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Convergence rate estimates for iterative solutions of the biharmonic equation *

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Abstract: A technique is developed whereby one can obtain asymptotic estimates of eigenvalues of first-order iteration matrices. The technique is applied to iteration matrices arising from the numerical solution of the 1- and 2-dimensional biharmonic equation. The eigenvalue estimates are computationally verified.

Keywords: Gauss–Siedel, SOR, Garabedian, biharmonic equation, eigenvalue estimates.

1. Introduction

Given a symmetric positive-definite matrix equation $Ax = b$, first order iterative methods for the computing the solution are defined by

$$Mx^{k+1} = Nx^k + b \tag{1.1}$$

where

$$A = M - N, \qquad x^0 = \text{"arbitrary" initial vector.} \tag{1.2}$$

A necessary and sufficient condition for convergence of first order schemes is that the spectral radius of the matrix $M^{-1}N$, $\rho(M^{-1}N)$, satisfies

$$\rho(M^{-1}N) < 1. \tag{1.3}$$

General conditions on the matrices M and N to assure (1.3) can be found in [1], [2], and [4] and although these conditions cover a wide number of situations, little is known of the actual rate of convergence of the iteration (1.1). This is disconcerting for it has been observed many times over that slight changes in the splitting (1.2) can decrease the number of iterations by several orders of magnitude and it would be useful to know ahead of time the number of iterations needed to

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reduce the norm of the error $\| x^k - x \|$, by a factor of, say, 10. A rough approximation to such a number is given by the quantity

$$-1/\log_{10}\left[\rho(M^{-1}N)\right] \tag{1.4}$$

and although this estimate can often be low, it nevertheless represents computational reality in many cases.

The disadvantage of formula in (1.4) is that it requires $\rho(M^{-1}N)$ which is, in general, unavailable. However, in the case where the matrix A originates from certain elliptic partial differential equations, approximations to the spectral radius are known. As an example, consider the numerical solution of the two-dimensional Laplace equation, $\Delta u = f$, on the unit square with zero-Dirichlet boundary conditions. Central difference approximations on a grid with uniform mesh space h yields the symmetric positive-definite matrix A given by

$$A = \frac{1}{h^2}\begin{bmatrix} C & B & & & \\ B & C & B & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & B \\ & & & B & C \end{bmatrix}$$

$$C = \begin{bmatrix} \ddots & & & & & \\ \ddots & \ddots & \ddots & & & \\ & 1 & -4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & \ddots & \end{bmatrix}, \qquad B = \begin{bmatrix} \ddots & & \\ & 1 & \\ & & \ddots \end{bmatrix}.$$

The following is known, cf. [4]:

(i) *Point Jacobi Method*:

$$M = \text{diagonal}(A), \qquad \rho(M^{-1}N) \cong 1 - \tfrac{1}{2}\pi^2 h^2.$$

(ii) *Point Gauss–areedel Method*:

$$D = \text{diagonal}(A), \qquad A = D - L - L^t, \qquad M = D - L;$$

$$\rho(M^{-1}N) \simeq 1 - \pi^2 h^2.$$

(iii) *Point SOR*:

$$D = \text{diagonal}(A), \qquad A = D - L - L^t,$$

$$M = D - wL, \quad 0 < w < 2,$$

$$\mathcal{L}_w = M^{-1}N, \qquad B = D^{-1}(L + L^t),$$

$$\rho(\mathcal{L}_{w_b}) \cong 1 - 2\pi h, \qquad w_b = \frac{2}{1 - \sqrt{1 - \rho(B)^2}}.$$

Consequently, if $h = 10^{-2}$, then (1.4) asserts that the number of iterations required by the point Jacobi method to decrease the error by a factor of 10 is 4664 whereas the number of iterations required by the point Gauss–Siedel method is approximately one-half the number of point

Jacobi iterations. Moreover, relaxing the Gauss–Siedel method to the optimal parameter of w_b achieves an SOR iterative method that requires 35 iterations to decrease the error by a factor of 10. Hence, one can see that eigenvalue estimates of the form are extremely important in determining the possible speed-up one might obtain by using a parameter $w > 1$.

Unfortunately, eigenvalue estimates such as those given in are difficult to obtain as they typically require knowledge of the eigenvectors of the matrix $\mathscr{L}_w$. However, for the purpose of iteration speed-up it is not important to have precise estimates of the eigenvalues of $\mathscr{L}_w$ but to have knowledge of how the eigenvalues behave with respect to the mesh size h. Using the "order" notation, (ii) and (iii) can be expressed as

(ii)$'$ $\rho(\mathscr{L}_1) \simeq 1 - \mathrm{O}(h^2)$,

(iii)$'$ $\rho(\mathscr{L}_{w_b}) \simeq 1 - \mathrm{O}(h)$,

and it is the decrease from 2 to 1 in the power of h that leads one to draw the conclusion that a considerable decrease in iteration count can be obtained by using w_b instead of $w = 1$.

In this paper, we develop a technique for obtaining $\mathrm{O}(h^\alpha)$ eigenvalue estimates of $M^{-1}N$ such as those given in (ii)$'$ and (iii)$'$ based upon an idea that was originally developed by Garabedian, [1]. We will demonstrate how this technique generates some of the classical $\mathrm{O}(h^\alpha)$ eigenvalue estimates for matrix splittings of the one-dimensional Laplace matrix and then apply the technique to obtain $\mathrm{O}(h^\alpha)$ eigenvalue estimates for matrix splittings arising from the numerical solution of the 1- and 2-dimensional biharmonic equation.

2. Basic approach

In [1], Garabedian observed that the optimum relaxation parameter w_b of the point relaxation method for solving the system of finite difference equations for the Laplacian can be derived by viewing the relaxation method as a time-differencing scheme. Specifically, for the Laplacian $\Delta\phi = 0$, the general relaxation method over a grid with mesh size $\Delta x = \Delta y = h$ is given by

$$\phi_{i,j}^{n+1} = (1 + w)\phi_{i,j}^n + \tfrac{1}{4}w\left[\phi_{i+1,j}^{n+1} + \phi_{i,j-1}^{n+1} + \rho_{i+1,j}^n + \phi_{i,j+1}^n\right]. \tag{2.1}$$

If we express w in the form

$$w = \frac{2}{1 + ch} \tag{2.2}$$

for any positive value of the constant c, then we can rearrange (2.1) to obtain

$$\frac{\phi_{i-1,j}^n + \phi_{i,j-1}^n + \phi_{i+1,j}^n + \phi_{i,j+1}^n - 4\phi_{i,j}^n}{h^2} = \frac{\phi_{i,j}^{n+1} - \phi_{ij}^n - \phi_{i-1,j}^{n+1} + \phi_{i-1,j}^n}{h^2} + \frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n - \phi_{i,j-1}^{n+1} + \phi_{i,j-1}^n}{h^2} + 2c\frac{\phi_{i,j}^{n+1} - \phi_{i,j}^n}{h}. \tag{2.3}$$

Using the familiar idea that the index n refers to a new time variable, t, and that (2.2) indicates the location of new net points spaced at time intervals equal to the original mesh size h, we recognize that (2.3) is the difference analogue of the hyperbolic partial differential equation

$$\phi_{xt} + \phi_{yt} + 2c\phi_t = \phi_{xx} + \phi_{yy}. \tag{2.4}$$

Thus for small values of h Garabedian observed that convergence of the iterative method (1.2) can be investigated by a Fourier analysis of the decay of time dependent terms in the solution of (2.4).

In this paper we approach the analysis of iterative methods much in the same way as Garabedian did except that a differential eigenvalue problem is obtained rather than a time dependent partial differential equation. Specifically, the iterative method for solving the matrix equation $Ax = b$ is expressed as a first order matrix iteration

$$Mx^{k+1} = Nx^k + b \tag{2.5}$$

and then using variations of the ideas set forth in Garabedian we will show that the convergence of (2.5) can be analyzed through a differential eigenvalue problem

$$Ru = \lambda Qu$$

(here R and Q are differential operators).

To illustrate the idea, let us consider the numerical solution of the 1-dimensional Laplace equation

$$u_{xx} = f(x), \quad 0 < x < 1, \tag{2.6}$$

with prescribed boundary conditions. Using central differences, (2.6) is approximated by the matrix equation $A\bar{u} = \bar{b}$ where

$$A = \frac{1}{h^2}\begin{bmatrix} \ddots & & & & \\ \ddots & \ddots & \ddots & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & & \ddots \end{bmatrix}. \tag{2.7}$$

Let $A = D - L - L^{\mathrm{t}}$ where D is a symmetric and positive definite matrix and consider the iteration

$$(D - L)\bar{u}^{n+1} = L^{\mathrm{t}}\bar{u}^n + \bar{b}. \tag{2.8}$$

It is well known that the convergence behavior of (2.8) is controlled by the magnitude of the eigenvalues of the generalized eigenvalue problem

$$(D - L)\bar{v} = \lambda L^{\mathrm{t}}\bar{v}. \tag{2.9}$$

In this paper we will always use a rearrangement of (2.9) and study the eigenvalue problem

$$A\bar{v} = \gamma(D - L)\bar{v}, \qquad \gamma = (1 - \lambda). \tag{2.10}$$

Letting $D = 2h^{-2}I$, (that is, (2.8) is the point-Gauss–Siedel method), then, away from the boundaries, (2.10) becomes

$$\begin{aligned} \frac{2v_i - (v_{i-1} + v_{i+1})}{h^2} &= (1 - \lambda)\left\{\frac{2v_i - v_{i-1}}{h^2}\right\} \\ &= (1 - \lambda)\left\{\frac{v_i}{h^2} + \frac{v_i - v_{i-1}}{h^2}\right\}. \end{aligned} \tag{2.11}$$

If we assume that the vector $\bar{v}$ is a discrete approximation to a smooth function v, then Taylor expansions yield

$$\frac{2v_i - (v_{i-1} + v_{i+1})}{h^2} = -(v_{xx})_i + \mathrm{O}(h^2)$$

and

$$\frac{v_i}{h^2} + \frac{(v_i - v_{i-1})}{h^2} = \frac{v_i}{h^2} + \frac{(v_x)_i}{h} + \mathrm{O}(1).$$

Consequently, if we ignore boundary conditions, (2.10) becomes

$$-v_{xx} + \mathrm{O}(h^2) = \frac{(1-\lambda)}{h^2}\{v + v_x h + \mathrm{O}(h^2)\}. \tag{2.12}$$

Now, for small h, $v_x h$ can be neglected as a lower order term, so that (2.12) is further approximated by the differential eigenvalue problem

$$-v_{xx} = \gamma v, \qquad \gamma \simeq (1-\lambda)/h^2. \tag{2.13}$$

That is, $\lambda \simeq 1 - \mathrm{O}(h^2)$. If (2.6) is augmented with zero Dirichlet boundary conditions, it is known, see [2], that for $w = 1$, the largest eigenvalue of (2.9) is precisely

$$(\cos \pi h)^2 = 1 - \pi^2 h^2 + \mathrm{O}(h^4) \tag{2.14}$$

so that the above technique does, in fact, yield the correct $\mathrm{O}(h^\alpha)$ eigenvalues estimates of (2.9).

Other well known eigenvalue and convergence rate estimates can be obtained for various block- and point-Jacobi and Gauss–Siedel methods when applied to the Laplacian $\Delta u = f$. In the following sections, we apply this technique to the biharmonic equation $\Delta^2 u = 0$ and obtain $\mathrm{O}(h^\alpha)$ eigenvalue estimates to a variety of classical splittings.

3. One-dimensional biharmonic equation

Consider the numerical solution of the one-dimensional biharmonic equation

$$\Delta^2 u = u_{xxxx} = g(x), \quad 0 < x < 1. \tag{3.1}$$

Using a standard central difference formula on a uniform mesh with mesh length h yields

$$h^4 \Delta^2 u_i = 6u_i - 4(u_{i-1} + u_{i+1}) + (u_{i-2} + u_{i+2}) + \mathrm{O}(h^2) \tag{3.2}$$

(here, the notation $u_\alpha = u(\alpha h)$ is used). If (3.1) is augmented by appropriate boundary conditions then a symmetric, positive-definite matrix equation

$$A\bar{u} = \bar{g} \tag{3.3}$$

arises where

$$A = \frac{1}{h^4}\begin{bmatrix} \ddots & & & & & & \\ & \ddots & & & & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & 1 & -4 & 6 & -4 & 1 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & & \ddots & \\ & & & & & & \ddots \end{bmatrix}. \tag{3.4}$$

For different splittings $A = M - N$, we will be obtaining eigenvalue estimates for the matrix eigenvalue problem

$$N\bar{v} = \lambda M\bar{v}. \tag{3.5}$$

For analysis purposes, we will use the more convenient equivalent form of (3.5) given by

$$A\bar{v} = \gamma M\bar{v}, \qquad \gamma = 1 - \lambda. \tag{3.6}$$

3.1. *Point–Jacobi method*

We first consider the point Jacobi method where

$$M = \frac{1}{h^4}\begin{bmatrix} \ddots & & \\ & 6 & \\ & & \ddots \end{bmatrix}. \tag{3.7}$$

Then, ignoring boundary conditions, (3.6) can be considered a consistent difference approximation to the differential eigenproblem

$$\Delta^2 v + O(h^2) = \frac{(1-\lambda)}{h^4} v. \tag{3.8}$$

Consequently, for small h, we obtain the approximate differential eigenproblem

$$\Delta^2 v = \gamma v \tag{3.9}$$

where

$$\gamma \simeq 6(1-\lambda)/h^4. \tag{3.10}$$

If we assume the boundary conditions

$$v(0) = v(1) = v_{xx}(0) = v_{xx}(1) = 0, \tag{3.11}$$

then the eigenvalues of (3.9) are

$$\gamma = (i\pi)^4, \quad i = 1, 2, \ldots, \tag{3.12}$$

so that

$$\lambda \simeq 1 - \tfrac{1}{6}(i\pi h)^4, \quad i = 1, 2, \ldots . \tag{3.13}$$

Table 1

n	$\lambda_{\max}$	$\lambda = 1 - \frac{1}{4}(\pi h)^4$	Rel. error
10	0.998900836	0.998891138	9.7×10^{-6}
20	0.999916774	0.999916522	5.5×10^{-7}
30	0.999982446	0.999982421	2.6×10^{-8}
40	0.999994259	0.999994254	5.0×10^{-9}

We now test the validity of the estimate (3.13). To do so, we assume the boundary conditions (3.11) on equation (3.1) so that the matrix A in (3.3) takes the form

$$A = \frac{1}{h^4}\begin{bmatrix} 5 & -4 & 1 & & & \\ -4 & 6 & \ddots & \ddots & & \\ 1 & \ddots & \ddots & & & 1 \\ & \ddots & & & 6 & -4 \\ & & & 1 & -4 & 5 \end{bmatrix}. \tag{3.14}$$

Letting $h = 1/n + 1$, the eigenvalues of (3.5) are computed via EISPAC for different values of n. Table 1 records the results where $\lambda_{\max} = \max\{\text{eigenvalue}(M^{-1}N)\}$ and the relative error is given by $|\lambda_{\max} - \lambda|/\lambda_{\max}$. We see that, for small h, the estimate (3.13) does, in fact, hold true.

3.2. Point-Gauss–Siedel method

We now consider the point Gauss–Siedel method and let

$$D = \frac{1}{h^4}\begin{bmatrix} \ddots & & \\ & 6 & \\ & & \ddots \end{bmatrix}. \tag{3.15}$$

Then,

$$M = D - L = \frac{1}{h^4}\begin{bmatrix} \ddots & & & & \\ & \ddots & & & \\ \ddots & \ddots & \ddots & & \\ & 1 & -4 & 6 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}, \tag{3.16}$$

so that, for sufficiently smooth v,

$$\begin{aligned}[] [Mv]_i &= [(D-L)v]_i \\ &= \frac{3v_i}{h^4} + \frac{3v_i - 4v_{i-1} + v_{i-2}}{h^4} = \frac{3v_i}{h^4} + \frac{2}{h^3}(v_x)_i + \mathrm{O}(h^{-2}). \end{aligned}$$

Ignoring boundary conditions, (3.6) is a consistent difference approximation to the differential eigenvalue problem

$$\Delta^2 v + \mathrm{O}(h^2) = \frac{(1-\lambda)}{h^4}\left[3v + 2hv_x + \mathrm{O}(h^2)\right].$$

Table 2

n	$\rho(M^{-1}N)$	$\lambda = 1 - \frac{1}{3}(\pi h)^4$	Rel. error
10	0.99779900	0.99778227	1.7×10^{-5}
20	0.99983350	0.99983304	4.6×10^{-7}
30	0.99996488	0.99996484	4.0×10^{-8}
40	0.99998851	0.99998851	0

Consequently, for small h, we obtain the approximate differential eigenvalue problem to obtain

$$\Delta^2 v = \gamma v \tag{3.17}$$

where $\gamma \simeq (1-\lambda)/h^4$. The eigenvalues of (3.17) are given by (3.12) so that

$$\lambda \cong 1 - \tfrac{1}{3}(i\pi h)^4, \quad i = 1, 2, \ldots, \tag{3.18}$$

i.e.,

$$\rho(M^{-1}N) = 1 - \mathrm{O}(h^4). \tag{3.19}$$

To test the validity of the eigenvalue estimate given by (3.18) we compute the spectral radius of (3.5) via EISPAC for different values of $h = 1/n+1$ using (3.15) and the relative error is given by $|\rho(M^{-1}N) - \lambda|/\rho(M^{-1}N)$. Table 2 lists the results where we see, that for small h, the estimate (3.18) does, in fact, hold true.

3.3. SOR-improvement

We now attempt to determine how much improvement in the convergence rate of the previous Gauss–Siedel method can be obtained by SOR. In this case,

$$M = (1/w)D - L \tag{3.20}$$

where D and L are as in (3.15) and (3.16). We now follow the idea put forth in Garabedian, [1], and set

$$w = 2/(1+ch), \quad 0 < c \leqslant h^{-1}. \tag{3.21}$$

Then (3.6) becomes

$$A\bar{v} = (1-\lambda)\{(\tfrac{1}{2}D - L) + \tfrac{1}{2}chD\}\bar{v}. \tag{3.22}$$

Now,

$$\tfrac{1}{2}D - L = \frac{1}{h^4}\begin{bmatrix} \ddots & & & & \\ & \ddots & & & \\ \ddots & \ddots & \ddots & & \\ & 1 & -4 & 3 & \\ & \ddots & \ddots & \ddots & \end{bmatrix},$$

so that for sufficiently smooth v,

$$[(\tfrac{1}{2}D - L)v]_i = (2v_x)_i/h^3 + \mathrm{O}(h^{-2}).$$

Consequently, we have that (3.6) is a discrete approximation to the differential eigenvalue problem

$$\Delta^2 v + \mathrm{O}(h^2) = \frac{1-\lambda}{h^3}[2v_x + 3cv + \mathrm{O}(h)]. \tag{3.23}$$

Hence, for small values of h and for $c \ll h^{-1}$, (3.23) is approximated by the differential eigenvalue problem

$$\Delta^2 v = \gamma[2v_x + 3cv],$$
$$\gamma \cong (1-\lambda)/h^3. \tag{3.24}$$

Table 3

c	$\rho(\mathscr{L}_w)$
10	0.9999528
2	0.9999676
0.92	0.9990460

That is,

$$\lambda \simeq 1 - \mathrm{O}(h^3). \tag{3.25}$$

hence, if w_b is such that

$$\rho(\mathscr{L}w_b) \leqslant \rho(\mathscr{L}_w), \quad 1 < w < 2,$$

then (3.25) implies that

$$\rho(\mathscr{L}w_b) \leqslant 1 - \mathrm{O}(h^3).$$

To verify this, we let $h = 1/41$ and vary the constant c until $w_b = 2/(1 + c_b h)$ is achieved. Table 3 lists the computational results when (3.14) is used. Hence, we see that as in the case for Laplace's equation, overrelaxation of the Gauss-Siedel method reduces the power of h in the eigenvalue estimates by 1.

3.4. $\mathrm{O}(h^2)$-estimates

We now ask the question as to whether it is possible to define a symmetric positive definite matrix D so that the eigenvalue estimates of $(M^{-1}N)$ where

$$A = D - L - L^{\mathrm{t}}, \qquad M = D - L, \qquad N = L^{\mathrm{t}},$$

are of the form $1 - \mathrm{O}(h^2)$.

To do so, we begin by letting

$$D = \frac{1}{h^4}\begin{bmatrix} \ddots & & & & \\ \ddots & \ddots & \ddots & & \\ & \beta_{i-1} & \alpha_i & \beta_i & \\ & & \ddots & \ddots & \ddots \\ & & & & \ddots \end{bmatrix}.$$

Then,

$$-L = \frac{1}{h^4}\begin{bmatrix} \ddots & & & & \\ & \ddots & & & \\ & & \ddots & & \\ \ddots & & \ddots & \ddots & \\ & 1 & -(4+\beta_{i-1}) & \frac{1}{2}(6-\alpha_i) & \\ & & & \ddots & \ddots \end{bmatrix}.$$

Hence, for a sufficiently smooth function v,

$$h^4[(\tfrac{1}{2}D - L)v]_i = 3v_i + \tfrac{1}{2}\beta_i v_{i+1} + (-4 - \tfrac{1}{2}\beta_{i-1})v_{i-1} + v_{i-2}. \tag{3.26}$$

Using the Taylor expansions

$$v_{i+1} = v_i + (v_x)_i h + \tfrac{1}{2}(v_{xx})_i h^2 + \tfrac{1}{6}(v_{xxx})_i h^3 + O(h^4),$$
$$v_{i-1} = v_i - (v_x)_i h + \tfrac{1}{2}(v_{xx})_i h^2 - \tfrac{1}{6}(v_{xxx})_i h^3 + O(h^4),$$
$$v_{i-2} = v_i - 2(v_x)_i h + 2(v_{xx})_i h^2 - \tfrac{4}{3}(v_{xxx})_i h^3 + O(h^4),$$

we get from (3.26) that

$$h^4[(\tfrac{1}{2}D - L)v]_i = \tfrac{1}{2}(\beta_i - \beta_{i-1})v_i + \tfrac{1}{2}(\beta_i + \beta_{i-1} + 4)(v_x)_i h + \tfrac{1}{4}(\beta_i - \beta_{i-1})(v_{xx})_i h^2 + \tfrac{1}{12}(\beta_i + \beta_{i-1} - 8)(v_{xxx})_i h^3 + O(h^4). \tag{3.27}$$

Consequently, if we assume that

$$\beta_i = -2 \quad \text{all } i,$$

then (3.27) simplifies to

$$h^4[(\tfrac{1}{2}D - L)v]_i = -(v_{xxx})_i h^3 + O(h^4). \tag{3.28}$$

In order that D be positive-definite, we assume

$$\alpha_i \geqslant 4, \quad \text{all } i.$$

Then

$$h^4[Dv]_i = \alpha_i v_i - 2(v_{i-1} + v_{i+1}) = (\alpha_i - 4)v_i - 2(v_{xx})_i h^2 + O(h^4)$$

and,

$$[(D-L)v]_i = [(\tfrac{1}{2}D - L)v + \tfrac{1}{2}Dv]_i = -(v_{xxx})_i h^{-1} + \tfrac{1}{2}(\alpha_i - 4)v_i h^{-4} - (v_{xx})_i h^{-2} + O(1).$$

Ignoring boundary conditions and assuming $\alpha_i = \alpha =$ constant, (3.6) is a consistent difference approximation to the eigenvalue problem

$$\Delta^2 v + O(h^2) = (1-\lambda)[-v_{xxx}h^{-1} + \tfrac{1}{2}(\alpha - 4)vh^{-4} - v_{xx}h^{-2} + O(1)]. \tag{3.29}$$

Thus for small h and $\alpha > 4$, (3.29) is approximated by the differential eigenproblem

$$\Delta^2 v = \gamma_\alpha v, \qquad \gamma_\alpha \simeq \tfrac{1}{2}(\alpha - 4)(1 - \lambda_\alpha)/h^4.$$

That is,

$$\lambda_\alpha \simeq 1 - \left(\frac{2}{\alpha - 4}\right)(i\pi h)^4, \quad i = 1, 2, \ldots. \tag{3.30}$$

However, for $\alpha = 4$, we have for small h, the eigenproblem

$$\Delta^2 v = \gamma_4 v_{xx}, \qquad \gamma_4 \simeq -(1 - \lambda_4)/h^2.$$

That is,

$$\lambda_4 \simeq 1 - (i\pi h)^2, \quad i = 1, 2, \ldots. \tag{3.31}$$

Table 4

n	$\rho(M^{-1}N)$	$\lambda = 1 - (\pi h)^4$	rel. error
10	0.99386454	0.99334682	5.2×10^{-4}
20	0.99961107	0.99949913	1.2×10^{-5}
30	0.99989571	0.99989452	1.2×10^{-6}
40	0.99996575	0.99996552	2.3×10^{-7}

To corroborate the estimate (3.30), we take

$$D = \frac{1}{h^4}\begin{bmatrix} 5 & -2 & & & \\ -2 & 6 & -2 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 6 & -2 \\ & & & -2 & 5 \end{bmatrix} \tag{3.32}$$

and calculate the spectral radius of $M^{-1}N$ where $M = D - L$, $A = D - L - L^t$, and A is given by (3.14). Table 4 records the results for various values of $h = 1/n + 1$.

A similar calculation is done to corroborate the estimate (3.31) where in this case

$$D = \frac{1}{h^4}\begin{bmatrix} 4 & -2 & & \\ -2 & \ddots & \ddots & \\ & \ddots & & -2 \\ & & -2 & 4 \end{bmatrix}. \tag{3.33}$$

Table 5 records the results.

3.5. SOR-improvement

We now show that SOR of the previous iteration achieves a reduction in the powers of h in the eigenvalue estimates. As before, let

$$D_\alpha = \frac{1}{h^4}\begin{bmatrix} \ddots & & & & \\ \ddots & \ddots & \ddots & & \\ & -2 & \alpha & -2 & \\ & & \ddots & \ddots & \ddots \\ & & & & \ddots \end{bmatrix}, \quad \alpha \geqslant 4,$$

and

$$A = D_\alpha - L_\alpha - L_\alpha^t$$
$$M_\alpha = \frac{1}{w} D_\alpha - L_\alpha, \quad 1 \leqslant w < 2.$$

Table 5

n	$\rho(M^{-1}N)$	$\lambda = 1 - (\pi h)^2$	Rel. error
10	0.(1956846	0.91843303	1.2×10^{-3}
20	0.97769651	0.97761994	7.8×10^{-5}
30	0.98974526	0.98977298	1.7×10^{-5}
40	0.99413364	0.99412870	4.9×10^{-6}

Table 6

c	$n=10$	30	50	70
100	0.9197	0.9683	0.9804	0.9876

We now deviate somewhat from the idea of Garabedian by letting

$$w = 2/(1 + ch^{\mu}) \tag{3.34}$$

where $\mu \geqslant 1$ and $0 < c \leqslant h^{-\mu}$. In this case,

$$\frac{1}{w} D_\alpha - L_\alpha = \left(\tfrac{1}{2} D_\alpha - L_\alpha\right) + \tfrac{1}{2} c h^{\mu} D_\alpha.$$

From (3.27) and (3.28), we have

$$\left[(1/wD_\alpha - L_\alpha) v\right]_i = -(v_{xxx})_i h^{-1} + \tfrac{1}{2}(\alpha - 4) c v_i h^{\mu-4} - c(v_{xx})_i h^{\mu-2} + \mathrm{O}(1),$$

so that (3.6) is a consistent difference approximation to

$$\Delta^2 v + \mathrm{O}(h^2) = (1 - \lambda)\left[-v_{xxx} h^{-1} + \tfrac{1}{2}(\alpha - 4) c v h^{\mu-4} - c v_{xx} h^{\mu-2} + \mathrm{O}(1)\right]. \tag{3.35}$$

We first consider the case $\alpha > 4$. Since we are trying to determine the existence of an w given by (3.34) that will minimize the power of h in the eigenvalue estimates, we see that a value of $\mu = 3$ in (3.34) will achieve this. Then for small h, (3.35) is approximated by the eigenproblem

$$\Delta^2 v = \gamma_\alpha\left[-v_{xxx} + \tfrac{1}{2}(\alpha - 4) c v\right], \qquad \gamma_\alpha \approx (1 - \lambda_\alpha)/h. \tag{3.36}$$

Consequently, for $\alpha > 4$ and $c \ll h^{-3}$,

$$\lambda_\alpha = 1 - \mathrm{O}(h). \tag{3.37}$$

Hence, for $\alpha > 4$, SOR reduces the power of h from 4 to 1. In order to verify this, we take D as in (3.32), $c = 100$, and determine w_b. Table 6 records the results where we see that (3.37) is, in fact, achieved.

In order to obtain a similar result for $\alpha = 4$, we let $\mu = 1$ in (3.35). Then, for small h, (3.35) is approximated by

$$\Delta^2 v = \gamma_4\left[-v_{xxx} - c v_{xx}\right], \qquad \gamma_4 \simeq (1 - \lambda_4)/h.$$

That is,

$$\lambda_4 \simeq 1 - \mathrm{O}(h).$$

Taking $h = 1/21$ we determine the $w_b(\alpha)$ for several values of $\alpha \geqslant 4$. In this situation

$$D = \frac{1}{h^4}\begin{bmatrix} \alpha & -2 & & \\ -2 & \ddots & \ddots & \\ & \ddots & \ddots & -2 \\ & & -2 & \alpha \end{bmatrix}.$$

Table 7 records the results where we see that $\alpha = 4$ is, in a sense, optimal.

Table 7

	$\alpha=10$	6	4.5	4.2	4	3.99
$w_b(\alpha)$	1.996	1.989	1.963	1.724	1.657	1.584
$\rho[\mathscr{L}_{w_b}(\alpha)]$	0.9039	0.9007	0.8899	0.8756	0.8234	0.8779

4. Two-dimensional biharmonic equation

We now consider the numerical solution of the two-dimensional biharmonic equation

$$\Delta^2 u = u_{xxxx} + 2u_{xxyy} + u_{yyyy} = f(x, y), \quad 0 \leqslant x, \ y \leqslant 1. \tag{4.1}$$

Using a standard central difference formula on a uniform mesh with mesh length $h = 1/n + 1$ yields

$$\begin{aligned}(\Delta^2 u)_{i,j} h^4 = 20u_{i,j} &- 8(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}) \\ &+ 2(u_{i-1,j} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}) \\ &+ (u_{i-2,j} + u_{i+2,j} + u_{i,j-2} + u_{i,j+2}) + \mathrm{O}(h^2).\end{aligned} \tag{4.2}$$

If (4.2) is augmented by appropriate boundary conditions, then a symmetric, positive-definite matrix equation

$$A\bar{u} = \bar{g}$$

where

$$A = \frac{1}{h^4}\begin{bmatrix} \ddots & & & & & & \\ & \ddots & \ddots & \ddots & \ddots & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & D & C & B & C & D & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & & \ddots & \\ & & & & & & \ddots \end{bmatrix}$$

arises where

$$B = \begin{bmatrix} \ddots & & & & & & \\ & \ddots & & & & & \\ \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & 1 & -8 & 20 & -8 & 1 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & & \ddots & \\ & & & & & & \ddots \end{bmatrix},$$

$$C = \begin{bmatrix} \ddots & & & & \\ \ddots & \ddots & \ddots & & \\ & 2 & -8 & 2 & \\ & & \ddots & \ddots & \ddots \\ & & & & \ddots \end{bmatrix}, \qquad D = \begin{bmatrix} \ddots & & \\ & 1 & \\ & & \ddots \end{bmatrix}.$$

We consider the splitting $A = D - L - L^{-t}$ where D is a symmetric positive-definite block tridiagonal matrix and analyze the matrix eigenvalue problem

$$A\bar{v} = (1-\lambda)\{D - L\}\bar{v}. \tag{4.3}$$

Let D be defined as

$$\begin{aligned}[][Du]_{ij} &= \alpha_{ij}v_{ij} + \beta_{i-1,j}v_{i-1,j} + \beta_{i+1,j}v_{i+1,j}\\ &\quad + \gamma_{i,j-1}v_{i,j-1} + \gamma_{i,j+1}v_{i,j+1}.\end{aligned} \tag{4.4}$$

Then

$$\begin{aligned}[][(\tfrac{1}{2}D - L)v]_{ij} &= 10v_{ij} + (\tfrac{1}{2}\beta_{i+1,j})v_{i+1,j} + (-8 - \tfrac{1}{2}\beta_{i-1,j})v_{i-1,j}\\ &\quad + (\tfrac{1}{2}\gamma_{i,j+1})v_{i,j+1} + (-8 - \tfrac{1}{2}\gamma_{i,j-1})v_{i,j-1}\\ &\quad + 2(v_{i-1,j-1} + v_{i+1,j-1}) + (v_{i-2,j} + v_{i,j-2}).\end{aligned}$$

Taylor expansions yield

$$\begin{aligned}[][(\tfrac{1}{2}D - L)v]_{ij} &= a_{ij}v_{ij} + b_{ij}(v_x)_{ij}h + c_{ij}(v_y)_{ij}h\\ &\quad + d_{ij}(u_{xx})_{ij}h^2 + e_{ij}(v_{xy})_{ij}h^2 + f_{ij}(v_{yy})_{ij}h^2\\ &\quad + g_{ij}(v_{xxx})_{ij}h^3 + p_{ij}(v_{xxy})_{ij}h^3 + q_{ij}(v_{yyx})_{ij}h^3 + r_{ij}(v_{yyy})h^3 + O(h^4)\end{aligned}$$

where

$$\begin{aligned} a_{ij} &= \tfrac{1}{2}(\beta_{i+1,j} - \beta_{i-1,j}) + \tfrac{1}{2}(\gamma_{i,j+1} - \gamma_{i,j-1}),\\ b_{ij} &= \tfrac{1}{2}(\beta_{i+1,j} + \beta_{i-1,j}) + 6, \qquad c_{ij} = \tfrac{1}{2}(\gamma_{i,j+1} + \gamma_{i,j-1}) + 2,\\ d_{ij} &= \tfrac{1}{4}(\beta_{i+1,j} - \beta_{i-1,j}), \qquad e_{ij} = 0, \qquad f_{ij} = \tfrac{1}{4}(\gamma_{i,j+1} - \gamma_{i,j-1}).\end{aligned}$$

We now construct the coefficients of the matrix D so that $a_{ij} = b_{ij} = c_{ij} = d_{ij} = e_{ij} = f_{ij} = 0$. This will happen when

$$\beta_{i+1,j} = \beta_{i-1,j} = -6 \quad \text{and} \quad \gamma_{i,j+1} = \gamma_{i,j-1} = -2.$$

It then follows that

$$g_{ij} = -1, \qquad p_{ij} = -2, \qquad q_{ij} = 0, \qquad r_{ij} = -1,$$

so that

$$h^4[(\tfrac{1}{2}D - L)v]_{ij} = -h^3[(v_{xxx})_{ij} + 2(v_{xxy})_{ij} + (v_{yyy})_{ij}] + \tfrac{1}{2}h^4(v_{xxxx})_{ij} + O(h^4) \tag{4.5}$$

and

$$h^4[Dv]_{ij} = (\alpha_{ij} - 16)v_{ij} - [6(v_{xx}) + 2(v_{yy})]_{ij}h^2 + O(h^4). \tag{4.6}$$

Combining (4.4) and (4.5), we see that (4.3) becomes (ignoring boundary conditions and assuming $\alpha_{ij} > \alpha = \text{constant}$)

$$\begin{aligned}\Delta^2 v + O(h^2) = (1-\lambda)\{&[v_{xxx} + 2v_{xxy} + v_{yyy}]h^{-1}\\ &+ \tfrac{1}{2}(\alpha - 16)vh^{-4} - \tfrac{1}{2}[6v_{xx} + 2v_{yy}]h^{-2} + O(1)\}.\end{aligned} \tag{4.7}$$

For $\alpha > 16$, we have for small h that (4.7) is approximated by the eigenproblem

$$\Delta^2 v = \gamma_\alpha v, \qquad \gamma_\alpha \cong \tfrac{1}{2}(\alpha - 16)(1 - \lambda_\alpha)/h^4. \tag{4.8}$$

If the boundary conditions

$$v(x, y) = \Delta v(x, y) = 0, \qquad (x, y) \in \text{Boundary}([0, 1] \times [0, 1]) \tag{4.9}$$

are used on (4.1) then

$$\lambda_\alpha \simeq 1 - (8/(\alpha - 16))(i\pi h)^4, \quad i = 1, 2, \ldots. \tag{4.10}$$

In the case $\alpha = 16$, (4.7) becomes for small h

$$\Delta^2 v = \gamma_{16}(-6v_{xx} - 2v_{yy}), \qquad \gamma_{16} \simeq (1 - \lambda_{16})/2h^2.$$

That is,

$$\lambda_{16} \cong 1 - (i\pi h)^2, \quad i = 1, 2, \ldots. \tag{4.11}$$

To corrobarate (4.10) and (4.11), we impose the boundary conditions (4.9) on (4.1). In this case, the matrix A takes the form

$$A = \frac{1}{h^4}\begin{bmatrix} \tilde{B}_1 & C_1 & D_1 & & \\ C_1 & B_2 & C_2 & \ddots & \\ D_1 & \ddots & \ddots & \ddots & D_{n-2} \\ & \ddots & & B_{n-1} & C_{n-1} \\ & & D_{n-2} & C_{n-1} & \tilde{B}_n \end{bmatrix}$$

$$\tilde{B}_1 = \tilde{B}_n = \begin{bmatrix} 18 & -8 & 1 & & & \\ -8 & 19 & -8 & 1 & & \\ 1 & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & & & & 1 \\ & & & & 19 & -8 \\ & & & 1 & -8 & 18 \end{bmatrix}$$

$$B_i = \begin{bmatrix} 19 & -8 & 1 & & & \\ -8 & 20 & -8 & 1 & & \\ 1 & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & & & & 1 \\ & & & & 20 & -8 \\ & & & 1 & -8 & 19 \end{bmatrix}, \quad 2 \leqslant i \leqslant n - 1,$$

$$C_i = \begin{bmatrix} -8 & 2 & & \\ 2 & \ddots & \ddots & \\ & \ddots & & 2 \\ & & 2 & -8 \end{bmatrix}, \quad 1 \leqslant i \leqslant n - 1,$$

$$D_i = I, \quad 1 \leqslant i \leqslant n - 2$$

To verify (4.10) we define

$$D = \frac{1}{h^4}\begin{bmatrix} D_1 & -2I & & \\ -2I & D_2 & \ddots & \\ & \ddots & \ddots & -2I \\ & & -2I & D_n \end{bmatrix}$$

$$D_1 = D_n = \begin{bmatrix} 18 & 6 & & & \\ -6 & 19 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 19 & -6 \\ & & & -6 & 18 \end{bmatrix}$$

$$D_i = \begin{bmatrix} 19 & -6 & & & \\ -6 & 20 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 20 & -6 \\ & & & -6 & 19 \end{bmatrix}, \quad 2 \leqslant i \leqslant n-1.$$

Again the spectral radii of $M^{-1}N = (D-L)^{-1}L^t$ for different values of $h = 1/n+1$ are computed with EISPAC and compared with (4.10). As before the relative errors are also computed. Table 8 records the results.

To verify (4.11), we define the off-diagonals of D as above and take the main diagonal of D to be the constant value 16. The spectral radii of $M^{-1}N = (D-L)^{-1}L^t$ for different values of $h = 1/n+1$ are computed via EISPAC and compared with (4.10). The relative errors are also computed. Table 9 records the results.

We now analyze the effect of SOR acceleration on (4.3). That is, we consider the matrix eigenvalue problem

$$Av = (1-\lambda)[(1/w)D - L]v$$

Table 8

n	$\rho(M^{-1}N)$	$\lambda = 1-2(\pi h)^4$	Rel. error
10	0.988623	0.986693	1.9×10^{-3}
20	0.999043	0.998998	4.5×10^{-5}

Table 9

n	$\rho(M^{-1}N)$	$\lambda = 1-(\pi h)^2$	Rel. error
10	0.920182	0.918433	1.9×10^{-3}
20	0.977748	0.977619	1.3×10^{-4}

and, as before, we assume

$$w = 2/(1 + ch^{\mu})$$

where $\mu \geqslant 1$ and $0 < c \leqslant h^{-\mu}$. Then

$$[(1/w)D - L] = [(\tfrac{1}{2}D - L) + \tfrac{1}{2}ch^{\mu}D].$$

Using (4.4) and (4.5) we get the differential problem

$$\Delta^2 v + O(h^2) = (1-\lambda)\{[v_{xxx} + 2v_{xxy} + v_{yyy}]h^{-1} + \tfrac{1}{2}(\alpha - 16)cvh^{\mu-4} - c(6v_{xx} + 2v_{yy})h^{\mu-2} + O(h^{\mu})\} \quad (4.12)$$

For $\alpha > 16$, we set $\mu = 3$ so that for small h, (4.11) is approximated by

$$\Delta^2 v = \gamma_{\alpha}\{v_{xxx} + 2v_{xxy} + v_{yyy} + \tfrac{1}{2}(\alpha - 16)cv\},$$

$$\gamma_{\alpha} \simeq (1 - \lambda_{\alpha})/h.$$

That is,

$$\lambda_{\alpha} \cong 1 - O(h).$$

For $\alpha = 16$, we take $\mu = 1$ so that for small h, (4.12) is approximated by

$$\Delta^2 v = \gamma_{16}\{v_{xxx} + 2v_{xxy} + v_{yyy} - c(6v_{xx} + 2v_{yy})\},$$

$$\gamma_{16} \cong (1 - \lambda_{16})/h,$$

so that

$$\lambda_{16} \cong 1 - O(h).$$

That is, as in the one-dimensional case, SOR reduces the power of h from either 4 (in the case $\alpha > 16$) or from 2 (in the case $\alpha = 16$) to 1.

5. Summary

In this paper we developed a technique for estimating the spectral radius of iteration matrices associated with the biharmonic equation. We have shown that the eigenvalue estimates are precise in many of the classical iteration schemes and can give informations on how much successive overrelaxation can improve the convergence rate. This technique also allows us to develop new iterative schemes where the convergence rate is considerably faster than the classical point-iteration method.

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