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A Lower Bound for the de Bruijn-Newman Constant Λ*

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Dedicated to the memory of our good friend Peter Henrici
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Summary. A lower bound is constructively found for the de Bruijn-Newman constant Λ, which is related to the Riemann Hypothesis. This lower bound is determined by explicitly exhibiting an associated Jensen polynomial with nonreal zeros.

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Introduction

The purpose of this paper is to obtain a constructive finite *lower bound* for the *de Bruijn-Newman constant* Λ, which is related to the Riemann Hypothesis. To our knowledge, no such lower bound for Λ appears in the literature.

For background, if $\zeta(z)$ denotes the Riemann zeta function, it is known that all zeros of $\zeta(z)$, except for simple zeros at $-2, -4, -6, \ldots$, lie in the "critical" strip defined by $0 \leqq \operatorname{Re} z \leqq 1$ (cf. Titchmarsh [15, p. 30]), and the Riemann Hypothesis is the statement that all the zeros of $\zeta(z)$ in this critical strip lie precisely on the line $\operatorname{Re} z = 1/2$. With Riemann's definition of his ξ-function, i.e.,

$$\xi(iz) := \frac{1}{2}\left(z^2 - \frac{1}{4}\right)\pi^{-\frac{z}{2}-\frac{1}{4}}\,\Gamma\left(\frac{z}{2}+\frac{1}{4}\right)\zeta\left(z+\frac{1}{2}\right), \tag{1}$$

it is known (cf. Henrici [5, p. 305]) that the *Riemann Hypothesis is equivalent to the statement that all zeros of* $\xi(z)$ *are real.* It is further known that ξ is an entire function of order one (cf. [15, p. 29]), and admits the integral representation (cf. Pólya [10, p. 11] or [15, p. 255])

$$\xi\left(\frac{x}{2}\right) = 8\int_0^\infty \Phi(t)\cos(xt)\,dt, \tag{2}$$

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where

$$\Phi(t) := \sum_{n=1}^{\infty} (2n^4\pi^2 e^{9t} - 3n^2\pi e^{5t}) \exp(-n^2\pi e^{4t}) \qquad (0 \leqq t < \infty). \tag{3}$$

It is also known (cf. Pólya [10] or Csordas, Norfolk, and Varga [4, Theorem A]) that Φ satisfies the following properties:

$$\begin{aligned}
&\text{i) } \Phi(z) \text{ is analytic in the strip } -\pi/8 < \operatorname{Im} z < \pi/8;\\
&\text{ii) } \Phi(t) = \Phi(-t), \text{ and } \Phi(t) > 0 \ (t \in \mathbb{R});\\
&\text{iii) for any } \varepsilon > 0, \lim_{t\to\infty} \Phi^{(n)}(t) \exp[(\pi-\varepsilon)e^{4t}] = 0, \quad \text{for each } n = 0, 1, 2, \ldots.
\end{aligned} \tag{4}$$

From (2), the Taylor series for ξ is given by

$$\frac{1}{8}\xi\left(\frac{x}{2}\right) = \sum_{m=0}^{\infty} \frac{(-1)^m \hat{b}_m x^{2m}}{(2m)!}, \tag{5}$$

where

$$\hat{b}_m := \int_0^{\infty} t^{2m} \Phi(t)\, dt \qquad (m = 0, 1, 2, \ldots). \tag{6}$$

(Note from (4 ii) that $\hat{b}_0 > 0$, so that $\xi(0) \neq 0$.) On setting $z = -x^2$ in (5), the function $F_0(z)$, defined by

$$F_0(z) := \sum_{m=0}^{\infty} \frac{\hat{b}_m z^m}{(2m)!}, \tag{7}$$

is then an entire function of order 1/2. Furthermore, from these definitions, x_0 is a real zero of $\xi\left(\frac{x}{2}\right)$ iff $z_0 := -x_0^2$ is a negative real zero of $F_0(z)$. From this, it follows that the *Riemann Hypothesis is equivalent to the statement that all zeros of $F_0(z)$ are real and negative.*

Continuing, let $H_0(x)$ be the Fourier transform of $\Phi(t)/2$, i.e., (cf. (4 ii))

$$H_0(x) := \tfrac{1}{2} \int_{-\infty}^{\infty} \Phi(t) e^{ixt}\, dt = \int_0^{\infty} \Phi(t) \cos(xt)\, dt, \tag{8}$$

so that $H_0(x) = \xi(x/2)/8$. Based on the properties of the function $\Phi(t)$ in (4), it can be shown, for any real λ, that the function $H_\lambda(x)$, defined by

$$H_\lambda(x) := \tfrac{1}{2} \int_{-\infty}^{+\infty} e^{\lambda t^2} \Phi(t) e^{ixt}\, dt = \int_0^{\infty} e^{\lambda t^2} \Phi(t) \cos(xt)\, dt \qquad (\lambda \in \mathbb{R}), \tag{9}$$

is an entire function of order one, with $H_\lambda(x)$ real for real x. More precisely, the sharper result is that $H_\lambda(x)$ is an entire function of order 1 *and of type* ∞ for any $\lambda \in \mathbb{R}$. Because this sharper result is not needed in what immediately follows, we have given the proof of this assertion in Appendix A.

Continuing, as a special case of a result of Pólya [11], if $H_0(x)$ has only real zeros, then the entire function $H_\lambda(x)$ also has only real zeros, provided that $\lambda \geqq 0$. Now, two results of de Bruijn [2] establish that

$$\begin{aligned} &\text{i) } H_\lambda(x) \text{ has only real zeros for any } \lambda \geqq 1/2; \\ &\text{ii) if } H_\lambda(x) \text{ has only real zeros for some real } \lambda, \text{ then } H_{\lambda'}(x) \\ &\quad \text{has only real zeros for any } \lambda' \geqq \lambda. \end{aligned} \tag{10}$$

In particular, it follows from (10) that if the Riemann Hypothesis is true, then $H_\lambda(x)$ must possess only real zeros for any $\lambda \geqq 0$.

It is interesting that Newman [7] has shown more recently that there exists a real number Λ, which in our notation satisfies $-\infty < \Lambda \leqq 1/2$, such that (cf. (9))

$$\begin{aligned} &H_\lambda(x) \text{ has only real zeros when } \lambda \geqq \Lambda, \text{ and} \\ &H_\lambda(x) \text{ has some nonreal zeros when } \lambda < \Lambda. \end{aligned} \tag{11}$$

Because of de Bruijn's earlier related work, it seems fitting to call the above constant Λ the *de Bruijn-Newman constant.*

We remark that Newman's method of proof for the existence of Λ in [7] is *nonconstructive*, and no finite lower bound for Λ appears in [7], or in any publications that we are aware of. Of course, from our discussion above, the truth of the Riemann Hypothesis would imply that $\Lambda \leqq 0$, while Newman [7] offers the complementary conjecture that this de Bruijn-Newman constant Λ satisfies $\Lambda \geqq 0$. Because of this connection of Λ to the Riemann Hypothesis, obtaining constructive lower bounds for Λ seems *desirable.*

Returning to $H_\lambda(x)$ of (9), it follows, in analogy with (5), that

$$H_\lambda(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \hat{b}_m(\lambda)\, x^{2m}}{(2m)!}, \tag{12}$$

where

$$\hat{b}_m(\lambda) := \int_0^\infty t^{2m} e^{\lambda t^2} \Phi(t)\, dt \qquad (m = 0, 1, 2, \ldots; \lambda \in \mathbb{R}). \tag{13}$$

On setting $z = -x^2$ in (12), we obtain the entire function

$$F_\lambda(z) := \sum_{m=0}^{\infty} \frac{\hat{b}_m(\lambda)\, z^m}{(2m)!} \qquad (\lambda \in \mathbb{R}), \tag{14}$$

which is of order 1/2, and (14) extends the definition of (7) to all real values of λ. Noting that $\hat{b}_m(\lambda) > 0$ from (13) and (4ii) for all $m \geqq 0$, then as before, $x_0(\lambda)$ is a real zero of $H_\lambda(z)$ iff $z_0(\lambda) = -x_0^2(\lambda)$ is a negative real zero of $F_\lambda(z)$. Thus, all the zeros of $H_\lambda(x)$ are real iff all the zeros of $F_\lambda(z)$ are real and negative, and from (11), the de Bruijn-Newman constant Λ can be equivalently determined by the conditions

$$\begin{aligned} &F_\lambda(z) \text{ has only real negative zeros when } \lambda \geqq \Lambda, \text{ and} \\ &F_\lambda(z) \text{ has some nonreal zeros when } \lambda < \Lambda. \end{aligned} \tag{15}$$

We next remark that since $F_\lambda(z)$ is a transcendental (i.e., nonpolynomial) entire function of order 1/2 for any real λ, it can be expressed (using the Hadamard factorization theorem) as

$$F_\lambda(z)=C(\lambda)\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n(\lambda)}\right), \tag{16}$$

where from (14), $C(\lambda):=\hat{b}_0(\lambda)>0$, and where $|z_n(\lambda)|>0$ for all $n\geqq 1$, with $\sum_{n=1}^{\infty}|z_n(\lambda)|^{-1}<\infty$. In particular, if the real number λ is such that $z_n(\lambda)<0$ for all $n\geqq 1$, then $F_\lambda(z)$ is seen to be an element of the *Laguerre-Pólya class* (written $F_\lambda\in\mathscr{L}-\mathscr{P}$), this class being defined as the collection of all entire functions $f(x)$ of the form

$$f(x)=C\,e^{-\alpha x^2+\beta x}\,x^n\prod_{j=1}^{\omega}\left(1-\frac{x}{x_j}\right)e^{x/x_j}\qquad(\omega\leqq\infty), \tag{17}$$

where $\alpha\geqq 0$, β and C are real numbers, n is a nonnegative integer, and the x_j's are real and nonzero with $\sum_{j=1}^{\omega}1/x_j^2<\infty$. Thus, we can express (15) alternately as

$$\begin{aligned}&F_\lambda\in\mathscr{L}-\mathscr{P}\text{ when }\lambda\geqq\Lambda,\text{ and}\\&F_\lambda\notin\mathscr{L}-\mathscr{P}\text{ when }\lambda<\Lambda.\end{aligned} \tag{18}$$

Next, it is convenient to set

$$u_m(\lambda):=(m!)\,\hat{b}_m(\lambda)/(2m)!\qquad(m=0,1,2,\ldots;\ \lambda\in\mathbb{R}), \tag{19}$$

so that, from (14), we can write

$$F_\lambda(z)=\sum_{j=0}^{\infty}\frac{u_m(\lambda)\,z^m}{m!}\qquad(\lambda\in\mathbb{R}). \tag{20}$$

The n^{th} *Jensen polynomial associated with* $F_\lambda(z)$ is then defined as

$$G_n(t;\lambda):=\sum_{k=0}^{n}\binom{n}{k}u_k(\lambda)\,t^k\qquad(n=0,1,2,\ldots;\ \lambda\in\mathbb{R}). \tag{21}$$

Now, it is known (cf. Pólya-Schur [12]) that $F_\lambda(z)\in\mathscr{L}-\mathscr{P}$ iff $G_n(t;\lambda)\in\mathscr{L}-\mathscr{P}$ $(n=0,1,2,\ldots)$. As a consequence, if $F_\lambda(z)\in\mathscr{L}-\mathscr{P}$ then the zeros of each $G_n(t;\lambda)$ must be *all real and negative* for any $n\geqq 1$. From this and (18), we immediately have

Proposition 1. *Suppose that there is a positive integer m and a real number* $\hat{\lambda}$ *such that* $G_m(t;\hat{\lambda})$ *possesses a nonreal zero. Then,*

$$\hat{\lambda}<\Lambda, \tag{22}$$

where Λ *is the de Bruijn-Newman constant.*

This will be used in the next section.

2. Determination of a Lower Bound for Λ

We can use Proposition 1 in a numerically rigorous fashion to obtain a *constructive* finite lower bound for Λ. The idea is the following. For any real λ, the associated moments $\{\hat{b}_m(\lambda)\}_{m=0}^{N}$, from (13), can be computed with a known precision, as was similarly done in [4]. Calling the approximate moments $\{\hat{\beta}_m(\lambda)\}_{m=0}^{N}$, the Jensen polynomials associated with these approximate moments are then defined (cf. (19) and (21)) by

$$g_n(t;\lambda) := \sum_{k=0}^{n} \binom{n}{k} \frac{(k!)\,\hat{\beta}_k(\lambda)}{(2k)!}\, t^k \qquad (n=1,2,\ldots,N), \tag{23}$$

which differ, because of errors in the $\hat{\beta}_m(\lambda)$'s, from the exact Jensen polynomials $G_n(t;\lambda)$ of (21). One then finds the zeros of the $\{g_n(t;\lambda)\}_{n=1}^{N}$ with a good high-precision root finder, such as the Jenkins algorithm [6].

To estimate the discrepancies between the zeros of $g_n(t;\lambda)$ and $G_n(t;\lambda)$, we make use of the following well-known result of Ostrowski [8] and [9, Appendix B]:

Proposition 2. *Let $f(z)=\sum_{j=0}^{n} a_j z^j$ and $h(z)=\sum_{j=0}^{n} b_j z^j$ be complex polynomials with $a_0 a_n \neq 0$, and let the zeros of $f(z)$ be $\{z_j\}_{j=1}^{n}$ (which are all nonzero, but multiple zeros can occur). Assume that there is a real number τ with $4n\tau^{1/n} \leqq 1$, such that*

$$|b_j - a_j| \leqq \tau\, |a_j| \qquad (0 \leqq j \leqq n). \tag{24}$$

Then, the n zeros $\{w_j\}_{j=1}^{n}$ of $h(z)$ can be ordered in such a way that

$$\left|\frac{w_j}{x_j} - 1\right| < 8n\tau^{1/n} \qquad (1 \leqq j \leqq n). \tag{25}$$

Our main result is the

Theorem. *Let Λ be the de Bruijn-Newman constant of* (11). *Then,*

$$-50 < \Lambda. \tag{26}$$

Remark. In our opinion, the importance of (26) is that a *constructive* method has been found to produce a finite lower bound for Λ, a bound which can be checked independently by others. No special significance is attached to the number -50. In fact, it can be expected that our lower bound can be *substantially increased*, at the expense of using even higher precision calculations of the moments $\{\hat{b}_m(\lambda)\}_{m=0}^{N}$, higher precision determination of zeros of higher-degree Jensen polynomials, and more computer time! In this vein, we mention that the recent impressive results of van de Lune, te Riele and Winter [16], show that the first 1,500,000,001 zeros of the Riemann zeta function in the critical strip are simple and do indeed satisfy $\operatorname{Re} z = 1/2$. These computations involved about 900 CPU hours on a CYBER 175/750 and about 1000 CPU hours on a CYBER 205.

Because of their use in establishing our main result of (26), we consider in detail the moments $\hat{b}_m(\lambda)$ of (13), and their numerical approximation. Since each moment $\hat{b}_m(\lambda)$ of (13) can be expressed as the sum

$$\hat{b}_m(\lambda) = \int_0^1 t^{2m} e^{\lambda t^2} \Phi(t)\, dt + \int_1^\infty t^{2m} e^{\lambda t^2} \Phi(t)\, dt \qquad (m=0, 1, 2, \ldots), \tag{27}$$

we first give an upper bound for the last integral of (27). Assuming that λ is *negative*, it is elementary to show that $t^{2m} e^{\lambda t^2}$ is strictly decreasing for all $t > 1$, provided that $0 \leqq m \leqq |\lambda|$. Thus,

$$\int_1^\infty t^{2m} e^{\lambda t^2} \Phi(t)\, dt < e^\lambda \int_1^\infty \Phi(t)\, dt < \frac{\pi}{2} \exp(\lambda + 5 - \pi e^4) \qquad (0 \leqq m \leqq |\lambda|), \tag{28}$$

the last inequality above being a consequence (cf. [4, Eq. (3.7)]) of the known inequality $\int_1^\infty \Phi(t)\, dt < \frac{\pi}{2} \exp(5 - \pi e^4)$. On setting

$$b_m^{(1)}(\lambda) := \int_0^1 t^{2m} e^{\lambda t^2} \Phi(t)\, dt \qquad (m=0, 1, \ldots), \tag{29}$$

it follows from (28) and the positivity of the integrands of (27) that when $\lambda = -50$,

$$b_m^{(1)}(-50) < \hat{b}_m(-50) < b_m^{(1)}(-50) + 1.45 \cdot 10^{-94} \qquad (0 \leqq m \leqq 50). \tag{30}$$

Thus, the absolute error obtained by replacing $\hat{b}_m(-50)$ by $b_m^{(1)}(-50)$ is at most $1.45 \cdot 10^{-94}$ for all $0 \leqq m \leqq 50$.

Next, if $\Phi_N(t)$ denotes the sum of the first N terms of (3), then it is known (cf. [4, Eq. (4.6)]) that

$$0 < \Phi(t) - \Phi_N(t) < \pi N^3 \exp(5t - \pi N^2 e^{4t}) \qquad (0 \leqq t < \infty). \tag{31}$$

But for fixed $N \geqq 1$, $\exp(5t - \pi N^2 e^{4t})$ is strictly decreasing for $t \geqq 0$, so that

$$0 < \Phi(t) - \Phi_N(t) < \pi N^3 e^{-\pi N^2} \qquad (0 \leqq t < \infty),$$

and on evaluating the above when $N = 9$, we have, in particular, that

$$0 < \Phi(t) - \Phi_9(t) < 7.1 \cdot 10^{-108} \qquad (0 \leqq t \leqq 1). \tag{32}$$

On defining

$$b_m^{(2)}(\lambda) := \int_0^1 t^{2m} e^{\lambda t^2} \Phi_9(t)\, dt \qquad (0 \leqq m \leqq |\lambda|), \tag{33}$$

we then have, in analogy with (30), that

$$0<b_m^{(2)}(-50)<\hat{b}_m(-50)<b_m^{(2)}(-50)+1.46\cdot 10^{-94} \qquad (0\leqq m\leqq 50). \tag{34}$$

For our problem at hand, the specific moments $\{b_m^{(2)}(-50)\}_{m=0}^{16}$ of (33) were numerically approximated, using the Romberg integration method.

Now, for each real λ and for each nonnegative integer m, we note that the integrand of the integral of (33) is, from (3), in $C^{2k+2}[0,1]$ for *any* positive integer k. This smoothness of this integrand implies that the $(k+1)$-st column $T_{i,k}^{(m)}$ of the *Romberg integration table*, associated with the integral of (33), is convergent to the *exact* value of that integral when step lengths $h_i := 2^{-i}$ $(i=0, 1, \dots)$ are used (cf. Stoer and Bulirsch [14, Eq. (3.4.7), p. 136]). As this is true for each positive integer k, then the extrapolation method of Romberg integration applied to the integral of (33), can produce arbitrarily accurate values of $b_m^{(2)}(\lambda)$.

For the Romberg method of integration applied to each of the moments $\{b_m^{(2)}(-50)\}_{m=0}^{16}$, all computations were performed in FORTRAN 77, using Brent's MP package [1] for extended-precision floating-point numbers and 110 digits of precision on a VAX-11/780 in the Institute for Computational Mathematics at Kent State University. On denoting the entries of the Romberg table, associated with the integral $b_m^{(2)}(-50)$, by $T_{i,k}^{(m)}$ $(0\leqq m\leqq 16)$, the actual iterations of the Romberg method (with step lengths $h_i=2^{-i}$) were terminated in all cases when *three* successive entries, in a single column of $T_{i,k}^{(m)}$, agreed to 65 significant digits in normalized floating-point representation. The *final* approximation to $b_m^{(2)}(-50)$, denoted by $\hat{\beta}_m(-50)$, was then defined to be the *largest* values of k and i in the Romberg table such that $T_{i-1,k}^{(m)}$ and $T_{i,k}^{(m)}$ agreed to 65 significant digits in normalized floating-point representation. These computed moments $\{\hat{\beta}_m(-50)\}_{m=0}^{16}$, truncated to 60 significant digits, are given in Table 1. In particular, this accuracy implies that

$$\left|\frac{\hat{\beta}_m(-50)}{b_m^{(2)}(-50)}-1\right|\leqq 10^{-64} \qquad (m=0, 1, \dots, 16),$$

and thus, with (34) and the magnitude of the numbers $\hat{\beta}_m(-50)$ from Table 1, we have the conservative bounds

$$\left|\frac{\hat{\beta}_m(-50)}{\hat{b}_m(-50)}-1\right|\leqq 10^{-60} \qquad (m=0, 1, \dots, 16). \tag{35}$$

As an added precaution in estimating the moments $\{b_m^{(2)}(-50)\}_{m=0}^{16}$, we used a technique, described in Bulirsch and Stoer [3] and in [14, p. 141], for determining *asymptotic upper and lower bounds* for the numbers $b_m^{(2)}(-50)$. Specifically, from the entries $T_{i,k}^{(m)}$ of the Romberg table associated with $b_m^{(2)}(-50)$, one can easily compute the numbers $U_{i,k}^{(m)}$ defined by

$$U_{i,k}^{(m)} := 2T_{i+1,k}^{(m)} - T_{i,k}^{(m)},$$

and these numbers $U_{i,k}^{(m)}$ build a *new* associated Romberg-type table for the sought mement $b_m^{(2)}(-50)$. Because of the smoothness of the integrand of (33),

Table 1. The computed moments

m	$\beta_m(-50)$	
0	4.19656 13339 29181 93310 73686 70252 18989 08075 14869 18579 39818 0690	(−2)
1	2.32011 71418 85536 82157 44972 44537 77375 47795 47788 78966 01105 0975	(−4)
2	3.77089 97677 30881 23738 16195 78125 66204 22778 11075 77741 16472 9297	(−6)
3	1.00227 69596 72457 01475 32275 01674 91380 18026 69849 21468 18990 4564	(−7)
4	3.66337 61735 70406 23425 94004 30741 73648 26291 73317 49864 18599 5080	(−9)
5	1.69252 58688 59480 57551 74685 29708 06315 55393 12181 30248 37182 8204	(−10)
6	9.40358 19102 26836 44833 49098 87957 05821 36909 15984 87893 65573 6861	(−12)
7	6.07928 26738 29840 63138 68635 44716 86713 97453 67318 45834 48369 0895	(−13)
8	4.46758 07993 72611 29511 35750 68116 37426 68930 24699 78052 39388 9235	(−14)
9	3.66774 22933 21191 60815 42487 62105 96955 97101 68171 08309 51339 1938	(−15)
10	3.31887 96848 26088 11013 54576 58364 37597 62937 04392 36885 19692 7259	(−16)
11	3.27495 66308 18925 42519 54660 46267 96438 12270 41617 07059 97861 9700	(−17)
12	3.49350 79889 03492 82136 66727 14098 12286 23184 09702 38293 93952 1310	(−18)
13	3.99970 36556 45639 31360 95058 35153 19042 31348 37182 39401 70449 4114	(−19)
14	4.88503 24854 26203 87448 03575 24970 26569 88256 82483 91503 16030 0221	(−20)
15	6.33188 05610 98341 63013 78223 19621 78670 63968 85444 40652 31370 0822	(−21)
16	8.67132 23144 43954 11051 39659 30824 92230 12635 52164 34153 16708 6153	(−22)

then for each nonnegative integer k, the first column (corresponding to the trapezoidal rule) of the Romberg table for $b_m^{(2)}(-50)$, can be written symbolically as

$$T(h) = b_m^{(2)}(-50) + \tau_1 h^2 + \tau_2 h^4 + \ldots + \tau_k h^{2k} + (\tau_{k+1} + o(1))\, h^{2k+2},$$

where the τ_k are independent of h. If $\tau_{k+1} \neq 0$, it is shown in [3] that the $(k+1)$-st columns $T_{i,k}^{(m)}$ and $U_{i,k}^{(m)}$ of these tables eventually converge *monotonically* to $b_m^{(2)}(-50)$, as $i \to \infty$, at asymptotically the same rate, but from *opposite* directions. Hence, these numbers thus provide asymptotically upper and lower bounds for $b_m^{(2)}(-50)$. Indeed, in all cases of our computations, the final elements of the first columns of $T_{i,0}^{(m)}$ and $U_{i,0}^{(m)}$ were respectively monotone increasing, and monotone decreasing, i.e.,

$$T_{7,0}^{(m)} < T_{8,0}^{(m)} < T_{9,0}^{(m)} \quad \text{and} \quad U_{7,0}^{(m)} > U_{8,0}^{(m)} \qquad (m = 0, 1, \ldots, 16).$$

This monotonicity suggests that each interval $[T_{9,0}^{(m)}, U_{8,0}^{(m)}]$ contains $b_m^{(2)}(-50)$, for all $m = 0, 1, \ldots, 16$.

While it might be interesting for the reader to peruse the bulk of digits of the tables $T_{i,k}^{(m)}$ and $U_{i,k}^{(m)} (m = 0, 1, \ldots, 16)$, we have opted instead to give in Table 2 the much briefer listing of the *number* of significant digits to which $T_{9,0}^{(m)}$ and $U_{8,0}^{(m)}$ actually agree. From Table 2, we note two interesting items. First, we note that the agreement between $T_{9,0}^{(m)}$ and $U_{8,0}^{(m)}$ in Table 2 start in the case $m = 0$ with the surprisingly high number 93, and second, that this agreement decays to 73 digits when $m = 16$. But, this 73 digits of agreement is still very conservative for what is claimed in (35).

Table 2. Significant digits of agreement between $T_{9,0}^{(m)}$ and $U_{8,0}^{(m)}$

m	Significant digits of agreement	m	Significant digits of agreement
0	93	9	79
1	91	10	79
2	89	11	78
3	88	12	76
4	85	13	75
5	85	14	74
6	82	15	74
7	82	16	73
8	81	–	–

Proof of the Theorem. From (21), set $G_{16}(t;\,-50):=\sum_{j=0}^{16} b_j t^j$, and from (23), set $g_{16}(t;\,-50):=\sum_{j=0}^{16} a_j t^j$. From (35), it follows that (24) is satisfied with $\hat{\tau}:=10^{-60}$, i.e.,

$$|b_j-a_j|\leqq 10^{-60}\,|a_j| \quad (0\leqq j\leqq 16). \tag{36}$$

Thus, from (25) of Proposition 2, the zeros $\{w_j\}_{j=1}^{16}$ of $G_{16}(t;\,-50)$ and the zeros $\{z_j\}_{j=1}^{16}$ of $g_{16}(t;\,-50)$ are coupled through

$$\left|\frac{w_j}{z_j}-1\right|<8n\hat{\tau}^{1/n}=8\cdot 16(10^{-60})^{1/16}<0.022762. \tag{37}$$

We remark that the Jenkins algorithm [4], implemented in MP with 110 digits of precision, was used to determine all zeros of the polynomial $g_{16}(t;\,-50)$. (The zeros of $g_{16}(t;\,-50)$, truncated to twenty decimal digits, all given in Table 3.) From Table 3, we see that $g_{16}(t;\,-50)$ has a complex zero

$$z_2:=-220.91911\ 17368\ 44952\ldots+i\,7.09256\ 52553\ 63889\ 67\ldots \tag{38}$$

whose modulus is

$$|z_2|=221.03293\ 51307\ 13450\ldots.$$

On multiplying by $|z_2|$ in (37) for the case $j=2$, we obtain

$$|w_2-z_2|<0.022762\cdot|z_2|<5.03116. \tag{39}$$

It is evident from (38) and (39) that

$$\operatorname{Im} w_2>2.0614, \tag{40}$$

whence $G_{16}(t;\,-50)$ has a nonreal zero. Applying Proposition 1 then gives the desired result of (26). $\square$

As a final remark, the specific numbers -50 (as a lower bound for Λ) and 16 (as the degree of the associated Jensen polynomial $G_{16}(t;\,-50)$ having some

Table 3. The computed zeros of $g_{16}(t;\ -50)$

Real part	Imaginary part
−138.94239 25728 27322 76	–
−220.91911 17368 44952 75	7.09256 52553 63889 6761
−220.91911 17368 44952 75	−7.09256 52553 63889 6761
−286.00232 55522 12961 55	–
−310.79782 67793 62579 98	–
−350.18851 27945 36537 80	–
−386.00014 13750 66719 39	–
−422.33256 42144 23880 51	–
−459.76615 10040 07194 18	–
−498.36874 57252 58991 53	–
−538.57501 58487 17219 65	–
−580.89029 00913 24826 13	–
−626.04149 21507 01991 05	–
−675.20466 35826 12683 01	–
−730.62228 63983 20152 69	–
−798.10604 06173 97392 18	–

nonreal zeros) appear somewhat *magically* in this section. To explain this further, our first high-precision attempt to constructively determine a lower bound for Λ began with $\lambda = -100$, and the associated approximate Jensen polynomials of (23), i.e., $\{g_n(t;\ -100)\}_{n=1}^{9}$, all had only real negative zeros, while $g_{10}(t;\ -100)$ possessed a nonreal zero. Now, the precision of these calculations were much more than adequate, in applying Proposition 2, to deduce the lower bound

$$-100 < \Lambda.$$

As this was almost too easy, the process was repeated with $\lambda = -50$. The associated approximate Jensen polynomials $\{g_n(t;\ -50)\}_{n=1}^{15}$ had only real negative zeros, while $g_{16}(t;\ -50)$ possessed a nonreal zero. This time, however, the precision of these calculations were just within the applicability of Proposition 2 to deduce that $G_{16}(t;\ -50)$ has a nonreal zero, whence

$$-50 < \Lambda.$$

These calculations support our optimism that further improved lower bounds for Λ are possible!

Appendix A

Our object here is to show that the entire function (cf. (9)) $H_\lambda(x)$, defined by

$$H_\lambda(x) := \frac{1}{2} \int_{-\infty}^{+\infty} e^{\lambda t^2}\, \Phi(t)\, e^{ixt}\, dt = \int_{0}^{\infty} e^{\lambda t^2}\, \Phi(t) \cos(xt)\, dt, \tag{A 1}$$

is of order 1, and of maximal type, (i.e., its type, σ_λ, satisfies $\sigma_\lambda=\infty$), for *all* $\lambda\in\mathbb{R}$. We remark that this is known to be true for the special case $\lambda=0$, since the Riemann ξ-function is of order one and of maximal type (cf. Titchmarsh [15, p. 29]) and since $H_0(x)=\xi(x/2)/8$. For other values of real λ, this may be a known result, but we are unaware of any references for this. We do know that related results can be derived from the fact that the following differential identity holds, namely

$$H_\lambda(x)=F_\lambda(D)\,H_0(x), \qquad \left(D:=\frac{d}{dx}\right)$$

where

$$F_\lambda(z):=\sum_{m=0}^{\infty}\frac{(-1)^m\lambda^m z^{2m}}{m!}.$$

Such results (cf. Sikkema [13, p. 100] and van der Steen [17, p. 62]) give the weaker results that $H_\lambda(x)$ is either of order 1 and type at most ∞, or of order less than 1, for any $\lambda\in\mathbb{R}$. In contrast with the above references, our technique of proof uses asymptotic series which may have independent interest.

To begin, for any $\lambda\in\mathbb{R}$, set $M_\lambda(r):=\max_{|z|=r}|H_\lambda(z)|$, so that $M_\lambda(r)$ is the maximum modulus function for $H_\lambda(z)$. It an be verified from (A 1) that

$$M_\lambda(r):=\int_0^\infty e^{\lambda t^2}\,\Phi(t)\cosh(rt)\,dt,$$

so that

$$\frac{1}{2}\int_0^\infty e^{\lambda t^2}\,\Phi(t)\,e^{rt}\,dt\leqq M_\lambda(r)\leqq\int_0^\infty e^{\lambda t^2}\,\Phi(t)\,e^{rt}\,dt, \tag{A 2}$$

for all $r\geqq 0$ and any $\lambda\in\mathbb{R}$. Next, from [4, eq. 3.41 and Thm. $A(i)$], we have

$$\pi(2\pi e^{4t}-3)\exp(5t-\pi e^{4t})<\Phi(t)<\frac{203}{202}\pi(2\pi e^{4t}-3)\exp(5t-\pi e^{4t}),$$

for all $t\geqq 0$, and as $(2\pi-3)\,e^{4t}<2\pi e^{4t}-3<2\pi e^{4t}$ for all $t>0$, then

$$\frac{\pi}{2}(2\pi-3)\int_0^\infty\exp[(9+r)t+\lambda t^2-\pi e^{4t}]\,dt<M_\lambda(r)<\left(\frac{203}{202}\right)2\pi^2\int_0^\infty\exp[(9+r)t+\lambda t^2-\pi e^{4t}]\,dt \tag{A 3}$$

for all $r\geqq 0$. Thus, we set

$$I(r,\lambda):=\int_0^\infty\exp[(9+r)t+\lambda t^2-\pi e^{4t}]\,dt=\int_0^\infty e^{h(t)}\,dt, \tag{A 4}$$

where

$$h(t):=h(t;r,\lambda):=(9+r)\,t+\lambda t^2-\pi e^{4t}\,(t\geqq 0). \tag{A 5}$$

We have

$$\begin{aligned} h'(t) &= 9+r+2\lambda t-4\pi e^{4t}, \\ h''(t) &= 2\lambda-16\pi e^{4t}, \\ h^{(m)}(t) &= -4^m \pi e^{4t}\,(m\geqq 3). \end{aligned} \tag{A6}$$

In what follows, λ is a fixed real number, and $r(\geqq 0)$ is a large parameter, tending to $+\infty$. Since our assertion that $H_\lambda(x)$ is of order 1 and maximal type is known to be true for $\lambda=0$, we may assume $\lambda\neq 0$.

Lemma 1. *Given any $\lambda\in\mathbb{R}$, then for each $r>0$ sufficiently large, there is a unique $\hat{t}=\hat{t}(r,\lambda)$ for which $h'(\hat{t})=0$.*

Proof. Since $h^{(3)}(t)=-64\pi e^{4t}<0$ all $t\geqq 0$, then $h''(t)$ is strictly decreasing for $t\geqq 0$, with $h''(t)\to-\infty$ as $t\to+\infty$. Next, $h''(0)=2\lambda-16\pi$. If $\lambda\leqq 8\pi$, set $\tilde{t}:=0$. If $\lambda>8\pi$, let $\tilde{t}>0$ be the unique value of $t>0$ such that $h''(\tilde{t})=0$. Thus, in all cases, $h''(t)<0$ for all $t>\tilde{t}$. Then, choose $r>0$ sufficiently large so that $h'(\tilde{t})>0$. Because $h'(t)$ is strictly decreasing (for $t>\tilde{t}$), with $h'(\tilde{t})>0$ and with $h'(t)\to-\infty$ as $t\to\infty$, there is evidently a unique $\hat{t}>\tilde{t}$ for which $h'(\hat{t})=0$. □

By construction, we note that $h''(\hat{t})<0$ for all r sufficiently large, so that the Taylor expansion of $h(t)$, about $\hat{t}$, can be written as

$$h(t)=h(\hat{t})-\frac{|h''(\hat{t})|}{2}(t-\hat{t})^2+\sum_{m=3}^{\infty}\frac{h^{(m)}(\hat{t})}{m!}(t-\hat{t})^m. \tag{A7}$$

We now estimate $\hat{t}=\hat{t}(r,\lambda)$. By definition, $h'(\hat{t})=0$ implies from (A6) that

$$4\pi e^{4\hat{t}}=9+r+2\lambda\hat{t}$$

Calling $4\hat{t}:=\hat{u}$, $(9+r)/4\pi=:\beta$, and $\lambda/8\pi=:\alpha$, the above equation becomes

$$e^{\hat{u}}=\beta+\alpha\hat{u}. \tag{A8}$$

Lemma 2. *For $\beta>0$ sufficiently large, the unique solution $\hat{u}$ of* (A8) *satisfies*

$$\log[\beta+\alpha\log\beta]<\hat{u}<\log\beta+\frac{\alpha\log\beta}{\beta}. \tag{A9}$$

Thus

$$\hat{t}=\frac{1}{4}\left\{\log\beta+\frac{\alpha\log\beta}{\beta}+O\left(\left(\frac{\log\beta}{\beta}\right)^2\right)\right\},\quad \text{as}\,\beta\to\infty. \tag{A10}$$

Proof. From (A6) and (A8), we have that $h'(t)/4\pi=\{-e^u+\beta+\alpha u\}=:g(u)$. If we set $\bar{u}:=\log\beta+\alpha\log\beta/\beta$, then a calculation shows that

$$g(\bar{u})=-\frac{\alpha^2\log^2\beta}{2\beta}\left\{1-\frac{2}{\log\beta}\right\}+O\left(\frac{\alpha^3\log^3\beta}{\beta^2}\right),\quad \beta\to+\infty.$$

Thus, since $\lambda\neq 0$ implies $\alpha\neq 0$, then for $\beta>0$ sufficiently large, we have $g(\bar{u})<0$, whence (from the monotonicity of Lemma 1) $\hat{u}<\bar{u}$. Similarly, with $\underline{u}:=$

$\log(\beta+\alpha\log\beta)$, a calculation shows that

$$g(\underline{u})=\alpha\log\left\{1+\frac{\alpha\log\beta}{\beta}\right\}=\alpha\left\{\frac{\alpha\log\beta}{\beta}-\frac{1}{2}\left(\frac{\alpha\log\beta}{\beta}\right)^2+O\left(\left(\frac{\alpha\log\beta}{\beta}\right)^3\right)\right\}$$

$$=\frac{\alpha^2\log\beta}{\beta}\left\{1-\frac{1}{2}\left(\frac{\alpha\log\beta}{\beta}\right)+O\left(\left(\frac{\alpha\log\beta}{\beta}\right)^2\right)\right\}>0,$$

for all β sufficiently large, whence $g(\underline{u})>0$. But as $g(\hat{u})=0$, the monotonicity of Lemma 1 gives that $\underline{u}<\hat{u}<\bar{u}$ for all β sufficiently large, which is (A 9). Finally,

$$\underline{u}:=\log(\beta+\alpha\log\beta)=\log\beta+\log\left[1+\frac{\alpha\log\beta}{\beta}\right]=\log\beta+\frac{\alpha\log\beta}{\beta}+O\left(\left(\frac{\alpha\log\beta}{\beta}\right)^2\right),$$

$\beta\to\infty$. This means that $\bar{u}-\underline{u}=O\left(\left(\frac{\alpha\log\beta}{\beta}\right)^2\right)$, and as $\alpha=\lambda/8\pi$ is a fixed nonzero constant, then $\hat{u}=4\hat{t}$ evidently satisfies (A 10). □

We now estimate the terms $\{h^{(m)}(\hat{t})\}_{m=0}^{\infty}$ of (A 7). A short calculation shows that

$$h(\hat{t})=\pi\beta\log\beta\left\{1-\frac{1}{\log\beta}+O\left(\frac{\log\beta}{\beta}\right)\right\},$$

$$h''(\hat{t})=-16\pi\beta\left\{1+O\left(\frac{\log\beta}{\beta}\right)\right\},$$

$$h^{(m)}(\hat{t})=-4^m\pi\beta\left\{1+O\left(\frac{\log\beta}{\beta}\right)\right\}, (m\geqq 3), \qquad \beta\to+\infty. \tag{A 11}$$

Next, from (A 7) and (A 4), we have that

$$I(r,\lambda)=e^{h(\hat{t})}\int_0^\infty \exp\left\{-\frac{|h''(\hat{t})|}{2}(t-\hat{t})^2+\sum_{m=3}^{\infty}\frac{h^{(m)}(\hat{t})}{m!}(t-\hat{t})^m\right\}dt.$$

Setting $t-\hat{t}=v$, then

$$I(r,\lambda)=e^{h(\hat{t})}\int_{-\hat{t}}^{\infty} e^{-\frac{|h''(\hat{t})|}{2}v^2}\, e^{Q(v)}\,dv,$$

where $Q(v):=\sum_{m=3}^{\infty} h^{(m)}(\hat{t})\, v^m/m!$. Setting

$$e^{Q(v)}:=1+\sum_{j=3}^{\infty}\mu_j v^j, \quad \text{where } \mu_j=\frac{h^{(m)}(\hat{t})}{m!} \quad \text{for } j=3,4,\text{ and } 5,$$

then we have, since $\hat{t}\to+\infty$ as $r\to\infty$, that (in the sense of asymptotic series, cf. Henrici [5, Ch. 11])

$$I(r,\lambda)=e^{h(\hat{t})}\int_{-\hat{t}}^{\infty} e^{-\frac{|h''(\hat{t})|}{2}v^2}\{1+\mu_3 v^3+\mu_4 v^4\,\ldots\}\,dv$$

$$\cong e^{h(\hat{t})}\left\{\int_{-\infty}^{+\infty} e^{-\frac{|h''(\hat{t})|}{2}v^2}dv+\mu_4\int_{-\infty}^{+\infty} e^{-\frac{|h''(\hat{t})|}{2}v^2}v^4\,dv+\dots\right\},$$

$$=e^{h(\hat{t})}\left\{\sqrt{\frac{2\pi}{|h''(\hat{t})|}}+\frac{h^{(4)}(\hat{t})}{8(h''(\hat{t}))^2}\sqrt{\frac{2\pi}{|h''(t)|}}+\dots\right\}$$

$$=e^{h(\hat{t})}\sqrt{\frac{2\pi}{|h''(\hat{t})|}}\left\{1+\frac{h^{(4)}(\hat{t})}{8(h''(\hat{t}))^2}+\dots\right\}=e^{h(\hat{t})}\sqrt{\frac{2\pi}{|h''(\hat{t})|}}\left\{1+O\left(\frac{1}{\beta}\right)\right\}.$$

Using the results of (A 11) in the above, we deduce, after some calculations, that

$$I(r,\lambda)=e^{\pi\beta\log\beta-\pi\beta+O(\log^2\beta)},\ r\to\infty. \tag{A 12}$$

Thus, upon recalling that $\beta:=(9+r)/4\pi)$, this gives that

$$\log I(r,\lambda)=\frac{r\log r}{4}+O(r), \tag{A 13}$$

as $r\to\infty$. From this, there follows

$$\log\log I(r,\lambda)=\log r+\log\log r+O(1),\qquad r\to\infty, \tag{A 14}$$

so that

$$\lim_{r\to\infty}\frac{\log\log I(r,\lambda)}{\log r}=1. \tag{A 15}$$

But since we have from (A 3) that

$$\frac{\pi}{2}(2\pi-3)\,I(r,\lambda)<M_\lambda(r)<\left(\frac{203}{202}\right)2\pi^2\,I(r,\lambda)\qquad(\text{all } r\geqq 0),$$

then on taking logarithms and using (A 13), we have

$$\lim_{r\to\infty}\frac{\log M_\lambda(r)}{r\log r}=4. \tag{A 16}$$

Hence, from (A 16), we then have that

$$\lim_{r\to\infty}\frac{\log\log M_\lambda(r)}{\log r}=1\qquad(\text{all } \lambda\in\mathbb{R}). \tag{A 17}$$

This proves that $H_\lambda(x)$ is an entire function of order 1 for *any* $\lambda\in\mathbb{R}$. Next, as $H_\lambda(x)$ is of order 1, its type, σ_λ, is defined as usual by

$$\overline{\lim_{r\to\infty}}\frac{\log M_\lambda(r)}{r}=:\sigma_\lambda.$$

But from (A 16), it is evident that $\sigma_\lambda=+\infty$ for *all* $\lambda\in\mathbb{R}$. $\square$

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References

1. Brent, R.P.: A FORTRAN multiple-precision arithmetic package. Assoc. Comput. Mach. Trans. Math. Software **4**, 57–70 (1978)
2. de Bruijn, N.G.: The roots of trigonometric integrals. Duke Math. J. **17**, 197–226 (1950)
3. Bulirsch, R., Stoer, J.: Asymptotic upper and lower bounds for results of extrapolation methods. Numer. Math. **8**, 93–104 (1966)
4. Csordas, G., Norfolk, T.S., Varga, R.S.: The Riemann Hypothesis and the Turán inequalities. Trans. Amer. Math. Soc. **296**, 521–541 (1986)
5. Henrici, P.: Applied and Computational Complex Analysis, vol. 2. New York: Wiley 1977
6. Jenkins, M.A.: Zeros of a real polynomial. Collected Algorithms from CACM, No. 493 (1974)
7. Newman, C.M.: Fourier transforms with only real zeros. Proc. Amer. Math. Soc. **61**, 245–251 (1976)
8. Ostrowski, A.M.: Recherches sur la méthode de Gräffe et les zéros des polynômes et des séries de Laurant. Acta. Math. **72**, 99–257 (1940)
9. Ostrowski, A.M.: Solution of Equations and Systems of Equations. New York: Academic Press 1960
10. Pólya, G.: Über die algebraisch-funktionentheoretischen Untersuchungen von J.L.W.V. Jensen. Kgl. Danske Vid Sel. Math.-Fys. Medd. **7**, 3–33 (1927)
11. Pólya, G.: Über trigonometrische Integrale mit nur reellen Nullstellen. J. Reine Angew. Math. **158**, 6–18 (1927)
12. Pólya, G., Schur, J.: Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen. J. Reine Angew. Math. **144**, 89–113 (1914)
13. Sikkema, P.C.: Differential Operators and Differential Equations of Infinite Order with Constant Coefficients. Groningen: Noorhoff 1953
14. Stoer, J., Bulirsch, R.: Introduction to Numerical Analysis. Berlin, Heidelberg, New York: Springer 1980
15. Titchmarsh, E.C.: The Theory of the Riemann Zeta-function, 2nd Edition (revised by D.R. Heath-Brown). Oxford: Oxford University Press 1986
16. van de Lune, J., te Riele, H.J.J., Winter, D.T.: On the zeros of the Riemann zeta function in the critical strip IV. Math. Comput. **46**, 667–681 (1986)
17. van der Steen, P.: On Differential Operators of Infinite Order. Delft: Uitgeverij Waltman 1968

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Errata for
"A Lower Bound for the de Bruijn-Newman Constant Λ",
by G. Csordas, T.S. Norfolk, and R.S. Varga

The following corrections should be made:

- p. 486, eq. 20. Read " $\sum_{m=0}^{\infty}$ " for " $\sum_{j=0}^{\infty}$ ".
- p. 493, line +16. Read "It can be..." for "It an be...".
- p. 494, line +15. Read "a unique $\hat{t} > \tilde{t}$ " for "a unique $\tilde{t} > \tilde{t}$...".
- p. 494, line +17. Read "about $\hat{t}$ " for "about $\tilde{t}$ ".
- p. 496, line +2. At the end of this display, read " $\sqrt{\frac{2\pi}{|h''(\hat{t})|}}$ " for " $\sqrt{\frac{2\pi}{|h''(t)}}$ ".