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ON HYBRID SEMI-ITERATIVE METHODS*

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Abstract. Given a large sparse system of linear algebraic equations in fixed point form $\mathbf{x} = T\mathbf{x}+\mathbf{c}$, one way to solve this system is to apply a *semi-iterative method* (SIM) to the basic iteration method $\mathbf{x}_m = T\mathbf{x}_{m-1} + \mathbf{c}$. It is known that if the spectrum, $\sigma(T)$, of T is contained in some compact subset Ω of the complex plane (where $1 \notin \Omega$), then there are *asymptotically optimal* SIMs, associated with the basic iteration and Ω, whose asymptotic rates of convergence are *best possible* for the class of all matrices T with $\sigma(T) \subseteq \Omega$. However, for a given compact set Ω, an asymptotically optimal SIM for Ω is usually not well suited for efficient numerical computation unless this SIM can be generated from a k-step recurrence formula.

In this paper, *hybrid semi-iterative methods* are investigated which consist of two independent steps: First, $\mathbf{x} = T\mathbf{x}+\mathbf{c}$ is transformed into a consistent linear system, namely $\mathbf{x} = \tilde{T}\mathbf{x}+\tilde{\mathbf{c}}$, where $\tilde{T} := t_n(T)$ and where $t_n(z)$ is a complex polynomial in z, and then asymptotically optimal SIMs are considered with respect to $\mathbf{x} = \tilde{T}\mathbf{x} + \tilde{\mathbf{c}}$ and $\tilde{\Omega} := t_n(\Omega)$. In Theorem 6, a geometrical characterization is given of those $t_n(z)$ for which the asymptotically optimal hybrid SIMs for $\tilde{\Omega}$ give the same effective asymptotic convergence rate as do asymptotically optimal SIMs applied to the original matrix problem $\mathbf{x} = T\mathbf{x}+\mathbf{c}$ and Ω.

Finally, three examples are given (one arising from neutron-transport theory) to show how specific hybrid SIMs can give asymptotically optimal rates of convergence, and also, that these associated hybrid SIMs, generated by k-step recurrence formulas, are numerically effective.

Key words. semi-iterative methods (SIMs), asymptotically optimal SIMs, maximally convergent polynomials, k-step recurrence formulas, Chebyshev SIMs, Green's functions

AMS(MOS) subject classifications. 65F10, 41A10

1. Background and terminology.

The motivation for our work comes from Davis and Hageman [2] (cf. also [9, §11.3]), who considered the numerical solution of the *neutron-transport equation in x-y geometry.* If the discretized form of this equation is written, in matrix fixed point form, as

$$\mathbf{x} = T\mathbf{x} + \mathbf{c},$$

then Davis and Hageman showed that the eigenvalues of T are either real or purely imaginary. More precisely, these eigenvalues of T are contained in a cross-shaped region of the form

$$C_{\alpha,\beta} := [-\alpha, \alpha] \bigcup [-i\beta, i\beta], \text{ where } 0 < \alpha < 1 \text{ and } \beta > 0.$$

With $\sigma(T)$ denoting the spectrum of T, then $\sigma(T) \subseteq C_{\alpha,\beta}$, and the following different iterative methods for solving the above equation can be considered.

Assuming (as in Davis and Hageman [2]) that $0 < \beta \leq \alpha < 1$, then the *Jacobi iterative method* for this problem, defined by

$$\mathbf{x}_{m+1} = T\mathbf{x}_m + \mathbf{c} \qquad (m \geq 0),$$

is necessarily convergent (for any starting vector $\mathbf{x}_0$) to the unique solution $\mathbf{x}$ of $\mathbf{x} = T\mathbf{x}+\mathbf{c}$, and the asymptotic convergence factor $\kappa(T)$ (cf. [5, §4]) for the Jacobi iterative method satisfies

$$\kappa(T) \leq \kappa_1 := \alpha < 1.$$

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Since $C_{\alpha,\beta}$ and thus, the spectrum of T, is contained in an elliptic region with foci $\pm\sqrt{\alpha^2-\beta^2}$ (cf. Fig. 1, where $\alpha = 0.8$ and $\beta = 0.6$ have been chosen), the well-known *Chebyshev acceleration method* (cf. Golub and Varga [6], Manteuffel [11]), applied to the Jacobi iteration, *improves* the speed of convergence substantially. It can be shown that the asymptotic convergence factor of the Chebyshev method is bounded above by

$$\kappa_2 := \frac{\alpha+\beta}{1+\sqrt{1-\alpha^2+\beta^2}}.$$

Note that κ_2 is strictly smaller than κ_1 if $\alpha \neq \beta$. Next, "embracing" $\sigma(T)$ by a stretched hypocycloid (cf. Fig. 1) and applying the resulting *stationary four-step method* $\mathbf{y}_m := \mu_0(T\mathbf{y}_{m-1}+c)+\mu_2\mathbf{y}_{m-2}+\mu_4\mathbf{y}_{m-4}, m \geq 4$ (the parameters $\mu_j, j = 0,2,4$, depending only on α and β; cf. Niethammer and Varga [12, §9]) leads to the asymptotic convergence factor

$$\kappa_4 := \rho, \text{ where } \rho \in (0,1) \text{ is uniquely determined by}$$

$$(1-2\lambda)\rho^4 - 2(1-3\lambda)\rho^2 - 4(\lambda/\alpha)\rho + 1 = 0, \text{ and } \lambda := \beta^2/(\alpha^2+2\beta^2).$$

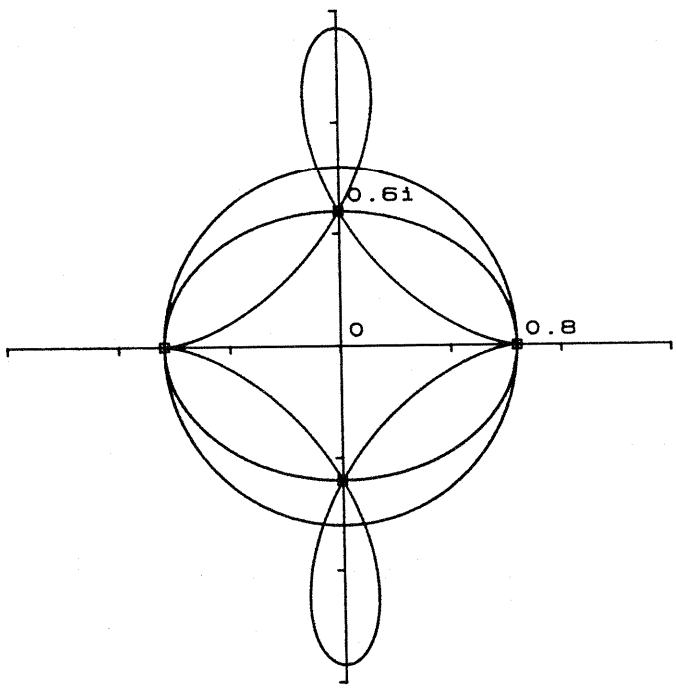

FIG. 1

Erratum

for "On hybrid semi-iterative methods" by
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p. 154, last column of Table 1. Read "$\tilde{K}$" for "tilde K"

p. 157, lines +16, +17. Read "on a certain regularity" for "on certain regularity"

p. 162, equation (4.5). Read "$\mu_1 \mathbf{y}_{m-1}$" for "$\mu_1 \mathbf{y}_m$".

There is another approach to iteratively solve $(I-T)\mathbf{x}=\mathbf{c}$. Obviously, $\mathbf{x}=T\mathbf{x}+\mathbf{c}$ is equivalent to $\mathbf{x}=T^2\mathbf{x}+(I+T)\mathbf{c}$, and from $\sigma(T)\subseteq C_{\alpha,\beta}$, there follows $\sigma(T^2)\subseteq[-\beta^2,\alpha^2]$. Thus, we can apply the optimal Chebyshev acceleration method (cf. [11]) to the transformed system $\mathbf{x}=T^2\mathbf{x}+(I+T)\mathbf{c}$, yielding the asymptotic convergence factor $\hat{\kappa}_2(\alpha,\beta)$. But since we have essentially doubled the amount of work per each iteration step (instead of multiplying by T we now have to multiply by T^2 in every step), we should consider

$$\tilde{\kappa}:=\sqrt{\hat{\kappa}_2(\alpha,\beta)}=\frac{\sqrt{1+\beta^2}-\sqrt{1-\alpha^2}}{\sqrt{\alpha^2+\beta^2}},$$

in order to compare the different methods fairly. Table 1 shows the convergence factors κ_2,κ_4 and $\tilde{\kappa}$ for some values of α and $\tilde{\kappa}$.

TABLE 1

α	β	κ_2	κ_4	$tilde\kappa$
0.8	0.5	0.72992	0.72541	0.54911
0.8	0.6	0.75736	0.68796	0.56619
0.8	0.7	0.78046	0.65705	0.58386
0.8	0.8	0.80000	0.63188	0.60159
0.9	0.5	0.84169	0.83056	0.66256
0.9	0.6	0.86172	0.80421	0.67516
0.9	0.7	0.87689	0.78109	0.68829
0.9	0.8	0.88957	0.76107	0.70151

These numbers indicate that the Chebyshev method applied to $\mathbf{x}=T^2\mathbf{x}+(I+T)\mathbf{c}$ is *superior* to the other two methods. We shall show that this method is even *asymptotically optimal* with respect to the given information that $\sigma(T)\subseteq C_{\alpha,\beta}$. In a more general framework, we shall further investigate conditions under which this technique, of transformating a linear system by a polynomial mapping, leads to a more efficient numerical iterative scheme.

To go beyond the previous specific example, we need some terminology. Given a linear system

$$A\mathbf{x}=\mathbf{b},\quad A\in\mathbb{C}^{N,N},\quad \mathbf{b}\in\mathbb{C}^N, \tag{1.1}$$

we assume that A is nonsingular. A splitting of the matrix A, i.e., $A=M-(M-A)$ with M nonsingular, leads us to the equivalent fixed point form

$$\mathbf{x}=T\mathbf{x}+\mathbf{c}, \tag{1.2}$$

where $T:=I-M^{-1}A$ and $\mathbf{c}:=M^{-1}\mathbf{b}$, and where 1 is not an eigenvalue of T. It is well known that the basic iteration

$$\mathbf{x}_{m+1}=T\mathbf{x}_m+\mathbf{c}\ (m\geq 0);\qquad \mathbf{x}_0=\mathbf{a}, \tag{1.3}$$

converges, for arbitrary $\mathbf{a}$, to the solution $\mathbf{x}$ of (1.1) and (1.2) if and only if the spectral radius of T, denoted by $\rho(T)$, satisfies $\rho(T)<1$.

For any infinite lower triangular matrix $P=[\pi_{m,i}]$, i.e.,

$$P=\begin{bmatrix}\pi_{0,0}&&&&\\ \pi_{1,0}&\pi_{1,1}&&&\\ \pi_{2,0}&\pi_{2,1}&\pi_{2,2}&&\\ \vdots&\vdots&\vdots&\ddots&\\ \cdot&\cdot&\cdot&&\cdot\\ \cdot&\cdot&\cdot&&\end{bmatrix} \tag{1.4}$$

with complex entries $\pi_{m,i}$ and row sums identically equal to 1, i.e.,

$$\sum_{i=0}^{m} \pi_{m,i} = 1 \quad (m \geq 0), \tag{1.5}$$

we can define the vector sequence

$$\mathbf{y}_m := \sum_{i=o}^{m} \pi_{m,i} \mathbf{x}_i \quad (m \geq 0). \tag{1.6}$$

Varga ([16], [17, §5.1]) calls (1.6) a *semi-iterative method* (SIM) *with respect to the basic iteration* (1.3). Other authors frequently use the term *polynomial acceleration method applied to* (1.3) (see, e.g., Hageman and Young [9, p. 40]).

For any $m \geq 0$, the associated *error vector* $\mathbf{e}_m := \mathbf{x} - \mathbf{y}_m$ for the vector $\mathbf{y}_m$ of (1.6) satisfies

$$\mathbf{e}_m = p_m(T)\mathbf{e}_0, \tag{1.7}$$

where p_m is the polynomial defined by

$$p_m(z) := \sum_{i=0}^{m} \pi_{m,i} z^i \ (m \geq 0). \tag{1.8}$$

Clearly from (1.5), $p_m(1) = 1$, for all $m \geq 0$.

Since the SIM (1.6) is completely determined by either the infinite matrix P of (1.4) or the polynomial sequence $\{p_m\}_{m\geq 0}$ of (1.8), we shall say that (1.6) is *induced* by P or by $\{p_m\}_{m\geq 0}$.

We now assume that we have a priori information that the spectrum $\sigma(T)$ of the iteration matrix T is contained in some compact subset Ω of the complex plane. Since $\lambda = 1$ cannot be an eigenvalue of T (otherwise the matrix A in (1.1) would be singular), we also assume that $1 \notin \Omega$.

We next define the root-convergence factor $\kappa(T, P)$ by

$$\kappa(T, P) := \overline{\lim_{m\to\infty}} \left\{ \sup_{\mathbf{e}_0 \neq 0} \left[\frac{\|\mathbf{e}_m\|}{\|\mathbf{e}_0\|} \right]^{1/m} \right\} \tag{1.9}$$

(which is independent from the chosen vector norm), which is a measure for the asymptotic decay of the norm of the error vectors (1.7). $\kappa(T, P)$ is also called the *asymptotic convergence factor* for the SIM induced by P. The *asymptotic convergence factor for* Ω is then defined by

$$\kappa(\Omega) := \inf_{P} \left\{ \sup \left[\kappa(T, P) : T \in \mathbb{C}^{N,N} \text{ for some } N \geq 1 \text{ and } \sigma(T) \subseteq \Omega \right] \right\}, \tag{1.10}$$

where the infinum is taken over all infinite matrices P of the form (1.4) which satisfy (1.5). For any SIM (induced by P), there is a matrix T with $\sigma(T) \subseteq \Omega$ for which $\kappa(T, P) \geq \kappa(\Omega)$. On the other hand, there exist infinite matrices $\hat{P}$ of the form (1.4) (generating SIMs) such that $\kappa(T, \hat{P}) \leq \kappa(\Omega)$ for any T with $\sigma(T) \subseteq \Omega$ (cf. Eiermann, Niethammer, and Varga [5]). If this is valid, or in other words, if the asymptotic convergence factor for the SIM generated by $\hat{P}$ with respect to the whole class $\{T : T \in \mathbb{C}^{N,N}$ for some $N \geq 1$ and $\sigma(T) \subseteq \Omega\}$ is as small as possible, the induced SIM will be called *asymptotically optimal with respect to* Ω.

For our later use, we give the following alternate characterization (cf. [5]) of $\kappa(\Omega)$ of (1.10):

$$\kappa(\Omega) = \lim_{m\to\infty} \gamma_m^{1/m}; \tag{1.11}$$

$$\gamma_m := \min\left\{ \max_{z\in\Omega} |p_m(z)| : p_m \in \Pi_m \text{ and } p_m(1) = 1 \right\},$$

where Π_m denotes the set of all complex polynomials of degree m. The minimization problem (1.11) has a *unique* solution for each $m \geq 0$ (see, e.g., Smirnov and Lebedev [15, p. 367]), provided that the *cardinality* of Ω (i.e., the number of distinct points of Ω) is infinite. (We henceforth assume that any Ω considered here satisfies this condition.) We remark that, with additional assumptions on Ω, the asymptotic convergence factor $\kappa(\Omega)$ for Ω can be nicely described in terms of a conformal mapping (cf. (3.3′) of §3), or more generally in terms of a Green's function for $\bar{\mathbb{C}}\backslash\Omega$ (cf. (3.3) of §3). This will be considered in detail in §3.

Besides the polynomials $\{p_m\}_{m\geq 0}$ of (1.8), we introduce another polynomial sequence, namely $\{q_{m-1}\}_{m\geq 0}$, associated with a given SIM, where

$$q_{m-1}(z) := \frac{1 - p_m(z)}{1 - z} \qquad (m \geq 0). \tag{1.12}$$

Since we have $p_m(1) = 1$, q_{m-1} is indeed a polynomial of degree $m-1$. Moreover, it can be shown that q_{m-1} is the uniquely determined Hermite interpolation polynomial interpolating the function $g(z) := 1/(1-z)$ at the zeros of p_m (cf. [5]). The iterates $\mathbf{y}_m$ of a SIM, as defined in (1.6), can now be expressed (cf. Eiermann and Niethammer [4]) as

$$\mathbf{y}_m = p_m(T)\mathbf{a} + q_{m-1}(T)\mathbf{c}. \tag{1.13}$$

A SIM is *asymptotically optimal* with respect to Ω if and only if the associated interpolating polynomials q_{m-1} *converge maximally* to g on Ω (cf. [5, Cor. 12]). Maximal convergence of polynomials to analytic functions is a well-known concept from complex approximation theory (see, e.g., Walsh [18, Chap. IV]). Since several classes of maximally convergent polynomials are known, this concept can be used to construct fast convergent SIMs, e.g., SIMs which are generated by orthogonal polynomials (see, e.g., Gragg and Reichel [7]), by Faber polynomials (cf. [5, §7]), or by generalized Chebychev polynomials (see, e.g., Opfer and Schober [13]).

We remark that, for a given set Ω, it may be difficult and computationally expensive to explicitly construct maximally convergent polynomials. Moreover, we are only interested in those polynomials which in addition satisfy some recurrence relation which then can be used to construct effective schemes for the computation of the vector sequence $\{\mathbf{y}_m\}_{m\geq 0}$ of (1.6) (cf. [5, §3]). We therefore consider a different approach to solve the problem of (1.2). This can be described roughly as follows.

We transform the system (1.2) *into an equivalent system*

$$\mathbf{x} = \tilde{T}\mathbf{x} + \tilde{\mathbf{c}}, \tag{1.14}$$

where $\tilde{T} := t_n(T)$ *with* $t_n \in \Pi_n$ *and* $t_n(1) = 1$.

Since $\sigma(T) \subseteq \Omega$, *we have* $\sigma(\tilde{T}) \subseteq \tilde{\Omega} := \{z = t_n(w) : w \in \Omega\}$. *We therefore seek to select the polynomial* t_n *in such a way that an asymptotically optimal* SIM, *with respect to* $\tilde{\Omega}$ (*or at least a* SIM *whose corresponding convergence factor is only slightly larger than* $\kappa(\tilde{\Omega})$), *is either known or can be constructed with a reasonable amount of work.*

Further, we also require that this SIM *permits an effective computation of the associated iterates* $\mathbf{y}_m$ *(cf.* (1.6)).

Then we apply this optimal, or "nearly optimal", SIM *to the transformed equation* (1.14).

We call such a composite acceleration scheme a *hybrid semi-iterative method with respect to the basic iteration* (1.3). It is obviously completely determined by a transformation polynomial $t_n \in \Pi_n$ (where $t_n(1) = 1$), and a SIM with respect to (1.14) which may be induced by an infinite lower triangular matrix P (cf. (1.4)), or equivalently, either one of the polynomial sequences $\{p_m\}_{m\geq 0}$ (cf. (1.8)) and $\{q_{m-1}\}_{m\geq 0}$ (cf. (1.12)).

In §2, we shall study some of the properties of hybrid SIMs and their relation to other procedures for solving linear systems. To judge the practical applicability of a hybrid SIM, it is necessary to compare the convergence factors associated with an "ordinary" SIM to those of a hybrid method. For that purpose, we will investigate in §2 the relationship between $\kappa(\Omega)$ and $[\kappa(\tilde{\Omega})]^{1/n}$, where $\tilde{\Omega} := t_n(\Omega)$.

One of the main results (cf. Theorem 6) of this paper is the result, which depends on certain regularity condition on Ω, that

$$\kappa(\Omega) \leq [\kappa(\tilde{\Omega})]^{1/n},$$

and that

$$\kappa(\Omega) = [\kappa(\tilde{\Omega})]^{1/n},$$

if and only if $z \notin \Omega$ implies $t_n(z) \notin \tilde{\Omega}$. This means that, for a given Ω and for a given polynomial $t_n(z)$, the associated hybrid SIM generates an asymptotically optimal SIM for Ω if and only if $z \notin \Omega$ implies $t_n(z) \notin \tilde{\Omega}$.

Finally, in §4 we shall illustrate the effectiveness of hybrid SIMs by several examples, one example having an application to the numerical solution of the neutron-transport equations.

2. Some properties of hybrid SIMs. Our first proposition gives conditions on the transformation polynomial t_n which imply that the transformed system (1.14) is consistent with the original one of (1.2).

PROPOSITION 1. *Let* $t_n \in \Pi_n$ *satisfy* $t_n(1) = 1$. *Then, the (uniquely determined) solution* $\mathbf{x}$ *of* (1.2) *is also a solution of*

$$\mathbf{x} = \tilde{T}\mathbf{x} + \tilde{\mathbf{c}}, \tag{2.1}$$

where $\tilde{T} := t_n(T), \tilde{\mathbf{c}} := u_{n-1}(T)\mathbf{c}$ *and* $u_{n-1}(z) := (1 - t_n(z))/(1 - z) \in \Pi_{n-1}$. *Moreover, if* $t_n(\lambda) \neq 1$ *for any eigenvalue* λ *of* T, *then the linear systems* (1.2) *and* (2.1) *are consistent, i.e., they both have the vector* $\mathbf{x}$ *as a unique solution.*

Proof. If x denotes the unique solution of (1.2), we have by induction that

$$\mathbf{x} = T^k\mathbf{x} + (I + T + \ldots + T^{k-1})\mathbf{c} = T^k\mathbf{x} + (I - T^k)(I - T)^{-1}\mathbf{c} \quad (k \geq 0).$$

Writing $t_n(z) := \sum_{k=0}^{n} c_k z^k$ where $t_n(1) = \sum_{k=0}^{n} c_k = 1$, then on multiplying the above equation by c_k, and summing on k, directly gives the first assertion. The second assertion is simply a consequence of the identity $\sigma(\tilde{T}) = \{t_n(\lambda) : \lambda \in \sigma(T)\}$. □

By definition, any pair (t_n, P) completely determines a hybrid SIM. Recall that t_n denotes an arbitrarily fixed polynomial of degree n (with $t_n(1) = 1$) which transforms the original linear system (1.2) into the linear system $\mathbf{x} = \tilde{T}\mathbf{x} + \tilde{\mathbf{c}}$ of (2.1), and that

the infinite lower triangular matrix P of (1.4) induces a SIM with respect to the corresponding basic iteration $\mathbf{x}_m = \tilde{T}\mathbf{x}_{m-1} + \tilde{\mathbf{c}}, \quad m \geq 1$. Such a SIM can also be generated by polynomial sequences $\{p_m\}_{m\geq 0}$ (cf. (1.8)) or $\{q_{m-1}\}_{m\geq 0}$ which implies that the iterates $\tilde{\mathbf{y}}_{m-1}$ of the given hybrid method can be expressed as

$$\tilde{\mathbf{y}}_m = p_m(\tilde{T})\mathbf{a} + q_{m-1}(\tilde{T})\tilde{\mathbf{c}} = p_m(t_n(T))\mathbf{a} + q_{m-1}(t_n(T))u_{n-1}(T)\mathbf{c} \tag{2.2}$$

(cf. (1.13) and the definitions in Proposition 1). With $\tilde{p}_{mn}(z) := p_m(t_n(z)) \in \Pi_{mn}$ (obviously, $\tilde{p}_{mn}(1) = 1$) and $\tilde{q}_{mn-1}(z) := (1 - \tilde{p}_{mn}(z))/(1 - z)) \in \Pi_{mn-1}$, it can be shown that $\tilde{q}_{mn-1}(z) = q_{m-1}(t_n(z))u_{n-1}(z)$ for any $m \geq 0$. Hence, it follows that (2.2) is equivalent to

$$\tilde{\mathbf{y}}_m = \tilde{p}_{mn}(T)\mathbf{a} + \tilde{q}_{mn-1}(T)\mathbf{c} \quad (m \geq 0). \tag{2.3}$$

In other words, the iterates $\tilde{\mathbf{y}}_m$ of the hybrid SIM can be considered as a subsequence of the approximants belonging to an "ordinary" SIM with respect to the basic iteration (1.3) (cf. also (1.13)).

If we approximate the solution of (1.2) by a SIM, we normally use a recursion of the form (cf. [5, §3])

$$\mathbf{y}_m = \mu_{m,0}(T\mathbf{y}_{m-1} + \mathbf{c}) + \mu_{m,1}\mathbf{y}_{m-1} + \ldots + \mu_{m,m}\mathbf{y}_0 \qquad (m \geq 1), \tag{2.4}$$

where $\mu_{m,0} \neq 0$ and $\sum_{i=0}^{m}\mu_{m,i} = 1$ for all $m \geq 0$. Besides the standard SIMs, other iterative schemes for the solution of linear systems (e.g., the conjugate gradient method) can be represented in the form (2.4), too. (Of course, the only methods (2.4) which are of practical interest are those for which there is an integer k such that $\mu_{m,i} = 0$ for any $m \geq i \geq k$. In this case, (2.4) reduces to a so-called *k-step method.*)

In applying (2.4), the main problem consists in determining parameters $\mu_{m,i}(m \geq i \geq 0)$ such that the sequence $\{y_m\}_{m\geq 0}$ tends rapidly to the solution of $\mathbf{x} = T\mathbf{x}+\mathbf{c}$ (1.2). To overcome this difficulty, we apply a method of the form (2.4) to the *transformed* system $\mathbf{x} = \tilde{T}\mathbf{x} + \tilde{\mathbf{c}}$ (2.1), i.e., we consider the iterates

$$\tilde{\mathbf{y}}_m = \tilde{\mu}_{m,0}(\tilde{T}\tilde{\mathbf{y}}_{m-1} + \tilde{\mathbf{c}}) + \tilde{\mu}_{m,1}\tilde{\mathbf{y}}_{m-1} + \ldots + \tilde{\mu}_{m,m}\tilde{\mathbf{y}}_0 \qquad (m \geq 1), \tag{2.5}$$

where we naturally try to choose the transformation in such a way that the problem of selecting the parameters $\{\tilde{\mu}_{m,i}\}_{0\leq i\leq m}$ is easier to solve than the original one. (It will be shown in §4 that there are indeed cases for which such transformations exist.)

Calculating $\tilde{\mathbf{y}}_m$ according to (2.5), we have to multiply by the transformed matrix $\tilde{T}$. If the original iteration matrix T is large and sparse, it certainly is not appropriate to compute and store $\tilde{T} = t_n(T)$ explicitly. One way to avoid this is described in the following proposition, which easily can be proven by induction on the degree of t_n.

PROPOSITION 2. *Let the transformation polynomial $t_n \in \Pi_n, t_n(1) = 1$, be factored, according to*

$$t_n(z) = \prod_{j=1}^{n}(z - \xi_j)/(1 - \xi_j), \tag{2.6}$$

and define

$$\alpha_j := 1/(1 - \xi_j) \quad (1 \leq j \leq n). \tag{2.7}$$

Then, the iterates $\tilde{\mathbf{y}}_m$ of (2.5) may be computed by

$$\text{(2.8a)}\qquad \tilde{\mathbf{y}}_{m-1}^{(0)} = \tilde{\mathbf{y}}_{m-1},\quad \tilde{\mathbf{y}}_{m-1}^{(j)} = \alpha_j(T\tilde{\mathbf{y}}_{m-1}^{(j-1)} + \mathbf{c}) + (1-\alpha_j)\tilde{\mathbf{y}}_{m-1}^{(j-1)} \quad (1 \le j \le n-1),$$

and

$$\text{(2.8b)}\qquad \tilde{\mathbf{y}}_m = \tilde{\mu}_{m,0}\alpha_n(T\tilde{\mathbf{y}}_{m-1}^{(n-1)} + \mathbf{c}) + \tilde{\mu}_{m,0}(1-\alpha_n)\tilde{\mathbf{y}}_{m-1}^{(n-1)} + \tilde{\mu}_{m,1}\tilde{\mathbf{y}}_{m-1} + \ldots + \tilde{\mu}_{m,m}\tilde{\mathbf{y}}_0,$$

for $m \ge 1$.

The algorithm proposed in (2.8) consists of an *inner iteration* (2.8a), which corresponds to $n-1$ steps of a *nonstationary first-order Richardson method,* and of an *outer iteration* (2.8b), which itself can be considered as one step of a SIM with respect to the basic iteration (1.3). Note further that (2.8) requires n (the degree of t_n) matrix-vector multiplications by T.

To conclude this section, we present a slightly different approach to hybrid SIMs, which establishes a connection to the so-called *polynomial preconditioned conjugate gradient methods* (see, e.g., Dubois,Greenbaum, and Rodrigue [3]). Recall that the basic iteration (1.3) was constructed via a splitting $A = M-(M-A)$ of the coefficent matrix A (cf. (1.1)). Roughly speaking, the "quality" of this splitting, i.e., the speed of convergence of the associated basic iteration (1.3), depends on how well M^{-1} "approximates" A^{-1}. Since $A^{-1} = [I-M^{-1}(M-A)]^{-1}M^{-1} = (I-T)^{-1}M^{-1}$, it may be possible to improve this quality by replacing $(I-T)^{-1}$ by a polynomial approximation $u_{n-1}(T)$, with $u_{n-1} \in \Pi_{n-1}$, i.e., by considering the new splitting

$$\text{(2.9)}\qquad A = \hat{M} - (\hat{M} - A),\ \text{where } \hat{M} := M[u_{n-1}(T)]^{-1}.$$

Now, $\hat{M}$ exists if and only if $u_{n-1}(\lambda) \neq 0$ for every $\lambda \in \sigma(T)$, or equivalently, if and only if $t_n(\lambda) := 1-(1-\lambda)u_{n-1}(\lambda) \neq 1$ for every $\lambda \in \sigma(T)$. Note that the relationship between u_{n-1} and t_n is exactly the one established in Proposition 1 and further, that the existence of $\hat{M}$ is equivalent to the consistency of (2.1) with (1.2). Then (2.9) leads to another fixed-point formulation of (1.1):

$$\text{(2.10)}\qquad \mathbf{x} = \hat{T}\mathbf{x} + \hat{\mathbf{c}},\ \text{where } \hat{T} := \hat{M}^{-1}(\hat{M}-A) \text{ and } \hat{\mathbf{c}} := \hat{M}^{-1}\mathbf{b}.$$

Using the definitions of $\hat{M}$ and t_n as well as the identity $A = M(I-T)$, we easily see that $\hat{T} = t_n(T) = \tilde{T}$ and $\hat{\mathbf{c}} = u_{n-1}(T)\mathbf{c} = \tilde{\mathbf{c}}$ (cf. Proposition 1). Thus, it is obvious that a SIM, with respect to (2.10), is just an hybrid SIM with respect to (1.3). Moreover, if the iterative procedure (2.4) represents the conjugate gradient method, the algorithm (2.8) is known as a *conjugate gradient method with polynomial preconditioning,* an algorithm which is of recent interest because of connections with parallel architectures (cf. Johnson, Micchelli, and Paul [10]).

3. The asymptotic convergence factor. In the previous section, we have assumed that the spectrum of the iteration matrix T is contained in some compact subset Ω of the complex plane, where $1 \notin \Omega$. From (2.8), we have seen that one step of a hybrid SIM with respect to (1.3) essentially requires n (the degree of t_n) matrix-vector multiplications by T. Further, we know from (1.10) that $\kappa(\tilde{\Omega})$ is the *best* possible convergence factor that we can obtain with this information on $\sigma(T)$. On the other hand, we have to perform only one matrix-vector multiplication by T if we apply a SIM with respect to (1.3) (cf. (2.4)), and $\kappa(\Omega)$ is then the associated optimal

convergence factor. Thus, to fairly compare both schemes, namely (2.4) and (2.8), we must study the relationship between $\kappa(\Omega)$ and $[\kappa(\tilde{\Omega})]^{1/n}$.

To do so, we need some further restrictions on Ω, namely, we require that Ω belongs to the following class $\mathbb{M}$:

$$\begin{aligned}\mathbb{M} := \{\Omega \subseteq \mathbb{C} \ : \ &\Omega \text{ is compact, } 1 \notin \Omega, \\ &\Omega \text{ has no isolated points} \\ &\text{and } \bar{\mathbb{C}}\backslash\Omega \text{ is of finite connectivity}\}.\end{aligned} \tag{3.1}$$

If $\Omega \in \mathbb{M}$, then from theorems of Lebesgue and Osgood (cf. Walsh [18, §4.1]), there exists a (uniquely determined) *Green's function* G for $\bar{\mathbb{C}}\backslash\Omega$ with pole at infinity (where $\bar{\mathbb{C}} := \mathbb{C} \bigcup\{\infty\}$). The function G is a positive real-valued function defined on $\mathbb{C}\backslash\Omega$ which is characterized (cf. [18, §4.1]) by the following three properties:

G is a harmonic function in $\mathbb{C}\backslash\Omega$,
$G(z) - \log|z|$ is harmonic in a neighborhood of infinity, and
$\lim_{z\to\zeta} G(z) = 0$ for every $\zeta \in \partial\Omega$,

where $\partial\Omega$ denotes the boundary of Ω.

From a generalized version of Bernstein's Lemma (cf. [18, §4.6]), the inequality

$$\max_{z\in\Omega}|p_m(z)| \geq [\exp(-G(1))]^m \tag{3.2}$$

is valid for any polynomial p_m of degree m which satisfies $p_m(1) = 1$. From (1.11), this implies, for any $\Omega \in \mathbb{M}$, that

$$\kappa(\Omega) \geq \exp(-G(1)).$$

Using the Theorem of Kalmár and Walsh (cf. [18, Chap. VII]), it can be shown (cf. [5, §4]) that equality holds above, i.e.,

$$\kappa(\Omega) = \exp(-G(1)). \tag{3.3}$$

If $\bar{\mathbb{C}}\backslash\Omega$ is simply connected, then by the Riemann Mapping Theorem, there exists a conformal mapping Φ from $\bar{\mathbb{C}}\backslash\Omega$ onto $\bar{\mathbb{C}}\backslash\{z : |z| \leq 1\}$ such that the points at infinity correspond to each other. In this case, the Green's function G of $\bar{\mathbb{C}}\backslash\Omega$ is given explicitly by $G(z) = \log|\Phi(z)|$ and therefore, we obtain (cf. [5, §4])

$$\kappa(\Omega) = \frac{1}{|\Phi(1)|}. \tag{3.3'}$$

For our later use, we need the following generalization (to the more general sets of class $\mathbb{M}$) of the "Comparison Theorem" of Niethammer and Varga ([12, Thm. 3]).

PROPOSITION 3. *If the sets Ω_1 and Ω_2 belong to $\mathbb{M}$ and if Ω_1 is a proper subset of Ω_2, then*

$$\kappa(\Omega_1) < \kappa(\Omega_2). \tag{3.4}$$

Proof. Let G_1 and G_2 denote, respectively, the Green's functions with pole at infinity for $\bar{\mathbb{C}}\backslash\Omega_1$ and $\bar{\mathbb{C}}\backslash\Omega_2$. Their difference $G_1 - G_2$ is harmonic in $\bar{\mathbb{C}}\backslash\Omega_2$ (with a removable singularity at ∞). Further, $G_1(z) - G_2(z)$ approaches a nonnegative value if z tends to the boundary of Ω_2, and for $\zeta \in \partial\Omega_2\backslash\partial\Omega_1$ (such points existing by hypothesis), we have

$$\lim_{z\to\zeta}(G_1(z) - G_2(z)) = G_1(\zeta) > 0.$$

The minimum principle for harmonic functions now implies $G_1(z) - G_2(z) > 0$ for any $z \in \mathbb{C}\backslash\Omega_2$. In particular, this holds for $z = 1$, and, in view of (3.3), then inequality (3.4) follows. □

The purpose of this section is to study the relationship between $\kappa(\Omega)$ and $\kappa(\tilde{\Omega})$, where $\tilde{\Omega} := t_n(\Omega)$ and $t_n \in \Pi_n$ with $t_n(1) = 1$. Assuming that Ω belongs to the class $\mathbb{M}$, then the condition $1 \notin \tilde{\Omega}$, or equivalently the condition

$$t_n(z) = 1 \Rightarrow z \notin \Omega \tag{3.5}$$

is necessary and sufficient for $\tilde{\Omega} \in \mathbb{M}$ to hold. Note that (3.5) is also sufficient for (1.2) and (2.1) to be consistent (cf. Proposition 1).

We first show that $\kappa(\Omega)$ is always a lower bound for $[\kappa(\tilde{\Omega})]^{1/n}$.

LEMMA 4. *Let $\Omega \in \mathbb{M}$ and assume that the nth degree polynomial t_n satisfies $t_n(1) = 1$ and condition* (3.5). *With $\tilde{\Omega} := t_n(\Omega)$, there holds*

$$\kappa(\Omega) \leq [\kappa(\tilde{\Omega})]^{1/n}. \tag{3.6}$$

Proof. If the polynomial sequence $\{\hat{p}_m\}_{m\geq 0}$ consists of the solutions of the minimization problem (1.11) for $\tilde{\Omega}$, there holds (cf. (1.10), (1.11))

$$\kappa(\tilde{\Omega}) = \overline{\lim_{m\to\infty}} \left\{ \max_{z\in\tilde{\Omega}} |\hat{p}_m(z)|^{1/m} \right\}.$$

But by (3.2) and (3.3), we obtain

$$\max_{z\in\tilde{\Omega}} |\hat{p}_m(z)| = \max_{z\in\Omega} |\hat{p}_m(t_n(z))| \geq \kappa(\Omega)^{mn} \quad (m \geq 0),$$

because $\hat{p}_m(t_n(z))$ is a polynomial of degree mn with $\hat{p}_m(t_n(1)) = 1$. This implies (3.6). □

Next, we give a condition which implies strict inequality in (3.6).

LEMMA 5. *Let $\Omega \in \mathbb{M}$ and assume that the nth degree polynomial t_n satisfies $t_n(1) = 1$ and condition* (3.5). *Further, we assume that there exists a point $z_0 \notin \Omega$ with $t_n(z_0) \in t_n(\Omega) = \tilde{\Omega}$. Then,*

$$\kappa(\Omega) < [\kappa(\tilde{\Omega})]^{1/n}. \tag{3.7}$$

Proof. Consider $\hat{\Omega} := \{z \in \mathbb{C} : t_n(z) \in \tilde{\Omega}\}$. Then, $\hat{\Omega}$ is certainly compact. Under the given assumptions on t_n, $\hat{\Omega}$ does not contain $z = 1$, and as a pre-image of a set with finitely many components under a polynomial transformation, it has also only finitely many components, none of which is an isolated point. All this implies that $\hat{\Omega}$ belongs to the class $\mathbb{M}$ (cf.(3.1)). By construction, $t_n(\hat{\Omega}) = t_n(\Omega) = \tilde{\Omega}$ and $\Omega \subseteq \hat{\Omega}$. But since $z_0 \in \hat{\Omega}\backslash\Omega$, then Ω is a proper subset of $\hat{\Omega}$.

From Proposition 3 and Lemma 4, we deduce that

$$\kappa(\Omega) < \kappa(\hat{\Omega}) \leq [\kappa(t_n(\hat{\Omega}))]^{1/n} = [\kappa(\tilde{\Omega})]^{1/n},$$

the desired result of (3.7). □

We are now in a position to prove the main result of this section.

THEOREM 6. *Let $\Omega \in \mathbb{M}$, and assume that the nth degree polynomial t_n satisfies $t_n(1) = 1$ and condition* (3.5). *With $\tilde{\Omega} := t_n(\Omega)$, then*

$$\kappa(\Omega) = [\kappa(\tilde{\Omega})]^{1/n} \tag{3.8}$$

if and only if the following condition is satisfied:

$$z \notin \Omega \Rightarrow t_n(z) \notin \tilde{\Omega}, \tag{3.9}$$

or, in other words, t_n induces not only a mapping from Ω onto $\tilde{\Omega}$, but also one from $\bar{\mathbb{C}}\backslash\Omega$ onto $\bar{\mathbb{C}}\backslash\tilde{\Omega}$.

Proof. From Lemma 5, it follows that (3.7) is necessary for $\kappa(\Omega) = [\kappa(\tilde{\Omega})]^{1/n}$ to hold. We have only to prove that it is also sufficient. Since $\tilde{\Omega} \in \mathbb{M}$ (this follows from (3.5)), there exists a Green's function $\tilde{G}$ for $\bar{\mathbb{C}}\backslash\tilde{\Omega}$ with pole at infinity. We define a positive real-valued function $\hat{G}$ by $\hat{G}(z) := (1/n)\tilde{G}(t_n(z))$ for all $z \in \mathbb{C}\backslash\Omega$. By a straightforward calculation, it can be verified that $\hat{G}$ is harmonic in $\mathbb{C}\backslash\Omega$, that $\hat{G}(z)$ approaches 0 if z tends to a boundary point of Ω, and that $\hat{G}(z) - \log|z| = 1/n[\tilde{G}(t_n(z)) - \log|t_n(z)|] + 1/n \log|t_n(z)/z^n|$ is harmonic in a neighborhood of ∞. Therefore, $\hat{G}$ must be the Green's function for $\bar{\mathbb{C}}\backslash\Omega$ with pole at infinity. Because of (3.3), this implies

$$\kappa(\Omega) = \exp(-\hat{G}(1)) = \exp\left(-\frac{1}{n}\tilde{G}(t_n(1))\right) = [\exp(-\tilde{G}(1))]^{1/n} = [\kappa(\tilde{\Omega})]^{1/n},$$

which is the desired result of (3.8). □

4. Examples. We now describe some examples where hybrid SIMs can be successfully applied to the matrix equation of (1.2). We shall first consider polynomial transformations $t_n \in \Pi_n$ of Ω, where $\Omega \in \mathbb{M}$ for which $t_n(\Omega)$ is a *real interval* $[\zeta, \eta]$, where $1 \notin [\zeta, \eta]$. It is known that the SIM induced by the translated and scaled Chebyshev polynomials

$$p_m(z) := T_m\left(\frac{2z - (\zeta+\eta)}{\eta - \zeta}\right) / T_m\left(\frac{2 - (\zeta+\eta)}{\eta - \zeta}\right) \qquad (m \geq 0) \tag{4.1}$$

is asymptotically optimal with respect to $[\zeta, \eta]$, i.e., its asymptotic convergence factor is given by (cf. Manteuffel [11])

$$\kappa := \kappa([\zeta, \eta]) = \begin{cases} (\eta - \zeta)/(\sqrt{1-\zeta} + \sqrt{1-\eta})^2 & \text{if } \zeta < \eta < 1, \\ (\eta - \zeta)/(\sqrt{\zeta - 1} + \sqrt{\eta - 1})^2 & \text{if } 1 < \zeta < \eta. \end{cases} \tag{4.2}$$

Moreover, the corresponding iterates $\{\mathbf{y}_m\}_{m\geq 0}$ satisfy a three-term recurrence relation of the form

$$\mathbf{y}_m = \mu_{m,0}(\tilde{T}\mathbf{y}_{m-1} + \tilde{\mathbf{c}}) + \mu_{m,1}\mathbf{y}_{m-1} + \mu_{m,2}\mathbf{y}_{m-2} \quad (m \geq 2) \tag{4.3}$$

(cf. [16, §5.1]), and these coefficients $\{\mu_{m,i}\}_{0\leq i\leq 2}$ in addition satisfy (cf. Golub and Varga [6])

$$\begin{aligned} &\lim_{m\to\infty} \mu_{m,0} = \mu_0 := \frac{4}{\eta - \zeta}\kappa, \quad \lim_{m\to\infty} \mu_{m,1} = \mu_1 := -2\frac{\eta + \zeta}{\eta - \zeta}\kappa, \\ &\lim_{m\to\infty} \mu_{m,2} = \mu_2 := -\kappa^2. \end{aligned} \tag{4.4}$$

In other words, if m tends to infinity, the recurrence relation (4.3) approaches the stationary two-step method

$$\mathbf{y}_m = \mu_0(\tilde{T}\mathbf{y}_{m-1} + \tilde{\mathbf{c}}) + \mu_1\mathbf{y}_m \quad + \mu_2\mathbf{y}_{m-2} \quad (m \geq 2), \tag{4.5}$$

the parameters $\mu_i (0 \leq i \leq 2)$ being defined by (4.4). This method is also *asymptotically optimal* with respect to $[\zeta, \eta]$, i.e., it has the convergence factor κ of (4.2) (cf. [12]).

Example 1. Returning to our introductory example (cf. §1), we assume that $\sigma(T)$, the spectrum of the iteration matrix T (cf. (1.3)), is contained in a cross-shaped region of the form

$$C_{\alpha,\beta} := [-\alpha, \alpha] \bigcup [-i\beta, i\beta], \text{ where } 0 < \alpha < 1 \text{ and } \beta > 0.$$

(Note that, in contrast to §1, β can now be any positive number.) Obviously, we have $C_{\alpha,\beta} \in \mathbb{M}$, and the polynomial $t_2(z) := z^2$, which maps $C_{\alpha,\beta}$ onto the interval $[-\beta^2, \alpha^2]$, satisfies the conditions of Theorem 6. Thus, we obtain from (4.2)

$$\kappa(C_{\alpha,\beta}) = \sqrt{\kappa([-\beta^2, \alpha^2])} = \frac{\sqrt{1+\beta^2} - \sqrt{1-\alpha^2}}{\sqrt{\alpha^2+\beta^2}}. \tag{4.6}$$

By Proposition 2, the application of the stationary two-step method (4.5) to the transformed linear system (2.1) is equivalent to the following procedure:

With arbitrary initial guesses $\tilde{\mathbf{y}}_0$ and $\tilde{\mathbf{y}}_1$, we compute the vector sequence $\{\tilde{\mathbf{y}}_m\}_{m\geq 0}$ according to the following recurrence relation:

$$\tilde{\mathbf{y}}_{m-1}^{(0)} = \tilde{\mathbf{y}}_{m-1}, \quad \tilde{\mathbf{y}}_{m-1}^{(1)} = T\tilde{\mathbf{y}}_{m-1}^{(0)} + \mathbf{c} \tag{4.7a}$$

and

$$\tilde{\mathbf{y}}_m = \mu_0(T\tilde{\mathbf{y}}_{m-1}^{(1)} + \mathbf{c}) + \mu_1\tilde{\mathbf{y}}_{m-1} + \mu_2\tilde{\mathbf{y}}_{m-2} \quad (m \geq 2), \tag{4.7b}$$

where

$$\mu_0 = \frac{4}{\alpha^2+\beta^2}\kappa, \quad \mu_1 = -2\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}\kappa, \quad \mu_2 = -\kappa^2, \tag{4.7c}$$

and where $\kappa := [\kappa(C_{\alpha,\beta})]^2$, as in (4.6).

In Table 2, the convergence factor κ_2 of the optimized Chebyshev SIM applied directly to (1.3) is compared to the convergence factor resulting from (4.7).

TABLE 2

α	β	κ_2	$\kappa(C_{\alpha,\beta})$
0.5	0.5	0.50000	0.35639
0.5	1.0	0.64575	0.49031
0.5	5.0	0.90542	0.84240
0.5	10.0	0.95131	0.91724

Example 2. We next consider a linear system (1.2), where $\sigma(T)$ is contained in the union of two disjoint intervals

$$U := [\alpha - \beta,\ \alpha - \epsilon\beta] \bigcup [\alpha + \epsilon\beta,\ \alpha + \beta], \tag{4.8}$$

where $\alpha \in \mathbb{R}$, $\beta > 0$, and $0 < \epsilon < 1$. (Similiar problems have recently been studied by de Boor and Rice [1] and Saad [14].)

Assuming $U \in \mathbb{M}$ (i.e., $1 \notin U$), we must distinguish between two different cases: If $\alpha \neq 1$, then $t_2(z) := ((z-\alpha)/(1-\alpha))^2$ maps U onto $[\epsilon^2\beta^2/(1-\alpha)^2, \beta^2/(1-\alpha)^2]$, and if $\alpha = 1$, then $t_n(z) := -z^2 + 2z$ maps U onto $[1-\beta^2,\ 1-\epsilon^2\beta^2]$. Since in each

case the assumptions of Theorem 6 are satisfied, we can proceed as in Example 1. The case $\alpha = 1$ is of special interest, because every Chebyshev-SIM applied to (1.3) will diverge in this situation, whereas the hybrid SIM consisting of the transformation $t_2(z) = -z^2 + 2z$ and the two-step method (4.5) not only converges but also has the best possible (i.e., optimal) asymptotic convergence factor

$$\kappa(U) := \sqrt{\frac{1-\epsilon}{1+\epsilon}}.$$

Two further remarks should be added. At first, it is of course possible to handle the case $\sigma(T) \subset U \in \mathbb{M}$ (cf. (4.8)) with *complex* parameters α and β in exactly the same way. (The reason for this is that such an optimal Chebyshev iteration for complex intervals is given in [12] and [5].) Secondly, if $U := [\alpha, \beta] \bigcup [\gamma, \delta] \in \mathbb{M}$, where $\alpha < \beta < \gamma < \delta$, but $\beta - \alpha \neq \delta - \gamma$, it is always possible to find a polynomial $t_2 \in \Pi_2$, with $t_2(1) = 1$, such that $t_2(U)$ is a real interval not containing $z = 1$. It is however impossible to choose a polynomial t_2 that fulfills the conditions of Theorem 6, i.e., for any $t_2 \in \Pi_2$, there exist complex numbers $z \notin U$ with $t_2(z) \in t_2(U)$. In other words, the hybrid SIM (4.7) (with appropriately chosen $\mu_i, 0 \leq i \leq 2$) converges in this case, too, but it is no longer asymptotically optimal.

Example 3. Suppose that the iteration matrix T of (1.3) is *weakly cyclic of index* p (cf. [16, p. 39]), i.e., there exists a permutation matrix $Q \in \mathbb{C}^{N,N}$ such that

$$QTQ^T = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & T_1 \\ T_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & T_3 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & T_p & 0 \end{bmatrix}, \tag{4.9}$$

where the diagonal submatrices are square. The eigenvalues of T are contained in the following star-shaped regions (cf. [16, §2.3] and Wild and Niethammer [19])

$$S_\beta^+ := \{z \in \mathbb{C} : z = \nu e^{2k\pi i/p}, \nu \in [0, \beta], 0 \leq k \leq p-1\} \tag{4.10a}$$

if $\sigma(T^p) \subset [0, \beta^p]$, where $\beta := \rho(T)$ (we shall refer to this case as the *nonnegative case*, cf. Fig. 2), or

$$S_\beta^- := \{z \in \mathbb{C} : z = \nu e^{(2k+1)\pi i/p}, \nu \in [0, \beta], 0 \leq k \leq p-1\} \tag{4.10b}$$

if $\sigma(T^p) \subset [-\beta^p, 0]$, where $\beta := \rho(T)$ (the *nonpositive case*, cf. Fig. 3).

Clearly, $S_\beta^+ \in \mathbb{M}$ (cf. (3.1)) if and only if $0 < \beta < 1$, and $S_\beta^- \in \mathbb{M}$ for any $\beta > 0$. Since $t_p(z) := z^p$ maps S_β^+ onto $[0, \beta^p]$ (and S_β^- onto $[-\beta^p, 0]$) and since the conditions of Theorem 6 are then fulfilled, (4.2) implies that the corresponding convergence factors are given by

$$\kappa(S_\beta^+) = \frac{\beta}{(1+\sqrt{1-\beta^p})^{2/p}} \text{ and } \kappa(S_\beta^-) = \frac{\beta}{(1+\sqrt{1+\beta^p})^{2/p}}. \tag{4.11}$$

Thus, the following hybrid SIM is *asymptotically optimal* in our situation: With arbitrary starting vectors $\tilde{\mathbf{y}}_0$ and $\tilde{\mathbf{y}}_1$ compute

$$\tilde{\mathbf{y}}_{m-1}^{(0)} = \tilde{\mathbf{y}}_{m-1}, \quad \tilde{\mathbf{y}}_{m-1}^{(j)} = T\tilde{\mathbf{y}}_{m-1}^{(j-1)} + \mathbf{c} \quad (1 \leq j \leq p-1) \tag{4.12a}$$

and

$$\tilde{\mathbf{y}}_m = \mu_0(T\tilde{\mathbf{y}}_{m-1}^{(p-1)} + \mathbf{c}) + \mu_1\tilde{\mathbf{y}}_{m-1} + \mu_2\tilde{\mathbf{y}}_{m-1} \quad (m \geq 2). \tag{4.12b}$$

The parameters $\mu_i (0 \leq i \leq 2)$ are given by

$$\mu_0 = \frac{4}{\beta^p}\kappa_\beta,\ \mu_2 = -2\kappa_\beta,\ \mu_4 = -(\kappa_\beta)^2 \tag{4.12c}$$

in the nonnegative case, and

$$\mu_0 = \frac{4}{\beta^p}\kappa_\beta,\ \mu_2 = 2\kappa_\beta,\ \mu_4 = -(\kappa_\beta)^2 \tag{4.12d}$$

in the nonpositive case, where (cf. (4.11))

$$\kappa_\beta := \begin{cases} [\kappa(S_\beta^+)]^p & \text{if } \sigma(T^p) \subset [0, \beta^p], \\ [\kappa(S_\beta^-)]^p & \text{if } \sigma(T^p) \subset [-\beta^p, 0]. \end{cases} \tag{4.12e}$$

The matrix QTQ^T of (4.9) is *consistently ordered* (cf. [16, p. 101]); we assume now that T is already given in this form and consider the *cyclic reduction* of the linear system (1.2) (cf. [16, §5.4]). If we partition the vectors $\mathbf{x}$ and $\mathbf{c}$ according to the partitioning of (4.9), (1.2) can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & T_1 \\ T_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & T_3 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & T_p & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_p \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_p \end{bmatrix}, \tag{4.13}$$

and thus, the transformed system $\mathbf{x} = T^p\mathbf{x} + \tilde{\mathbf{c}}$ $(\tilde{\mathbf{c}} := (I + T + \ldots + T^{p-1})\mathbf{c})$ is block-diagonal:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} \tilde{T}_1 & 0 & 0 & \cdots & 0 \\ 0 & \tilde{T}_2 & 0 & & 0 \\ 0 & 0 & \tilde{T}_3 & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{T}_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_p \end{bmatrix} + \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \tilde{c}_3 \\ \vdots \\ \tilde{c}_p \end{bmatrix}, \tag{4.14}$$

where

$$\tilde{T}_j = T_jT_{j-1}\ldots T_1T_pT_{p-1}\ldots T_{j+1} \quad (1 \leq j \leq p). \tag{4.15}$$

Thus, (1.3) is therefore equivalent to the p uncoupled equations (4.14), and in view of (4.13), it is sufficient to solve the reduced matrix problem

$$\mathbf{x}_p = \tilde{T}_p\mathbf{x}_p + \tilde{\mathbf{c}}_p. \tag{4.16}$$

To solve (4.16), we can again make use of (4.12), because the nonzero eigenvalues of $\tilde{T}_p$ and T^p are the same:

$$\tilde{\mathbf{y}}_{m-1}^{(0)} = \tilde{\mathbf{y}}_{m-1},\ \tilde{\mathbf{y}}_{m-1}^{(j)} = T_j\tilde{\mathbf{y}}_{m-1}^{(j-1)} + \mathbf{c}_j \quad (1 \leq j \leq p-1), \tag{4.17a}$$

$$\tilde{\mathbf{y}}_m = \mu_0(T_p\tilde{\mathbf{y}}_{m-1}^{(p-1)} + \mathbf{c}_p) + \mu_1\tilde{\mathbf{y}}_{m-1} + \mu_2\tilde{\mathbf{y}}_{m-2} \quad (m \geq 2), \tag{4.17b}$$

where the parameters $\mu_i (0 \leq i \leq 2)$ are chosen according to (4.12c) or (4.12d). In comparison to (4.12), the amount of work has been reduced substantially. In each step of (4.17), we have to multiply only *once* by each of the submatrices $T_1, T_2, \ldots, T_p$ which corresponds to one multiplication by T. Thus, the convergence factor of the iteration (4.17) is given by

$$\tilde{\kappa}_\beta := \begin{cases} \beta^p/(1+\sqrt{1-\beta^p})^2 & \text{if } \sigma(T^p) \subset [0, \beta^p], \\ \beta^p/(1+\sqrt{1+\beta^p})^2 & \text{if } \sigma(T^p) \subset [-\beta^p, 0]. \end{cases} \tag{4.18}$$

If the block Jacobi matrix T of the coefficient matrix A of (1.1) is a consistently ordered cyclic matrix of index p, the *block successive-overrelaxation* (SOR) *method* (cf. [16, Chap. 4]) is often used to solve (1.1) and (1.2), respectively. Using the notation of (4.13), the SOR method is described by

$$\begin{aligned} \mathbf{x}_1^{(m)} &= (1-\omega)\mathbf{x}_1^{(m-1)} + \omega\, T_1 \mathbf{x}_p^{(m-1)} &&+\omega\, \mathbf{c}_1 \\ \mathbf{x}_2^{(m)} &= (1-\omega)\mathbf{x}_2^{(m-1)} + \omega\, T_2 \mathbf{x}_1^{(m)} &&+\omega\, \mathbf{c}_2 \\ &\cdots && \qquad (m \geq 1), \\ \mathbf{x}_p^{(m)} &= (1-\omega)\mathbf{x}_p^{(m-1)} + \omega\, T_p\, \mathbf{x}_{p-1}^{(m)} &&+\omega\, \mathbf{c}_p \end{aligned} \tag{4.19}$$

where $\omega (0 < \omega < 2)$ is the associated relaxation parameter. Comparing (4.19) to (4.17), we find that the amount of work, for both the hybrid SIM and the block SOR method, is the same. To decide whether (4.19) or (4.17) yields faster convergence, we compare $\tilde{\kappa}_\beta$ to the spectral radius $\rho(L_{\omega^*})$ of the block SOR iteration matrix with optimal relaxation factor ω^*. In the nonnegative case, there holds $\rho(L_{\omega^*}) < 1$ if and only if $\beta < 1$. Here, ω^* is the (unique) solution of

$$(p-1)^{p-1}\omega^p\beta^p - p^p|\omega - 1| = 0 \tag{4.20}$$

which is contained in the open interval $(1, p/(p-1))$ (cf. [16, §4.3]). In the nonpositive case, we have $\rho(L_{\omega^*}) < 1$ if and only if $0 < \beta < p/(p-2)$ $(0 < \beta < \infty$ for $p = 2)$. Now ω^* is the (unique) solution of (4.20) which lies in $((p-2)/(p-1), 1)$ (cf. Hadjidimos, Li, and Varga [8] and Wild and Niethammer [19]). Further, there holds in each of both cases

$$\rho(L_{\omega^*}) = (p-1)|\omega^* - 1|. \tag{4.21}$$

From (4.21) and (4.18), we easily conclude that

$$\tilde{\kappa}_\beta = \rho(L_{\omega^*}) \text{ iff } p = 2$$

and, for $|\beta| > 0$,

$$\tilde{\kappa}_\beta < \rho(L_{\omega^*}) \text{ iff } p \geq 3.$$

In Fig. 4, $\tilde{\kappa}_\beta$ and $\rho(L_{\omega^*})$ are plotted, for $p = 5$, as a function of $\beta (-\infty < \beta < 1)$.

It is worth noting that in the nonpositive case, the block SOR method diverges if $\beta \geq p/(p-2)$ $(p > 2)$, whereas the hybrid method (4.17) will converge for every β. There is a more *intuitive* reason for this better performance of the hybrid methods (4.17). These methods are indeed asymptotically optimal for the star-shaped regions S_β^+ and S_β^- (cf. (4.10) and Figs. 2, 3). The p-cyclic block SOR method however can be considered as a certain p-step relaxation (cf. [19]) which is only optimal for the closed interior of certain hypocycloids shown in Figs. 2 and 3.

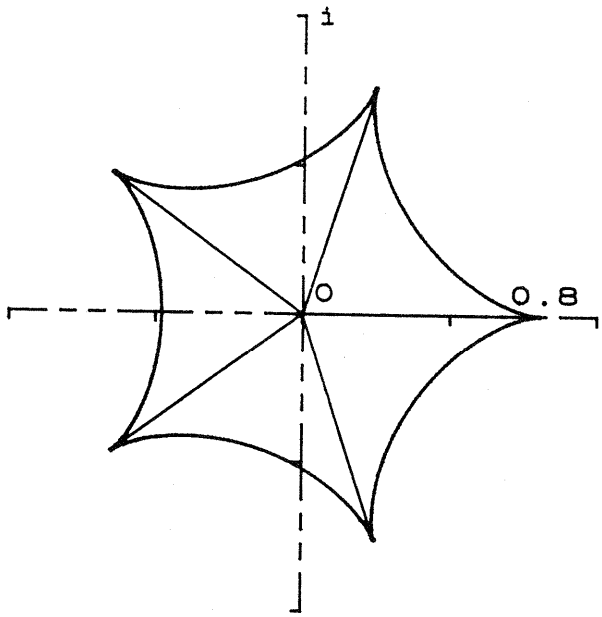

FIG. 2

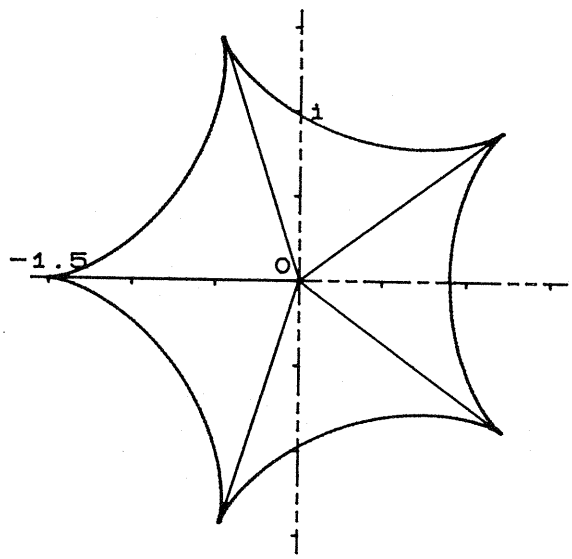

FIG. 3

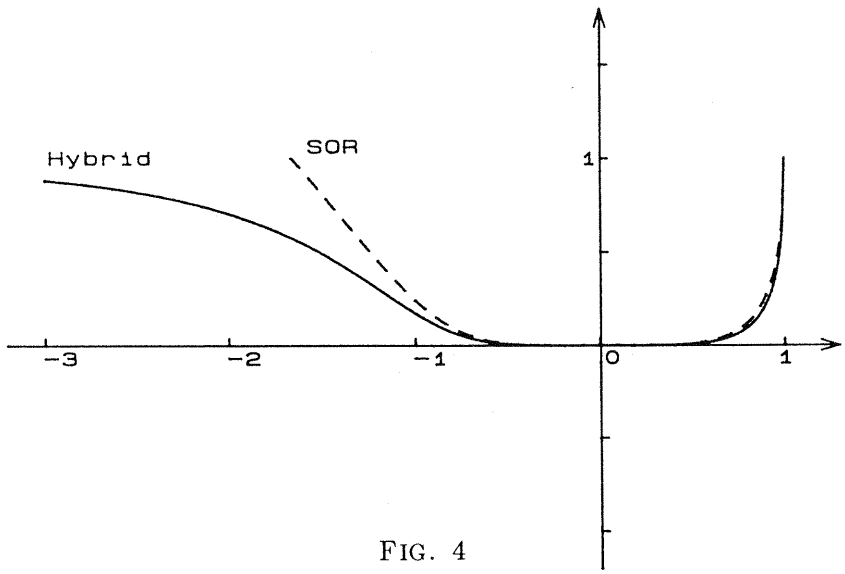

FIG. 4

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