On a New Proof and Sharpenings of a Result of Fejér on Bounded Partial Sums

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ABSTRACT

In this paper, we give a new proof, based on matrix theory, and sharpenings of a result of Fejér on the boundedness of partial sums of functions in H^∞.

1. INTRODUCTION AND STATEMENT OF RESULTS

Consider any function $f(z)$ in H^∞, i.e. (cf. Duren [1, p. 2]), any function $f(z) = \sum_{j=0}^{\infty} a_j z^j$ which is analytic in $|z| < 1$, and for which $\|f\|_\infty := \sup_{|z|<1}|f(z)| < \infty$. If $s_n(z)$ denotes its nth partial sum, i.e.,

$$s_n(z) := \sum_{j=0}^{n} a_j z^j \qquad (n \geqslant 0), \tag{1.1}$$

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then evidently $\|s_0\|_\infty = |f(0)| \leqslant \|f\|_\infty$. However, for the remaining partial sums $s_n(z)$, $\|s_n\|_\infty$ need *not* be bounded by $\|f\|_\infty$ for all $n \geqslant 1$. With

$$k_n(r;\theta) := \tfrac{1}{2} + r\cos\theta + \cdots + r^n \cos n\theta \tag{1.2a}$$

$$= \frac{1 - r^2 - 2r^{n+1}\{\cos[(n+1)\theta] - r\cos[n\theta]\}}{2|1 - re^{i\theta}|^2} \tag{1.2b}$$

(for all $0 \leqslant r < 1$, all real θ), it is well known (cf. Titchmarsh [7, §7.7]) that $s_n(z)$ has the integral representation

$$s_n(re^{i\theta}) = \frac{1}{\pi}\int_0^{2\pi} f(\tau e^{i(\theta-\phi)}) k_n\left(\frac{r}{\tau};\phi\right) d\phi \qquad (0 \leqslant r < \tau < 1). \tag{1.3}$$

As shown by Fejér [3], the triangle inequality applied to (1.2b) gives

$$k_n(r;\theta) \geqslant \frac{1 - r^2 - 2r^{n+1} - 2r^{n+2}}{2|1 - re^{i\theta}|^2} \qquad (0 \leqslant r < 1). \tag{1.4}$$

Thus, if ρ_n is defined to be the unique positive root (from Descartes's rule of signs) of

$$1 - \rho^2 - 2\rho^{n+1} - 2\rho^{n+2} = 0 \qquad (n \geqslant 1), \tag{1.5}$$

then $0 < \rho_n < 1$, and from (1.4), $k_n(r;\theta) \geqslant 0$ for all $0 \leqslant r \leqslant \rho_n$ and all θ. Now, this positivity of $k_n(r;\theta)$ implies, using (1.3) and (1.2a), that

$$|s_n(z)| \leqslant \|f\|_\infty \quad \text{for all } |z| \leqslant \rho_n \qquad (n \geqslant 1). \tag{1.6}$$

Next, from (1.5), it easily follows that

$$\rho_1 = \tfrac{1}{2}, \tag{1.7i}$$

$$1 > \rho_{n+1} > \rho_n \qquad \text{for all} \quad n \geqslant 1, \tag{1.7ii}$$

$$\lim_{n\to\infty} \rho_n = 1. \tag{1.7iii}$$

Hence, for any $n \geqslant 1$, (1.6) and (1.7ii) give Fejér's result

$$|s_m(z)| \leqslant \|f\|_\infty \quad \text{for all } |z| \leqslant \rho_n \qquad (\text{all} \quad m \geqslant n). \tag{1.8}$$

In particular, as $\rho_1 = 1/2$, the special case $n = 1$ of (1.8) is

$$|s_m(z)| \leqslant \|f\|_\infty \quad \text{for all } |z| \leqslant \tfrac{1}{2} \qquad (\text{all} \quad m \geqslant 1). \tag{1.9}$$

It is interesting to remark that Fejér's result (1.9) is known to be *sharp* (cf. [7, §7.73]), in the sense that the constant $\frac{1}{2}$ in (1.9) is the *largest* number for which (1.9) is valid for *all* $f(z)$ in H^∞.

In this paper, we present in Section 2 a new proof of Fejér's result (1.8) which is based on connections with linear algebra. In particular, we use the classical notion of *diagonal dominance* from matrix theory to show how (1.5) arises in a very natural way. We also obtain the apparently new observation that Fejér's result (1.8) is *sharp* for any odd positive integer n, and is not sharp for any even positive integer n. For convenience, we state below this extension of Fejér's result (1.8) as Proposition 1, whose proof is given in Section 2.

PROPOSITION 1. *For any $f(z)$ in H^∞ and for any positive integer n, the partial sums $s_m(z)$ of $f(z)$ satisfy* (1.8), *where ρ_n is defined in* (1.5). *Moreover*, (1.8) *is sharp (in the sense that ρ_n is the largest number for which* (1.8) *holds for all $f(z)$ in H^∞) iff n is an odd positive integer.*

It is, however, possible to *reformulate* Fejér's result (1.8) in a way which can be shown, again using matrix theory, to be sharp for *every* $n \geqslant 1$. To this end, consider the numerator of $k_n(r;\theta)$ of (1.2b), and, for each positive integer n, set

$$\hat{\rho}_n := \max\{ r \geqslant 0 : 1 - r^2 - 2r^{n+1}\cos[(n+1)\theta] + 2r^{n+2}\cos[n\theta] \geqslant 0 \text{ for all } \theta \}. \tag{1.10}$$

From (1.4) and (1.5), it is evident that $\hat{\rho}_n \geqslant \rho_n$, and from (1.10) that $1 > \hat{\rho}_n$. We shall show in Section 3 that the numbers $\{\hat{\rho}_n\}_{n=1}^\infty$ also satisfy the associated properties of (1.7). Thus, from (1.10) and (1.2b), we see that $k_n(r;\theta) \geqslant 0$ for all $0 \leqslant r \leqslant \hat{\rho}_n$ and all θ. In analogy with (1.6), (1.7ii), and (1.8), this positivity of $k_n(r;\theta)$ similarly gives

$$|s_m(z)| \leqslant \|f\|_\infty \quad \text{for all } |z| \leqslant \hat{\rho}_n \qquad (\text{all} \quad m \geqslant n). \tag{1.11}$$

Our new result, which improves upon Proposition 1, is Proposition 2, whose proof is given in Section 3.

PROPOSITION 2. *For any $f(z)$ in H^∞ and for any positive integer n, the partial sums $s_m(z)$ of $f(z)$ satisfy* (1.11), *where $\hat{\rho}_n$ is defined in* (1.10). *Moreover,* (1.11) *is sharp (in the sense that $\hat{\rho}_n$ is the largest number for which* (1.11) *holds for all $f(z)$ in H^∞) for every $n \geqslant 1$.*

Finally, we conclude this paper with a tabulation in Table 1 in Section 3 of the values of $\{\rho_n\}_{n=1}^{10}$ and $\{\hat{\rho}_n\}_{n=1}^{10}$, truncated to six decimal digits.

2. PROOF OF PROPOSITION 1

As usual, let π_n denote the collection of all complex polynomials of degree at most n. For any $g(z)=\sum_{j=0}^{n} b_j z^j$ in π_n, and for a fixed $h(z)=\sum_{j=0}^{\infty} a_j z^j$ in H^∞, define the convolution operator T_h by

$$(T_h g)(z) := (h * g)(z) := \sum_{j=0}^{n} a_j b_j z^j, \tag{2.1}$$

so that T_h maps π_n into π_n. Then, this operator T_h is said (cf. Ruschweyh [6]) to be *bound preserving* on π_n if

$$\|T_h g\|_\infty = \|h * g\|_\infty \leqslant \|g\|_\infty \qquad (\text{all} \quad g \in \pi_n). \tag{2.2}$$

Now, Fejér's result (1.6) (after a change of scale) is just

$$\left\| \sum_{j=0}^{n} a_j \rho^j z^j \right\|_\infty \leqslant \|f\|_\infty \qquad (\text{all} \quad 0 \leqslant \rho \leqslant \rho_n), \tag{2.3}$$

for any $f(z)=\sum_{j=0}^{\infty} a_j z^j$ in H^∞. Since we can write $\sum_{j=0}^{n} a_j \rho^j z^j = (g_n * f)(z)$, where $g_n(z) := \sum_{j=0}^{n} \rho^j z^j$, then (2.3) is equivalent to

$$\|T_{g_n} f\|_\infty = \|g_n * f\|_\infty \leqslant \|f\|_\infty \qquad (\text{all } f \in H^\infty; \quad 0 \leqslant \rho \leqslant \rho_n). \tag{2.3$'$}$$

As any polynomial is necessarily in H^∞, (2.3′) implies

$$\|T_{g_n} f\|_\infty = \|g_n * f\|_\infty \leqslant \|f\|_\infty \qquad (\text{all } f \in \pi_k; \quad 0 \leqslant \rho \leqslant \rho_n), \tag{2.3$''$}$$

for any k, so that T_{g_n} is bound preserving on π_k for any k. On the other hand, it is easily seen [since, for any $f(z)$ in H^∞, its partial sums converge uniformly to $f(z)$ on compact subsets of $|z|<1]$ that (2.3″) conversely *implies* (2.3′), and (2.3) and (2.3″) are thus equivalent with Fejér's result (1.6).

Our goal is to show, using matrix theory, that ρ_n in (2.3″) necessarily satisfies (1.5). This will then give a new proof of Fejér's result (1.6), and with the results of (1.7), a new proof of Fejér's result (1.8). The following lemma shows how (2.3″) can be reduced to a problem in matrix theory.

LEMMA 1 (cf. Ruscheweyh [6, Chapter 4, and [4]). *Let* $h(z)=:1+\sum_{j=1}^{\infty}h_jz^j$. *Then, the associated operator* T_h *(cf.* (2.1)*) is bound preserving on* π_n *iff the* $(n+1)\times(n+1)$ *Hermitian matrix*

$$\begin{bmatrix} 1 & h_1 & \cdots & h_n \\ \bar{h}_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & h_1 \\ \bar{h}_n & \cdots & \bar{h}_1 & 1 \end{bmatrix} \tag{2.4}$$

is positive semidefinite.

We remark that the matrix in (2.4) is of course, by its structure, a *Toeplitz matrix* (cf. [5, p. 27]). Next, we see from Lemma 1 that if $h(z)$ is in π_n, then the operator T_h is bound preserving on π_k, $k \geqslant n$, iff the $(k+1)\times(k+1)$ banded Hermitian matrix

$$\begin{bmatrix} 1 & h_1 & \cdots & h_n & & 0 \\ \bar{h}_1 & \cdot & \cdot & & \ddots & \\ \vdots & \cdot & \cdot & \cdot & & h_n \\ \bar{h}_n & & \cdot & \cdot & \cdot & \vdots \\ & \ddots & & \cdot & \cdot & h_1 \\ 0 & & \bar{h}_n & \cdots & \bar{h}_1 & 1 \end{bmatrix} \quad (k \geqslant n) \tag{2.5}$$

is positive semidefinite. This can be used as follows.

Fixing n in (2.3″), the associated operator T_{g_n} [where $g_n(z):=\sum_{j=0}^{n}\rho^jz^j$], viewed as a mapping of π_k into π_k, is bound preserving on π_k iff the

$(k+1)\times(k+1)$ Hermitian matrix B_{k+1}, defined by

$$B_{k+1} := \begin{bmatrix} 1 & \rho & \cdots & \rho^n & & 0 \\ \rho & \cdot & \cdot & & \ddots & \\ \vdots & \cdot & \cdot & \cdot & & \rho^n \\ \rho^n & & \cdot & \cdot & \cdot & \vdots \\ & \ddots & & \cdot & \cdot & \rho \\ 0 & & \rho^n & \cdots & \rho & 1 \end{bmatrix} \quad \text{if} \quad k \geqslant n+1, \tag{2.6$'$}$$

and by

$$B_{k+1} := \begin{bmatrix} 1 & \rho & \cdot & \cdot & \cdot & \rho^k \\ \rho & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \rho \\ \rho^k & \cdot & \cdot & \cdot & \rho & 1 \end{bmatrix} \quad \text{if} \quad 0 \leqslant k \leqslant n, \tag{2.6$''$}$$

is positive semidefinite, where $\rho \geqslant 0$. From, Lemma 1 and (2.3″), we thus seek the largest value of $\rho \geqslant 0$ such that the matrices B_{k+1} of (2.6) are Hermitian and positive semidefinite for *all* k.

Next, consider the $(k+1)\times(k+1)$ (nonsingular) upper bidiagonal matrix

$$P := \begin{bmatrix} 1 & -\rho & & & 0 \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & 0 & & \cdot & -\rho \\ & & & & 1 \end{bmatrix}, \tag{2.7}$$

where $\rho \geqslant 0$. Then, a calculation shows that the real symmetric congruence transformation $P^T B_{k+1} P$ is given by the diagonal matrix

$$P^T B_{k+1} P = \begin{bmatrix} 1 & & & \\ & 1-\rho^2 & & 0 \\ & 0 & \ddots & \\ & & & 1-\rho^2 \end{bmatrix} \quad \text{if} \quad 0 \leqslant k \leqslant n, \tag{2.8$'$}$$

and by

$$P^T B_{k+1} P$$

$$= \begin{bmatrix} 1 & & & -\rho^{n+1} & & 0 \\ & 1-\rho^2 & & +\rho^{n+2} & \ddots & \\ & & 1-\rho^2 & & \ddots & -\rho^{n+1} \\ & & & \ddots & 0 & \rho^{n+1} \\ -\rho^{n+1} & \rho^{n+2} & 0 & & \ddots & \\ & \ddots & \ddots & & & \ddots \\ 0 & & -\rho^{n+1} & \rho^{n+2} & & 1-\rho^2 \end{bmatrix}$$

$$\text{if} \quad k \geqslant n+1, \tag{2.8$''$}$$

where $-\rho^{n+1}$ in the first row of the above matrix is its $(1, n+2)$ element. Since quadratic forms are invariant under such congruence transformations (cf. Birkhoff and MacLane [2, p. 251]), then B_{k+1} is positive semidefinite iff $P^T B_{k+1} P$ is positive semidefinite.

We now recall the following familiar result from matrix theory, based on the old and useful notion of *diagonal dominance* [cf. (2.9)].

LEMMA 2 (cf. [8, p. 23, Exercise 4]). *Let $A = [a_{i,j}]$ be an $l \times l$ Hermitian diagonally dominant matrix, i.e., $a_{i,j} = \bar{a}_{j,i}$ $(1 \leqslant i, j \leqslant l)$ and*

$$|a_{i,i}| \geqslant \sum_{\substack{j=1 \\ j \neq i}}^{l} |a_{i,j}| \qquad (1 \leqslant i \leqslant l). \tag{2.9}$$

If A in addition possesses nonnegative diagonal entries (i.e., $a_{i,i} \geqslant 0$ *for* $1 \leqslant i \leqslant l$), *then A is positive semidefinite.*

We apply Lemma 2 to the real symmetric matrix $P^T B_{k+1} P$ of (2.8). Note that for $0 \leqslant \rho < 1$, the diagonal entries of $P^T B_{k+1} P$ are all positive and greater than or equal to $1-\rho^2$, and each row of this matrix contains at most four nonzero nondiagonal entries, namely $-\rho^{n+1}$, ρ^{n+2}, ρ^{n+2}, and $-\rho^{n+1}$. Thus, $P^T B_{k+1} P$ is diagonally dominant, and hence positive semidefinite from Lemma 2 for all k, if $\rho \geqslant 0$ satisfies

$$1 - \rho^2 \geqslant 2\rho^{n+1} + 2\rho^{n+2}. \tag{2.10}$$

But from (1.5), the above inequality holds iff $0 \leqslant \rho \leqslant \rho_n$. Hence, we have shown that if ρ satisfies $0 \leqslant \rho \leqslant \rho_n$, then the matrices B_{k+1} of (2.6) are Hermitian positive semidefinite for all k. Thus, (2.3″) is valid for any k, and we have a new proof of Fejér's result (1.6), namely that

$$|s_n(z)| \leqslant \|f\|_\infty \qquad \text{for all} \quad |z| \leqslant \rho_n, \tag{2.11}$$

for any $f(z)$ in H^∞. As previously remarked, Fejér's result (1.8) then follows from (2.11) and (1.7).

We now deduce, using matrix theory, the apparently new result of the *sharpness* of the constant ρ_n in (2.11) or (1.8) for any *odd* positive integer n. [Recall that this is known (cf. [7, §7.73]) for the case $n = 1$.] Assume that the matrices B_{k+1} of (2.6) are positive semidefinite for *all* k, and consider the $(k+1)\times(k+1)$ matrix of (2.8″) with n a fixed odd positive integer and with $k \geqslant 2n+2$. Consider the vector ξ (with $k+1$ components) given by $\xi :=$ $[1, -1, +1, -1, \ldots, (-1)^{k+2}]^{\mathrm{T}}$, and compute $\xi^T P^T B_{k+1} P \xi$, noting that $\xi^T\xi$ $= k+1$. Now, because n is odd, it follows from (2.8″) that

$$\left(P^T B_{k+1} P \xi\right)_j = \left(1 - \rho^2 - 2\rho^{n+1} - 2\rho^{n+2}\right)\xi_j, \tag{2.12}$$

provided that $n+1 < j \leqslant k+1-(n+1)$. The remaining components $(P^T B_{k+1} P\xi)_j$, for $1 \leqslant j \leqslant n+1$ and $k+1-(n+1) < j \leqslant k+1$, are $2n+2$ terms, each of which is bounded above in modulus for any choice of ρ in $[0,1]$. As n is a fixed (odd) integer, it follows that

$$\begin{aligned} \mu = \mu(n,k,\rho) &:= \frac{\xi^T P^T B_{k+1} P \xi}{\xi^T \xi} \\ &= 1 - \rho^2 - 2\rho^{n+1} - 2\rho^{n+2} + O\left(\frac{1}{k+1}\right) \end{aligned} \tag{2.13}$$

as $k \to \infty$. Since μ is a Raleigh quotient for the matrix $P^T B_{k+1} P$, μ necessarily lies between the largest and smallest eigenvalues of this matrix (cf. Horn and Johnson [5, p. 176]). As B_{k+1} is assumed to be positive semidefinite for all $k \geqslant 2n+2$, so is $P^T B_{k+1} P$, and thus

$$\mu = \mu(n,k,\rho) \geqslant 0 \qquad (k \geqslant 2n+2). \tag{2.14}$$

Letting $k \to \infty$ in (2.13) gives that

$$1 - \rho^2 - 2\rho^{n+1} - 2\rho^{n+2} \geqslant 0. \tag{2.15}$$

Thus, combining with the result of the previous paragraph, we have shown that when n is odd, the matrices B_{k+1} of (2.6) are Hermitian positive semidefinite for all k iff ρ satisfies $0 \leqslant \rho \leqslant \rho_n$. In particular, ρ_n is the largest constant for which (1.8) is valid when n is *odd*.

One may naturally ask if (1.8) is *sharp* for n any even positive integer. This turns out to be false for *every* even n. Recalling that the *positivity* of $k_n(r;\theta)$ is the key to establishing (1.6), we wish to show now that the triangle inequality, used in deducing (1.4), is always too pessimistic in the cases when n is even. More precisely, for $n = 2l$, we know that [cf. (1.4) and (1.5)]

$$k_{2l}(\rho_{2l};\theta) \geqslant \frac{1-\rho_{2l}^2 - 2\rho_{2l}^{2l+1} - 2\rho_{2l}^{2l+2}}{2|1-\rho_{2l}e^{i\theta}|^2} = 0 \qquad (\text{all real } \theta). \quad (2.16)$$

Now, suppose that equality holds throughout above for some real $\hat{\theta}$. Then (2.16) implies from (1.2b) that

$$\{1-\cos[(2l+1)\theta]\} + \rho_{2l}\{1+\cos[2l\theta]\} \geqslant 0 \qquad (\text{all real } \theta), \quad (2.17)$$

with equality holding for $\hat{\theta}$. As both expressions in braces in (2.17) are nonnegative and as $\rho_{2l} > 0$, then equality can hold in (2.17) iff $\cos[(2l+1)\hat{\theta}] = 1$ and $\cos[2l\hat{\theta}] = -1$, which is impossible when $n = 2l$ is an even positive integer. Thus, $k_{2l}(\rho_{2l};\theta) > 0$ for all θ, which implies that neither (1.6) nor (1.8) could be sharp when n is even. This completes the proof of Proposition 1 of section 1. □

3. PROOF OF PROPOSITION 2

We now turn to the proof of Proposition 2 of Section 1, based on the definition of $\hat{\rho}_n$ in (1.10). As previously mentioned, (1.2b) and (1.10) give that $k_n(r;\theta) \geqslant 0$ for all $0 \leqslant r \leqslant \hat{\rho}_n$ and all θ, so that from (1.3),

$$|s_n(z)| \leqslant \|f\|_\infty \quad \text{for all } |z| \leqslant \hat{\rho}_n \qquad (\text{all } \quad n \geqslant 1). \quad (3.1)$$

Next, it is evident from (1.4), (1.5), and (1.10) that

$$1 > \hat{\rho}_n \geqslant \rho_n \qquad (\text{all } n \geqslant 1), \quad (3.2)$$

and, because of the sharpness portion of Proposition 1, there necessarily follows

$$\hat{\rho}_n = \rho_n \qquad (\text{all } n \text{ odd}, \quad n \geqslant 1). \tag{3.3}$$

[This can also be seen by taking $\theta = \pi$ in (1.10).]

For the ρ_n's defined in (1.5), the statement in (1.7ii) that $\rho_{n+1} > \rho_n$ is *immediate*, but the analogous statement for the $\hat{\rho}_n$'s, i.e.,

$$\hat{\rho}_{n+1} > \hat{\rho}_n \qquad (\text{all} \quad n \geqslant 1), \tag{3.4}$$

now requires proof. But *assuming* that (3.4) is valid, then from (3.1),

$$|s_m(z)| \leqslant \|f\|_\infty \quad \text{for all } |z| \leqslant \hat{\rho}_n \qquad (\text{all} \quad m \geqslant n),$$

which is the basis [cf. (1.11)] for Proposition 2 of Section 1. We further note that, with (3.4) and (3.2), the numbers $\{\hat{\rho}_n\}_{n=1}^{\infty}$ similarly satisfy the associated properties of (1.7).

We now establish (3.4). First, for n an *odd* positive integer, say $n = 2l+1$ $(l \geqslant 0)$, (3.4) is true, since from (3.2), (3.3), and (1.7ii),

$$\hat{\rho}_{2l+1} = \rho_{2l+1} < \rho_{2l+2} \leqslant \hat{\rho}_{2l+2}.$$

Thus, to establish (3.4), it remains to show that

$$\hat{\rho}_{2l} < \hat{\rho}_{2l+1} \qquad (\text{all} \quad l \geqslant 1). \tag{3.5}$$

In Table 1 at the end of this section, we give numerical values for $\{\rho_n\}_{n=1}^{10}$ and $\{\hat{\rho}_n\}_{n=1}^{10}$. From this Table 1, we see that $\hat{\rho}_2 = 0.612372\ldots$ is less than $\hat{\rho}_3 = 0.647798\ldots$, so that it suffices to establish (3.5) for every $l \geqslant 2$.

From (1.2a), we see that

$$k_n(r;\theta) = \operatorname{Re}\left\{\tfrac{1}{2} + z + \cdots + z^n\right\} \qquad (z = re^{i\theta}), \tag{3.6}$$

so that $k_n(r;\theta)$ is a nonconstant harmonic function in the disk $|z| < 1$, for any $n \geqslant 1$. On taking the numerator of $k_n(r;\theta)$ in (1.2b) and on employing the definition (1.10), we have

$$1 - r^2 - 2r^{n+1}\cos[(n+1)\theta] + 2r^{n+1}\cos[n\theta] \geqslant 0$$
$$(0 \leqslant r \leqslant \hat{\rho}_n; \quad \theta \text{ real}), \tag{3.7}$$

with equality holding for some $\hat{\theta}_n$ when $r=\hat{\rho}_n$. But with the *minimum modulus principle* applied to $k_n(r;\theta)$, we further have

$$1-r^2-2r^{n+1}\cos[(n+1)\theta]+2r^{n+2}\cos[n\theta]>0 \qquad (0\leqslant r<\hat{\rho}_n, \quad \theta \text{ real}) \tag{3.8}$$

for every $n\geqslant 1$. On choosing $n=2l+1$ and $r=\hat{\rho}_{2l}$ in (3.8), suppose that

$$1-(\hat{\rho}_{2l})^2-2(\hat{\rho}_{2l})^{2l+2}\cos[(2l+2)\theta]+2(\hat{\rho}_{2l})^{2l+3}\cos[(2l+1)\theta]>0 \qquad (\text{all } \theta \text{ real}). \tag{3.9}$$

Then, it would follow from (3.8) that $\hat{\rho}_{2l}<\hat{\rho}_{2l+1}$, the desired result of (3.5). Thus, (3.9) is sufficient to establish (3.5). Now, the global minimum of the left side of (3.9), regarded as a function of θ, occurs when $\theta=\pi$, so that (3.9) holds iff

$$1-(\hat{\rho}_{2l})^2-2(\hat{\rho}_{2l})^{2l+2}-2(\hat{\rho}_{2l})^{2l+3}>0 \qquad (l\geqslant 2). \tag{3.10}$$

Next, on choosing $r=\hat{\rho}_{2l}$, $\theta=\pi+\pi/(2l)$, and $n=2l$ in (3.7), we obtain

$$1-(\hat{\rho}_{2l})^2-2(\hat{\rho}_{2l})^{2l+1}\cos\left(\frac{\pi}{2l}\right)-2(\hat{\rho}_{2l})^{2l+2}\geqslant 0. \tag{3.11}$$

On comparing (3.11) and (3.10), it is evident that the truth of

$$\hat{\rho}_{2l}<\left(\cos\frac{\pi}{2l}\right)^{1/2} \qquad (l\geqslant 2) \tag{3.12}$$

implies the truth of (3.10), so that establishing (3.12) will give the desired result of (3.5).

To establish (3.12), insert $\hat{r}:=\{\cos[\pi/(2l)]\}^{1/2}$, $\theta=\pi+\pi/(2l)$, and $n=2l$ in the left side of (3.7), which gives

$$\omega(l):=1-\cos\frac{\pi}{2l}-2\cos\left(\frac{\pi}{2l}\right)^{l+3/2}-2\left(\cos\frac{\pi}{2l}\right)^{l+1}. \tag{3.13}$$

Since $\cos(\pi/2l) > 1 - \pi^2/8l^2$ for all $l \geqslant 2$, then

$$\omega(l) < \frac{\pi^2}{8l^2} - 2\left(1 - \frac{\pi^2}{8l^2}\right)^{l+3/2} - 2\left(1 - \frac{\pi^2}{8l^2}\right)^{l+1}. \tag{3.14}$$

By elementary inequalities, it can be shown that the right side of (3.14) is negative for all $l \geqslant 2$, i.e.,

$$\omega(l) < 0 \qquad (l \geqslant 2). \tag{3.15}$$

But, as $\omega(l)$ just a specific evaluation of the left side of (3.7), then (3.15) implies, from (3.7), that $\hat{r} \dashv \cos[\pi/(2l)]\}^{1/2} > \hat{\rho}_{2l}$, which establishes both (3.12) and (3.5).

To complete the proof of Proposition 2, it remains to show that (1.11) is *sharp* for *each* $n \geqslant 1$. Following the lines of the proof of Proposition 1, assume that the $(k+1)\times(k+1)$ Hermitian matrix B_{k+1} of (2.6) *positive semidefinite* for all $k \geqslant 2n+2$, and let P be the $(k+1)\times(k+1)$ matrix of (2.7). For any real number θ, consider the vector $\xi := [e^{i\theta}, e^{2i\theta}, \ldots, e^{i(k+1)\theta}]^T$, and compute $\xi^* P^T B_{k+1} P \xi$, noting that $\xi^*\xi = k+1$. Similar to (2.12), we now find that

$$(P^T B_{k+1} P\xi)_j = \{1 - \rho^2 - 2\rho^{n+1}\cos[(n+1)\theta] + 2\rho^{n+2}\cos[n\theta]\}\,\xi_j, \tag{3.16}$$

provided that $n+1 < j < k+1-(n+1)$. The remaining components of $P^T B_{k+1} P\xi$ are again $2n+2$ terms, each of which is bounded above in modulus by a constant for any ρ in $[0,1]$ and for any real θ. As n is again a fixed integer, it follows that

$$\mu := \frac{\xi^* P^T B_{k+1} P \xi}{\xi^*\xi}$$

$$= 1 - \rho^2 - 2\rho^{n+1}\cos[(n+1)\theta] + 2\rho^{n+2}\cos[n\theta] + O\left(\frac{1}{k+1}\right) \tag{3.17}$$

as $k \to \infty$. But as μ is a Rayleigh quotient for the matrix $P^T B_{k+1} P$, where

B_{k+1} is assumed to be positive semidefinite, then

$$\mu := 1 - \rho^2 - 2\rho^{n+1}\cos[(n+1)\theta] + 2\rho^{n+2}\cos[n\theta] + O\left(\frac{1}{k+1}\right) \geqslant 0$$

for all $k \geqslant 2n+2$, and letting $k \to \infty$ gives that

$$1 - \rho^2 - 2\rho^{n+1}\cos[(n+1)\theta] + 2\rho^{n+2}\cos[n\theta] \geqslant 0.$$

But as θ can be *any* real number, we see from (3.7) that ρ must satisfy $0 \leqslant \rho \leqslant \hat{\rho}_n$.

Conversely, assume that $\tilde{\rho} > \hat{\rho}_n$. Applying the minimum modulus principle again to $k_n(r; \theta)$, it follows from (3.7) that there is a real $\tilde{\theta}$ for which

$$1 - \tilde{\rho}^2 - 2\tilde{\rho}^{n+1}\cos[(n+1)\tilde{\theta}] + 2\tilde{\rho}^{n+2}\cos[n\tilde{\theta}] < 0. \tag{3.18}$$

For the vector $\tilde{\xi} := [e^{i\tilde{\theta}}, e^{2i\tilde{\theta}}, \ldots, e^{i(k+1)\tilde{\theta}}]^T$ and for the matrices $B_{k+1}(\tilde{\rho})$ of (2.6) and $P(\tilde{\rho})$ of (2.7) (where $\tilde{\rho}$ replaces ρ), a calculation similar to that of (3.17) shows, using (3.16) and (3.18), that

$$\tilde{\mu} := \frac{\tilde{\xi}^* P^T(\tilde{\rho}) B_{k+1}(\tilde{\rho}) P(\tilde{\rho}) \tilde{\xi}}{\tilde{\xi}^* \tilde{\xi}} < 0$$

for all k sufficiently large, so that the matrices $B_{k+1}(\tilde{\rho})$ are *not* all Hermitian positive semidefinite. This established that $\hat{\rho}_n$ the largest constant for which (3.1) or (1.11) valid for all $f(z)$ in H^∞. This completes the proof for Proposition 2. □

We complete our discussion of Proposition 2 with several additional remarks. First, on considering the definition of (1.10), it clear from the sharpness of Proposition 2 that for each positive integer n, there a real θ_n in $[0, \pi]$ for which

$$1 - (\hat{\rho}_n)^2 - 2(\hat{\rho}_n)^{n+1}\cos[(n+1)\theta_n] + 2(\hat{\rho}_n)^{n+2}\cos[n\theta_n] = 0. \tag{3.19}$$

What is interesting to note is that θ_n is in fact *uniquely determined* in $[0, \pi]$ from (3.19). Indeed, for n an odd positive integer, it is clear that $\theta_n = \pi$, while for n an even positive integer, it can be shown (we omit the proof) that θ_n is unique and lies in $(\pi - \pi/n, \pi - \pi/(n+1))$. This observation can be used in the following way to give a direct construction of the sharpness of $\hat{\rho}_n$

TABLE 1

n	ρ_n	$\hat{\rho}_n$
1	0.500000	0.500000
2	0.589754	0.612372
3	0.647798	0.647798
4	0.689139	0.694572
5	0.720412	0.720412
6	0.745071	0.747177
7	0.765116	0.765116
8	0.781794	0.782826
9	0.795930	0.795930
10	0.808091	0.808673

of (1.11). Specifically, as in [7, §7.73], consider $f_a(z) := (z-a)/(az-1)$, which is an element of H^∞ for any $0 < a < 1$. For any positive integer n, let $s_n(z; f_a)$ denote the nth partial sum of $f_a(z)$. Then, for any $\rho > \hat{\rho}_n$, it can be shown (we omit the proof) that

$$\left|s_n\left(\rho e^{i\theta_n}; f_a\right)\right| > \|f_a\|_\infty \tag{3.20}$$

for all $0 < a < 1$ with a sufficiently close to unity. Obviously, (3.20) directly gives the sharpness of $\hat{\rho}_n$ of (1.11).

In Table 1 we list the values of $\{\rho_n\}_{n=1}^{10}$ and $\{\hat{\rho}_n\}_{n=1}^{10}$, truncated to six decimal digits. Each ρ_n ($n \geqslant 1$) of Table 1 is, of course, the unique positive zero of the polynomial $1-\rho^2-2\rho^{n+1}-2\rho^{n+2}$ from (1.5). To describe how $\hat{\rho}_n$ was determined, suppose $n=2$ and consider [cf. (1.10)]

$$g_2(r;\theta) := 1 - r^2 - 2r^3\cos 3\theta + 2r^4\cos 2\theta. \tag{3.21}$$

Then,

$$\frac{\partial g_2}{\partial \theta} = 2r^3\sin\theta\left(12\cos^2\theta - 4r\cos\theta - 3\right), \tag{3.22}$$

which vanishes only for $\theta = 0$, π, and $\theta_\pm := \cos^{-1}\{(r \pm \sqrt{r^2+9})/6\}$, where $0 < r < 1$. The minimum of $g_2(r;\theta)$, evaluated at these four values of θ, is then the global minimum in θ of $g_2(r;\theta)$. Then, by a simple bisection procedure on the variable r, one finds the unique value r $(=\hat{\rho}_2)$ for which this global minimum exactly zero. (A similar procedure applies for all $n > 2$.)

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The following correction should be made:

P. 248, line +11. Read "... of (2.6) is positive" for "... of (2.6) positive"