac{75}{2\pi} t\, e^{-8t}\right\} \qquad (t \geq 0), \tag{3.97}$$

and observe that (3.96) is equivalent to the following two inequalities:

$$-\frac{1}{2\pi} - 2t\, e^{4t} \leq A(t) \qquad (t \geq 0) \tag{3.98}$$

and

$$A(t) \leq 0 \qquad (t \geq 0). \tag{3.99}$$

Since the inequalities (3.98) and (3.99) can be readily established with the aid of the calculus, we conclude that (3.96) holds, and thus the proof of lemma is complete. ■

Lemma 3.10. *With* $g(t)$ *defined by* (3.1),

$$g(t) > 0 \qquad (t \in I_3 := [0.056, \infty)). \tag{3.100}$$

Proof. By (3.66) and (3.67) we have

$$g(t) \geq T_1(t) + \sum_{j=2}^{5} T_j(t) \qquad (t \geq 0). \tag{3.101}$$

Since $\theta_1(t) > 0$ for $t \geq 0.04623$ (cf. (3.68) and (3.70)), we can express (3.101) in the form

$$g(t) \geq T_1(t)(1 + E_1(t)) \qquad (t \geq 0.04623), \tag{3.102}$$

where

$$E_1(t) := \frac{1}{T_1(t)} \sum_{j=2}^{5} T_j(t) \qquad (t \geq 0.04623). \tag{3.103}$$

Then, on combining the bounds for $|T_j(t)|$ for $0 \leq j \leq 5$ (cf. (3.68), (3.85)-(3.87), and (3.91)), we obtain

$$|E_1(t)| \leq \frac{1}{T_1(t)} \sum_{j=2}^{5} |T_j(t)| = \frac{\exp(4t - 3\pi e^{4t})}{\theta_1(t)} (c\pi t + d\, e^{-4t}) \tag{3.104}$$

for $t \geq 0.04623$, where

$$c := \frac{9{,}040}{16} + (1.0362)2^{10} + 2^7 = 1{,}754.0688 \tag{3.105}$$

and

$$d := (1.109)2^6 + 2^5 = 102.976. \tag{3.106}$$

Since $\theta_1(t)$ is strictly increasing for $t \geq 0.056$ (cf. (3.71)), it follows that, for $t \geq 0.056$,

$$|E_1(t)| < \frac{\exp(4t - 3\pi e^{4t})}{\theta_1(0.056)} (c\pi t + d\, e^{-4t}) := E_2(t), \tag{3.107}$$

where the constants c and d are given by (3.105) and (3.106), respectively. Now, in light of (3.102) and (3.107), it suffices to prove that

$$E_2(t) < 1 \qquad \text{for all} \quad t \geq 0.056. \tag{3.108}$$

To this end we compute $E_2'(t)$, for $t \geq 0.056$, and obtain

$$E_2'(t) = \frac{\exp(4t - 3\pi e^{4t})}{\theta_1(0.056)} E_3(t),$$

where

$$E_3(t) := 4c\pi t - 12\pi^2 ct\, e^{4t} - 12\pi d + c\pi. \tag{3.109}$$

Since $E_3(0.056) < -11{,}691.67$ and

$$E_3'(t) = 4c\pi - 12\pi^2 c\, e^{4t} - 48\pi^2 ct\, e^{4t} < 0 \qquad (t \geqslant 0), \tag{3.110}$$

it follows that $E_2'(t)<0$ for all $t \geq 0.056$, so that $E_2(t)$ is positive and strictly decreasing for $t \geq 0.056$. Finally, a calculation shows that

$$\theta_1(0.056) > 0.003\,970\ldots$$

and

$$E_2(0.056) < 0.932\,545,$$

which gives (3.108) and the desired result of (3.100). ■

Theorem 3.11. *With the definition of* (3.1),

$$g(t) > 0 \quad \textit{for all} \quad t \in (0, \infty). \tag{3.111}$$

Proof. Since $I_1 \cup I_2 \cup I_3 = (0, \infty)$, where $I_1(0, 0.03]$, $I_2 := [0.03, 0.06]$, and $I_3 = [0.056, \infty)$, then (3.111) is an immediate consequence of Lemmas 3.4, 3.6, and 3.10. ■

As a final remark we mention that the paper of Matiyasevich [6] came to our attention in Mathematical Reviews (MR 85g: 11079), after our work [4] was in press. Matiyasevich claimed in [6] a solution of Pólya's conjecture [13, p. 16], based on his triple-integral representation of the Turán differences $D_m(0)$ of (2.12). Specifically, he observed in [6] (from (2.14)-(2.15) and (2.19)-(2.20)) in the case $\lambda = 0$ that it was sufficient to show that $g(t)$, defined in (2.9), satisfies $g(t) > 0$ on $(0, \infty)$, to deduce that $D_m(0) > 0$ for all $m \geq 1$. It was claimed in [6], "from interval computations that are as powerful as a proof," that $g(t) > 0$ on $(0, \infty)$. Although we have not verified his computer results, his claim that $g(t) > 0$ on $(0, \infty)$ is of course *true*, as the results of Section 3 (cf. Theorem 3.11) give a *constructive* proof of this. We emphasize here that our approach in this paper is based on the independent notion that $\log(\Phi(\sqrt{t}))$ is *strictly concave* on $(0, \infty)$, which led us to show that $g(t) > 0$ on $(0, \infty)$. While Matiyasevich also mentions in [6] the possibility of deriving more general moment inequalities from $g(t) > 0$ on $(0, \infty)$, these are different from the concrete results of Theorems 2.4 and Corollary 2.5.

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References

1. R. E. Barlow, A. W. Marshall, F. Proschan (1963): *Properties of probability distributions with monotone hazard rate.* Ann. Math. Statist., **34**:375–389.
2. R. P. Boas (1954): Entire Functions. New York: Academic Press.
3. N. G. de Bruijn (1950): *The roots of trigonometric integrals.* Duke Math. J., **17**:197–226.
4. G. Csordas, T. S. Norfolk, R. S. Varga (1986): *The Riemann Hypothesis and the Turán inequalities.* Trans. Amer. Math. Soc., **296**:521–541.
5. E. K. Haviland (1945): *On the asymptotic behavior of the Riemann ξ-function.* Amer. J. Math., **67**:411–416.

6. Yu. V. MATIYASEVICH (1982): *Yet another machine experiment in support of Riemann's conjecture*, Kiebernetika (Kiev), no. 6, 10, 22 (Russian, English summary). (Translated in: Cybernetics, **18**(6) (1983), 705-707.)
7. A. W. MARSHALL, I. OLKIN (1979): Inequalities: Theory of Majorization and Its Applications. New York: Academic Press.
8. C. M. NEWMAN (1976): *Fourier transforms with only real zeros.* Proc. Amer. Math. Soc. **61**: 245-251.
9. G. PÓLYA (1918): *Über die Nullstellen gewisser ganzer Funktionen.* Math. Z., **2**:352-383.
10. G. PÓLYA (1926): *Bemerkung über die Integraldarstellung der Riemannschen ξ-Funktion.* Acta Math., **48**:305-317.
11. G. PÓLYA (1926): *On the zeros of certain trigonometric integrals.* J. London Math. Soc., **1**:98-99.
12. G. PÓLYA (1927): *Über trigonometrische Integrale mit nur reellen Nullstellen.* J. Reine Angew. Math., **158**:6-18.
13. G. PÓLYA (1927): *Über die algebraisch-funktionen theoretischen Intersuchungen von J. L. W. V. Jensen.* Kgl. Danske Vid. Sel. Math.-Fys. Medd., **7**:3-33
14. G. PÓLYA (1974): *Location of zeros.* In: Collected Papers, vol. 2 (R. P. Boas, ed.). Cambridge, MA: MIT Press.
15. G. PÓLYA, J. SCHUR (1914): *Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen.* J. Reine Angew. Math. **144**:89-113.
16. C. L. PRATHER (1981): *Final sets for operators on classes of entire functions representable as a Fourier integral.* J. Math. Anal. Appl. **82**:200-220.
17. E. C. TITCHMARSH (1951): The Theory of the Riemann Zeta Function. Oxford: Oxford University Press.

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