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Integral Transforms and the Laguerre–Pólya Class

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1. INTRODUCTION

The problem of characterizing functions $K(t)$ such that the Fourier transform of $K(t)$,

$$F(z) := F(z; K) = \int_{-\infty}^{\infty} K(t) e^{izt}\, dt, \tag{1.1}$$

is an entire function with only real zeros, is a long-standing open problem. The interest in this question stems, in part, from the well-known fact that the Riemann Hypothesis is equivalent to the assertion that the Fourier transform of $\Phi(t)$, where

$$\Phi(t) := \sum_{n=1}^{\infty} (2n^4\pi^2 e^{9t} - 3n^2\pi e^{5t}) \exp(-n^2\pi e^{4t}) \qquad (0 \leqslant t < \infty),$$

has only real zeros (cf. [24] or [8]). Since today there are only a few general methods known for studying the distribution of zeros of (1.1) (cf. Theorem 2.2 and Theorem 2.3), we propose here to reexamine this question in light of our recent results ([8], [9]). Our goal here is to investigate the above problem by studying the interplay between properties of $K(t)$, its Fourier transform, the Laguerre–Pólya class, multiplier sequences, the moments of $K(t)$, universal factors, Turán-type inequalities, and k-times positive sequences.

In Section 2, we review the algebraic and transcendental characterizations of functions in the Laguerre–Pólya class, and we illustrate by examples some of the results pertaining to this class.

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In Section 3, we focus our attention on the relationship between properties of the moments of $K(t)$, multiplier sequences, totally positive sequences and k-times positive sequences. In the course of the supplementary discussions and new observations, we state several open problems. One of the new results in this section asserts that if $\log(K(\sqrt{t}))$ is strictly concave for $t>0$, where $K(t)$ is subject to certain regularity conditions, then the normalized moments of $K(t)$ satisfy the Turán inequalities (Theorem 3.12). This result, in conjunction with Pólya's characterization of universal factors, enables us to establish a large class of new necessary conditions for (1.1) to have only real zeros (Theorem 3.15).

In a subsequent paper, we will study the Turán inequalities for the Jensen polynomials to establish necessary *and* sufficient conditions for (1.1) to have only real zeros.

2. FOURIER TRANSFORMS AND FUNCTIONS IN THE LAGUERRE–PÓLYA CLASS

In this section, we first recall some results concerning kernels, $K(t)$, whose Fourier transforms are entire functions having only real zeros. Second, we review some of those properties of real entire functions having only real zeros, which will be used in the sequel.

Throughout this section, we will assume that the kernel $K(t)$, with $K: \mathbb{R} \to \mathbb{C}$, satisfies the following properties:

$$\left.\begin{array}{ll}\text{(i)} & K \text{ and } |K| \text{ are integrable over } \mathbb{R}, \\ \text{(ii)} & K(t) = \overline{K(-t)} \qquad (t \in \mathbb{R}), \qquad \text{and} \\ \text{(iii)} & K(t) = O(\exp(-|t|^{2+\alpha})), \alpha > 0, \text{as } t \to \infty.\end{array}\right\} \tag{2.1}$$

Moreover, we will denote the Fourier transform of $K(t)$ by

$$F(z) := F(z; K) := \int_{-\infty}^{\infty} K(t) e^{izt}\, dt. \tag{2.2}$$

To begin, we list some known properties of $F(z)$ defined by (2.2).

PROPOSITION 2.1 ([23, p. 9]) *With* (2.1) *and* (2.2), *the function* $F(z)$ *is a real entire function of order* $\rho = \rho(F)$, *where*

$$\rho < \frac{\alpha+2}{\alpha+1} < 2 \qquad (\alpha > 0). \tag{2.3}$$

There are two remarkable results, one due to Pólya [23], the other due to de Bruijn [3] (see also Ilieff [12]) which provide sufficient conditions on $K(t)$, such that its Fourier transform has only real zeros.

THEOREM 2.2 ([23, p. 7]) *Suppose the function* $K_1: \mathbb{R} \to \mathbb{R}$, *is real analytic on an interval about the origin; that is,*

$$K_1(t) := \sum_{k=0}^{\infty} c_k t^k, \qquad (t \in (-r, r),\ r > 0,\ c_k \in \mathbb{R},\ k = 0, 1, 2, \ldots), \tag{2.4}$$

and $K_1(t)$ *satisfies* (2.1(i)) *and* (2.1(iii)). *Then the integral*

$$H(z) := \int_0^{\infty} t^{z-1} K_1(t)\, dt \qquad (z \in \mathbb{C}), \tag{2.5}$$

represents a meromorphic function. If $H(z)$ *has only real negative zeros, then the entire function*

$$F_q(z) := \int_{-\infty}^{\infty} K_1(t^{2q}) e^{izt}\, dt \qquad (q = 1, 2, 3, \ldots), \tag{2.6}$$

has only real zeros.

THEOREM 2.3 ([3, Theorem 1]) *Let* $h(t)$ *be an entire function such that its derivative is the uniform limit, on compact subsets of* $\mathbb{C}$, *of a sequence of polynomials, all of whose zeros lie on the imaginary axis. If* $h(t)$ *is non-constant with* $h(t) = h(-t)$, *and if* $h(t) \geq 0$ $(t \in \mathbb{R})$, *then the entire function*

$$F_2(z) := \int_{-\infty}^{\infty} K_2(t) e^{izt}\, dt, \tag{2.7}$$

where $K_2(t) := \exp(-h(t))$, *has only real zeros.*

We illustrate the foregoing results by means of the following examples.

Example 2.1 Set

$$F(z; p) := \int_0^{\infty} \exp(-t^p) \cos(zt)\, dt \qquad (p > 1). \tag{2.8}$$

Then Pólya [22] has shown that

$$F(z; p) = \frac{1}{p} \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma\left(\dfrac{2n+1}{p}\right)}{\Gamma(2n+1)} z^{2n} \qquad (p > 1),$$

where $\Gamma(z)$ denotes the gamma function, is an entire function of order $p/(p-1)$. In particular,

$$F(z; 2) = \left(\frac{\pi}{4}\right)^{1/2} \exp(-z^2/4)$$

has no zeros at all. If $p = 4, 6, 8, \ldots$, then $F(z; p)$ has only real zeros; in fact, infinitely many. If p is *not* an even integer, then $F(z; p)$ has infinitely many *non-real* zeros!

Example 2.2 With $K(t) := \exp(-a \cosh t)$, $a > 0$, set

$$F(z; K) := \int_{-\infty}^{\infty} K(t) e^{izt}\, dt. \tag{2.9}$$

Then, it follows from Theorem 2.3 that the entire function $F(z; K)$ has only real zeros. We remark that in [23] Pólya gives an *ad hoc* proof of the fact that the function defined by (2.9) has only real zeros.

Both the theoretical as well as the historical background for the study of the Laguerre–Pólya class stems from the investigation of linear operators which preserve the reality of the zeros of certain entire functions. We will now proceed to define two classes of these linear operators.

Definition 2.1 Let $\Gamma := \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers. Then for an arbitrary polynomial or transcendental entire function $p(x) := \sum_{k=0}^{\infty} a_k x^k$, we define $\Gamma[p(x)]$ by

$$\Gamma[p(x)] := \sum_{k=0}^{\infty} a_k \gamma_k x^k, \tag{2.10}$$

whenever this series converges.

Definition 2.2 A sequence $\Gamma := \{\gamma_k\}_{k=0}^{\infty}$ of real numbers is called a *multiplier sequence of the first kind*, if Γ takes every real polynomial $p(x)$, with only real zeros, into a polynomial $\Gamma[p(x)]$ (defined by (2.10)) having also only real zeros. A sequence $\Gamma := \{\gamma_k\}_{k=0}^{\infty}$ of real numbers is called a *multiplier sequence of the second kind*, if Γ takes every real polynomial $p(x)$, all of whose zeros are real and of the same sign, into a polynomial, $\Gamma[p(x)]$, all of whose zeros are real.

In a celebrated paper, Pólya and Schur [25] have provided the following algebraic and transcendental characterizations of these sequences.

THEOREM 2.4 ([25]) *Let $\Gamma := \{\gamma_k\}_{k=0}^{\infty}$ be a sequence of real numbers. Then Γ is a multiplier sequence of the first kind if and only if the zeros of the polynomials*

$$g_n(x) := \Gamma[(1+x)^n] = \sum_{k=0}^{n} \binom{n}{k} \gamma_k x^k \qquad (n = 1, 2, 3, \ldots), \tag{2.11}$$

are all real and of the same sign. The sequence Γ is a multiplier sequence of the second kind if and only if the zeros of the polynomials (2.11) *are all real.*

THEOREM 2.5 ([25]) *Let $\Gamma := \{\gamma_k\}_{k=0}^{\infty}$, $\gamma_0 \neq 0$, be a sequence of real numbers. Then, in order that Γ be a multiplier sequence of the first kind, it is necessary and sufficient that the series*

$$f(x) := \Gamma[e^x] = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \tag{2.12}$$

converge in the whole complex plane, and that the entire function $f(x)$ *or* $f(-x)$ *can be represented in the form*

$$f(x) = ce^{\sigma x} \prod_{n=1}^{\infty} (1 + x/x_n), \tag{2.13}$$

where $\sigma \geqslant 0$, $x_n > 0$, $c \in \mathbb{R}$ *with* $\sum_{n=1}^{\infty} x_n^{-1} < \infty$. *In order that* Γ *be a multiplier sequence of the second kind, it is necessary and sufficient that the series*

$$f(x) := \Gamma[e^x] = \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \tag{2.14}$$

converge in the whole complex plane, and that the entire function $f(x)$ *of* (2.14) *can be represented in the form*

$$f(x) = ce^{-\alpha x^2 + \beta x} \prod_{n=1}^{\infty} (1 - x/x_n) e^{x/x_n}, \tag{2.15}$$

where $\alpha \geqslant 0$, $\beta, c \in \mathbb{R}$, *and each* x_n *is real and non-zero with* $\sum_{n=1}^{\infty} x_n^{-2} < \infty$.

Definition 2.3 A real entire function $h(x)$ which can be represented in the form

$$h(x) := x^m f(x), \tag{2.16}$$

where m is a non-negative integer and $f(x)$ is of the form (2.15), is said to be in the *Laguerre–Pólya class*, written $h(x) \in \mathscr{L} - \mathscr{P}$.

Remarks (a) By Theorem 2.5, we see that the two types of multiplier sequences give rise to two types of functions in the Laguerre–Pólya class. If m is a non-negative integer, and if $f(x)$ is of the form (2.13), then Pólya and Schur [25] termed the function $h(x) := x^m f(x)$ as a function of *type* I in the Laguerre–Pólya class, written $h(x) \in \mathscr{L} - \mathscr{P}(\mathrm{I})$. Similarly, functions of the form (2.16) where $f(x)$ is of the form (2.15) were termed, by these authors, as functions of *type* II in the Laguerre–Pólya class, written $h(x) \in \mathscr{L} - \mathscr{P}(\mathrm{II})$. Now it is clear that (i) if $h \in \mathscr{L} - \mathscr{P}(\mathrm{II})$, then $h \in \mathscr{L} - \mathscr{P}$ and (ii) if $h \in \mathscr{L} - \mathscr{P}(\mathrm{I})$, then $h \in \mathscr{L} - \mathscr{P}(\mathrm{II})$.

(b) The significance of the Laguerre–Pólya class in the theory of entire functions (see, for example, [14, Chapter 8] or [17]) is natural, since functions in $\mathscr{L} - \mathscr{P}$ *and only those*, are the uniform limits, on compact subsets of $\mathbb{C}$, of polynomials with only real zeros (cf. [19]).

(c) Consider the sequence $\Gamma := \{k\}_{k=0}^{\infty}$, corresponding to the entire function $f(x) := xe^x \in \mathscr{L} - \mathscr{P}(\mathrm{I})$. Then for any polynomial $p(x)$, we have $\Gamma[p(x)] = xp'(x)$. For this reason, the operators Γ have been studied as generalized forms of differentiation (see, for example, [5], [6], and the references cited therein).

Example 2.3 (a) Let $Q(x)$ be an arbitrary real polynomial with only real negative zeros. Let $\Gamma := \{\gamma_k\}_{k=0}^{\infty}$, where $\gamma_k := Q(k)$ $(k = 0, 1, 2, \ldots)$. Then, it follows from a classical theorem of Laguerre [17, p. 6], that Γ is a multiplier sequence of the first kind. Moreover,

$$\Gamma[e^x] = \sum_{k=0}^{\infty} \frac{Q(k)}{k!} x^k \in \mathscr{L} - \mathscr{P}(\mathrm{I}). \tag{2.17}$$

(b) Set $\Gamma := \{\gamma_k\}_{k=0}^{\infty}$, where $\gamma_k := (-1)^m$ if $k = 2m$ and $\gamma_k := 0$, if $k = 2m+1$ $(k = 0, 1, 2, \ldots)$. Then $\Gamma[e^x] = \cos(x) \in \mathscr{L} - \mathscr{P}(\mathrm{II})$ and so, by Theorem 2.5, Γ is a multiplier sequence of the second kind.

(c) Let m be a fixed positive integer and set $\Gamma := \{\gamma_k\}_{k=0}^{\infty}$, where

$$\gamma_k := \gamma_k(m) := \frac{k!}{(mk)!} \qquad (k = 0, 1, 2, \ldots). \tag{2.18}$$

Then, it is known that Γ is a multiplier sequence of the first kind (cf. [17, p. 44]). Thus, it follows from Theorem 2.5, that the particular Mittag–Leffler function

$$E_m(x) := \sum_{k=0}^{\infty} \frac{x^k}{(mk)!} \qquad (m = 1, 2, 3, \ldots), \tag{2.19}$$

is a function of type I in the Laguerre–Pólya class (see also [21]).

In terms of the foregoing notations and results, our goal in the subsequent sections is to investigate $K(t)$ (cf. (2.1)) such that its Fourier transform will enjoy some of the properties of functions in the $\mathscr{L} - \mathscr{P}$ class. To see that such an undertaking is feasible, we remark that de Bruijn [4, Theorem 4] has shown that if $f(z)$ is any function in $\mathscr{L} - \mathscr{P}$, then there is a *uniquely determined continuous function* $K(t) := K(t; f)$ such that

$$\exp(-z^2/2) \cdot f(z) = \int_{-\infty}^{\infty} K(t) e^{izt}\, dt. \tag{2.20}$$

However, as we have noted above, the complete characterization of $K(t)$ is *not* known.

3. MOMENTS AND PROPERTIES OF MULTIPLIER SEQUENCES

Henceforth, we shall assume that the kernel function, $K(t)$, satisfies (2.1) and the following properties:

$$\left.\begin{array}{l}
\text{(i) } K(t) > 0 \ (t \in \mathbb{R}), \\
\text{(ii) } K'(t) < 0 \ (t > 0), \text{ and} \\
\text{(iii) there is a positive number } \tau := \tau(K) \text{ such that } K(t) \text{ is analytic in the strip} \\
\qquad S(\tau) := \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \tau\}.
\end{array}\right\} \tag{3.1}$$

LEMMA 3.1 *With* (3.1), *set*

$$F(x):=F(x;K):=\int_{-\infty}^{\infty} K(t)e^{izt}\,dt=:\sum_{k=0}^{\infty}(-1)^k\frac{b_k}{(2k)!}x^{2k}, \tag{3.2}$$

where

$$b_k:=\int_{-\infty}^{\infty} t^{2k}K(t)\,dt=2\int_{0}^{\infty} t^{2k}K(t)\,dt \qquad (k=0,1,2,\ldots). \tag{3.3}$$

Set

$$F_c(z):=F_c(z;K):=\int_{-\infty}^{\infty} K(t)\cosh(t\sqrt{z})\,dt=:\sum_{k=0}^{\infty}\gamma_k\frac{z^k}{k!}, \tag{3.4}$$

where

$$\gamma_k:=\frac{k!}{(2k)!}b_k \qquad (k=0,1,2,\ldots). \tag{3.5}$$

Then, $F(x)\in\mathscr{L}-\mathscr{P}(\mathrm{II})$ *if and only if* $F_c(z)\in\mathscr{L}-\mathscr{P}(\mathrm{I})$. *Moreover,* $F_c(z)\in\mathscr{L}-\mathscr{P}(\mathrm{I})$ *if and only if* $\{\gamma_k\}_{k=0}^{\infty}$ *is a multiplier sequence of the first kind.*

Proof By Proposition 2.1, $F(x)$ is an even, real entire function of order less than 2. Also, it is easy to see that, for any choice of a branch of the square root, $F_c(z)$ is well defined. Next, by using the change of variables $z=-x^2$, we deduce that $F_c(z)$ is a real entire function of order less than 1. The same change of variables shows that a complex number w_0 is a zero of $F(x)$ if and only if $z_0=-w_0^2$ is a zero of $F_c(z)$. Since $K(t)>0$, for all t in $\mathbb{R}$, we observe that $\gamma_k>0$ $(k=0,1,2,\ldots)$, and that any real zero of $F_c(z)$ is necessarily negative. But if a function in $\mathscr{L}-\mathscr{P}$ has only positive Taylor coefficients, then it is necessarily a function of type I in $\mathscr{L}-\mathscr{P}$ (see, for example, [25]). Combining these observations, it follows that $F(x)\in\mathscr{L}-\mathscr{P}$ if and only if $F_c(z)\in\mathscr{L}-\mathscr{P}(\mathrm{I})$. Since $\gamma_k>0$ $(k=0,1,2,\ldots)$, the last assertion of the lemma is a consequence of Theorem 2.5. ■

Remarks (a) By (3.1) we have

$$F_c(z)=\int_{-\infty}^{\infty} K(t)\exp\{t\sqrt{z}\}\,dt=\int_{-\infty}^{\infty} K(t)\exp\{-t\sqrt{z}\}\,dt. \tag{3.6}$$

(b) We recall that the *Hadamard product* (or *composition*) of two sequences $\{\alpha_k\}_{k=0}^{\infty}$ and $\{\beta_k\}_{k=0}^{\infty}$, of complex numbers is defined by

$$\{\alpha_k\}_{k=0}^{\infty}*\{\beta_k\}_{k=0}^{\infty}:=\{\alpha_k\beta_k\}_{k=0}^{\infty}. \tag{3.7}$$

Now, by Example 2.5(c) (cf. (2.18)), the sequence $\left\{\frac{k!}{(2k)!}\right\}_{k=0}^{\infty}$ is a multiplier sequence of the first kind. Thus, by Lemma 3.1, $F_c(z)\in\mathscr{L}-\mathscr{P}(\mathrm{I})$ if and only if the Hadamard product $\left\{\frac{k!}{(2k)!}\right\}_{k=0}^{\infty}*\{b_k\}_{k=0}^{\infty}$ is a multiplier sequence of the first kind, where b_k is defined by (3.3).

The following known result provides a collection of necessary conditions for a function to be in the Laguerre–Pólya class.

THEOREM 3.2 ([13], [18], [25]) *Set*

$$f(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k. \tag{3.8}$$

If $f(x) \in \mathscr{L}-\mathscr{P}$, *then*

(a)
$$\gamma_k^2 \geqslant \gamma_{k-1}\gamma_{k+1} \qquad (k=1,2,3,\ldots), \tag{3.9}$$
and

(b)
$$\left.\begin{aligned} \Delta(n;m) := \sum_{k=0}^{2n} \frac{(-1)^{k+n}}{(2n)!} \binom{2n}{k} \gamma_{m+k}\gamma_{2n+m-k} \geqslant 0, \\ (\textit{for all } m, n = 0, 1, 2, \ldots). \end{aligned}\right\} \tag{3.10}$$

Remarks (a) Inequality (3.9) is frequently termed as the *Turán inequality* associated with $f(x)$ of (3.8). Since, in the special case $n=1$, (3.10) reduces to (3.9), i.e. $\Delta(1; m-1) = \gamma_m^2 - \gamma_{m-1}\gamma_{m+1} \geqslant 0$, $m = 1, 2, 3, \ldots$, Patrick [18] termed the inequalities (3.10) as the *extended Turán inequalities.*

(b) The extended Turán inequalities were first introduced in 1913 by Jensen [13]. Unaware of this work of Jensen, Patrick [18] in 1973 independently rediscovered inequalities (3.10). We hasten to note here that, while Jensen [13] did not provide detailed verification of (3.10), the proof of (3.10) presented in [18] is rigorous and complete.

(c) If γ_k is defined by (3.5), then the extended Turán inequalities, with $m=0$, take the form

$$\Delta(n;0) = \frac{4(-1)^n}{(4n)!} \int_0^\infty \int_0^\infty K(u)K(v)\,\mathrm{Re}[(u+iv)^{4n}]\,du\,dv \geqslant 0 \qquad (n=0,1,2,\ldots) \tag{3.11}$$

(d) In general, for a real entire function with non-negative Taylor coefficients, it is possible for (3.9) to hold, while it is possible for *some* of the extended Turán inequalities (3.10) to fail. As an example, set

$$f_4(x) := 1 + \frac{x}{1!} + \frac{1}{2}\frac{x^2}{2!} + \frac{1}{4}\frac{x^3}{3!} + \frac{1}{8}\frac{x^4}{4!}. \tag{3.12}$$

In this case, $\gamma_0 := 1$, $\gamma_1 := 1$, $\gamma_k := 2^{1-k}$, if $k = 2, 3, 4$, and $\gamma_k := 0$ if $k = 5, 6, 7, \ldots$. Thus, we see that

$$\gamma_k^2 \geqslant \gamma_{k-1}\gamma_{k+1} \qquad (k=1,2,3,\ldots),$$

but

$$\Delta(2;0) = -\frac{1}{96} \qquad \text{and} \qquad \Delta(2;1) = -\frac{1}{192}.$$

PROPOSITION 3.3 *With* (3.1) *and* (3.2), *if* $F(x) \in \mathscr{L}-\mathscr{P}$ *then there is a positive integer* m_0 *such that*

$$\int_{-\infty}^{\infty} t^{2k} q_k(t) K(t)\, dt \geqslant 0 \qquad (k \geqslant m_0), \tag{3.13}$$

where

$$q_k(t) := \frac{t^4}{4(2k+3)(2k+1)} - \frac{t^2}{(2k+1)} + 1 \qquad (k=0,1,2,\ldots). \tag{3.14}$$

Proof If $F(x) \in \mathscr{L}-\mathscr{P}$, then by definition, $F(x) \in \mathscr{L}-\mathscr{P}(\mathrm{II})$. Thus, by Lemma 3.1, $F_c(z) \in \mathscr{L}-\mathscr{P}(\mathrm{I})$ (cf. (3.4)). Consequently, by Theorem 3.2 the sequence $\{\gamma_k\}_{k=0}^{\infty}$, where γ_k is defined by (3.5), satisfies the Turán inequalities (3.9). But then by a known result [7, Theorem 4.6], there is a positive integer m_0, such that the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is convex, that is,

$$\gamma_{k+2} - 2\gamma_{k+1} + \gamma_k \geqslant 0 \qquad (k \geqslant m_0). \tag{3.15}$$

Therefore, if we express (3.15) in terms of (3.3) we obtain the desired result (3.13). ∎

Remarks (a) With (3.5), suppose the sequence $\{\gamma_k\}_{k=0}^{\infty}$ satisfies the Turán inequalities (3.9), and, in addition, $\gamma_0 \geqslant \gamma_1$, so that $1 \geqslant \frac{\gamma_1}{\gamma_0} \geqslant \frac{\gamma_2}{\gamma_1} \geqslant \frac{\gamma_3}{\gamma_2} \geqslant \cdots$. Then it directly follows that the sequence $\{\gamma_k\}_{k=0}^{\infty}$ is monotone decreasing; that is

$$\gamma_k \geqslant \gamma_{k+1} \qquad (k=0,1,2,\ldots). \tag{3.16}$$

In particular, in terms of (3.3), inequality (3.16) becomes

$$b_k \geqslant \frac{1}{2(2k+1)} b_{k+1} \qquad (k=0,1,2,\ldots), \tag{3.17}$$

or

$$\int_0^{\infty} t^{2k}\left(1 - \frac{t^2}{2(2k+1)}\right) K(t)\, dt \geqslant 0 \qquad (k=0,1,2,\ldots). \tag{3.18}$$

(b) If we apply the Cauchy–Schwarz inequality to (3.3), then we obtain

$$b_k^2 \leqslant b_{k-1} b_{k+1} \qquad (k=1,2,3,\ldots). \tag{3.19}$$

On the other hand, if the γ_k's defined by (3.5) satisfy the Turán inequalities, then

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} = \left(\frac{k!}{(2k)!}\right)^2 \left[b_k^2 - \frac{2k-1}{2k+1} b_{k-1} b_{k+1}\right] \geqslant 0 \qquad (k=1,2,3,\ldots), \tag{3.20}$$

which is nearly the reverse of inequality (3.19).

(c) The assumptions on $K(t)$ (cf. (3.1)) imply that there is a non-negative integer k_0 such that the sequence of moments, $\{b_k\}_{k=0}^{\infty}$ (cf. (3.3)) is increasing for $k \geqslant k_0$. Indeed,

$$
\begin{aligned}
b_{k+1}-b_k &= 2\int_0^\infty t^{2k}(t^2-1)K(t)\,dt \\
&= 2\int_1^\infty t^{2k}(t^2-1)K(t)\,dt - 2\int_0^1 t^{2k}(1-t^2)K(t)\,dt \\
&\geqslant 2\int_1^\infty (t^2-1)K(t)\,dt - K(0)\frac{4}{(2k+1)(2k+3)} > 0,
\end{aligned}
$$

if k is sufficiently large. In fact, an elementary argument shows that the function

$$
f(z) := \sum_{k=0}^{\infty} \frac{1}{b_k} z^k, \tag{3.21}
$$

where b_k is given by (3.3), is an entire function [11, Lemma 1].

We next consider the problem of completely characterizing functions

$$
\left.
\begin{aligned}
f(x) := \sum_{k=0}^{\infty} a_k x^k := 1 + \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \\
\left(a_k := \frac{\gamma_k}{k!} \geqslant 0, k = 1, 2, 3, \ldots, a_0 = \gamma_0 = 1 \right),
\end{aligned}
\right\} \tag{3.22}
$$

of type I in the Laguerre–Pólya class, normalized by $f(0) = 1$, in terms of the coefficients a_k. To this end, we recall here the following definition.

Definition 3.1 A real sequence $\{a_k\}_{k=0}^{\infty}$, $a_0 := 1$, is said to be *totally positive*, if the infinite lower triangular matrix

$$
A := [a_{i-j}] := \begin{bmatrix} a_0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad (i, j = 1, 2, 3, \ldots), \tag{3.23}
$$

where $a_m := 0$ for $m < 0$, is totally positive; that is, all the minors of A, of all orders, are non-negative.

In [1], Aissen, Edrei, Schoenberg and Whitney characterized the generating functions $f(x) := \sum_{k=0}^{\infty} a_k x^k$ of totally positive sequences. Their result is

THEOREM 3.4 ([1, p. 306]) *Suppose $f(x)$ defined by* (3.22) *is an entire function. Then $\{a_k\}_{k=0}^{\infty}$ is a totally positive sequence if and only if $f(x) \in \mathscr{L}-\mathscr{P}$(I).*

An immediate consequence of Theorem 3.4 is the following

COROLLARY 3.5 ([1, p. 306]) *Set*

$$
p(x) := a_0 + a_1 x + \cdots + a_n x^n \qquad (a_0 = 1, a_k \geqslant 0, k = 1, 2, \ldots, n). \tag{3.24}
$$

Then, $p(x) \in \mathscr{L}-\mathscr{P}$(I) if and only if the sequence $a_0, a_1, \ldots, a_n, 0, 0, 0, \ldots$ is totally positive.

In light of Theorem 2.5, another consequence of Theorem 3.4 is the following corollary.

COROLLARY 3.6 *Let* $\{\gamma_k\}_{k=0}^{\infty}$ *be a sequence of non-negative real numbers, normalized by* $\gamma_0 := 1$. *Then,* $\{\gamma_k\}_{k=0}^{\infty}$ *is a multiplier sequence of the first kind if and only if the sequence* $\left\{\frac{\gamma_k}{k!}\right\}_{k=0}^{\infty}$ *is totally positive.*

The foregoing results lead to the formulation of the following

Open Problem 3.1 With (3.1) and (3.3), characterize the kernel $K(t)$ such that the sequence $\left\{\frac{b_k}{(2k)!}\right\}_{k=0}^{\infty}$ is totally positive.

We briefly pause here to further elucidate upon the ideas developed so far. By (3.19) the moments b_k of (3.3) necessarily satisfy

$$b_k^2 \leqslant b_{k-1}b_{k+1} \qquad (k=1,2,3,\ldots). \tag{3.25}$$

Now, if in (3.25) the inequality reduces to equality for all k $(k=1,2,3,\ldots)$, then it follows that

$$b_k = b_0(b_1/b_0)^k \qquad (k=0,1,2,\ldots). \tag{3.26}$$

But using (3.2), (3.26) implies that $F(x;K) = b_0 \cos(x\sqrt{b_1/b_0})$ and this is *incompatible* with our assumptions (3.1). Thus, for at least one integer k, $k \geqslant 1$, there is a strict inequality in (3.25). But then by Theorem 3.2 (cf. (3.9)), the entire function $\sum_{k=0}^{\infty} \frac{b_k}{k!} x^k$ is *not* in the Laguerre–Pólya class. Hence by Corollary 3.6, the sequence $\left\{\frac{b_k}{k!}\right\}_{k=0}^{\infty}$ is *not* a totally positive sequence.

While Open Problem 3.1 appears to be very difficult, we propose here some related, but tractable, questions which arise when we generalize the concept of totally positive sequences. Following Fekete and Pólya [10] and Schoenberg [26], we recall here the following

Definition 3.2 Let k be a positive integer. Let $\{a_n\}_{n=0}^{\infty}$, with $a_0 := 1$, be a sequence of non-negative real numbers such that the series $\sum_{n=0}^{\infty} a_n$ is convergent. Then the sequence $\{a_n\}_{n=0}^{\infty}$ is said to be *k-times positive*, or *k-positive*, provided that the matrix $A := [a_{i-j}]$ of (3.23) has no negative minors of order $\leqslant k$.

If we denote by $\mathscr{P}_k$ the class of k-times positive sequences, then it is evident that

$$\mathscr{P}_1 \supset \mathscr{P}_2 \supset \cdots \supset \mathscr{P}_k \supset \cdots, \tag{3.27}$$

whose intersection

$$\mathscr{P}_\infty := \bigcap_{k=1}^{\infty} \mathscr{P}_k \tag{3.28}$$

is the class of totally positive sequences, denoted by $\mathscr{P}_\infty$.

For the various properties of k-times positive sequences we refer the reader to Schoenberg [26]. We single out here the following useful device for generating k-times positive sequences.

PROPOSITION 3.7 ([26]) *Set*

$$p(x) := \sum_{j=0}^{m} p_j x^j, \tag{3.29}$$

$$q(x) := \sum_{j=0}^{n} q_j x^j, \tag{3.30}$$

and

$$r(x) := p(x)q(x) = \sum_{j=0}^{m+n} r_j x^j. \tag{3.31}$$

(a) *If the coefficients of both polynomials $p(x)$ and $q(x)$ are k-times positive, then the coefficients of the product* (3.31) *are also k-times positive.*

(b) *Let t_0 be a fixed positive number. Then, the coefficients of $p(x)$ are k-times positive if and only if the coefficients of $p(xt_0)$ are k-times positive.*

We next establish a new relationship between 2-times positive sequences and sequences which satisfy the Turán inequalities (3.8).

PROPOSITION 3.8 *Let $\{\gamma_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers, normalized by $\gamma_0 := 1$. If*

$$\gamma_n^2 - \gamma_{n-1}\gamma_{n+1} \geqslant 0 \qquad (n = 1, 2, 3, \ldots), \tag{3.32}$$

then $\left\{\dfrac{\gamma_n}{n!}\right\}_{n=0}^{\infty} \in \mathscr{P}_2$.

Proof By (3.32), we have the inequalities

$$\frac{\gamma_1}{\gamma_0} \geqslant \frac{\gamma_2}{\gamma_1} \geqslant \cdots \geqslant \frac{\gamma_{n+1}}{\gamma_n} \geqslant \cdots, \tag{3.33}$$

and therefore

$$\lim_{n\to\infty} \frac{\gamma_{n+1}}{\gamma_n} =: R_0 \geqslant 0. \tag{3.34}$$

Consequently,

$$f(z) := \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} z^n \tag{3.35}$$

is an entire function and thus the requirement that (3.35) converges for $z = 1$ (cf. Definition 3.2) is satisfied. Next, it follows from (3.33) that $\gamma_m/\gamma_{m-p} \geqslant \gamma_{m+q}/\gamma_{m+q-p}$ ($p \geqslant 0$, $m \geqslant p$, $q \geqslant 0$), whence

$$\det\begin{bmatrix} \gamma_m & \gamma_{m-p} \\ \gamma_{m+q} & \gamma_{m+q-p} \end{bmatrix} \geqslant 0 \qquad (\gamma_{-n} := 0, \text{ if } n \geqslant 1,\ p, q = 1, 2, 3, \ldots, m = 0, 1, 2, \ldots). \tag{3.36}$$

In addition, if $m - p \geqslant 0$, then

$$\det\begin{bmatrix} \dfrac{\gamma_m}{m!} & \dfrac{\gamma_{m-p}}{(m-p)!} \\ \dfrac{\gamma_{m+q}}{(m+q)!} & \dfrac{\gamma_{m+q-p}}{(m+q-p)!} \end{bmatrix} \geqslant 0 \quad (m = 1, 2, 3, \ldots). \tag{3.37}$$

Therefore, by Definition 3.2, $\left\{\dfrac{\gamma_n}{n!}\right\}_{n=0}^{\infty} \in \mathscr{P}_2$. ■

Remarks (a) It is easy to construct examples which show that the converse of Proposition 3.8, in general, is *false*. As an example, consider $\gamma_n := ((n+1)e^n)^{-1}$ $(n = 0, 1, 2, \ldots)$. Then $\gamma_0 = 1$, $\sum_{n=0}^{\infty} \dfrac{\gamma_n}{n!} < \infty$, $\gamma_n^2 - \gamma_{n-1}\gamma_{n+1} < 0$, $n = 1, 2, 3, \ldots$, while a calculation shows that inequality (3.37) holds. Thus, $\left\{\dfrac{\gamma_n}{n!}\right\}_{n=0}^{\infty} \in \mathscr{P}_2$, but (3.32) does not hold.

(b) A geometric characterization of the class $\mathscr{P}_2$ is as follows (cf. [26]). Let $\{a_n\}_{n=0}^{\infty}$, $a_0 := 1$, be a sequence of positive numbers with $\sum_{n=0}^{\infty} a_n < \infty$. Then $\{a_n\}_{n=0}^{\infty} \in \mathscr{P}_2$ if and only if the polygonal line in the xy-plane, with vertices (x, y), where $x := n$ and $y := \log a_n$, $n = 0, 1, 2, \ldots$, is *concave*, i.e., $\log a_{n+1} \geqslant (\log a_{n+2} + \log a_n)/2$ for $n \geqslant 0$.

Open Problem 3.2 Find a geometric characterization of the class of k-times positive sequences, $\mathscr{P}_k$, where $k = 3, 4, 5, \ldots$.

Open Problem 3.3 With (3.1) and (3.3), characterize $K(t)$, such that the sequence $\left\{\dfrac{b_m}{(2m)!}\right\}_{m=0}^{\infty}$ is k-times positive, for $k = 2, 3, 4, \ldots$.

While we are unable to solve Open Problem 3.3, even in the special case when $k = 2$, we will provide below two different conditions, such that if $K(t)$ fulfills either of these conditions, then the sequence $\left\{\dfrac{m!}{(2m)!}\, b_m\right\}_{m=0}^{\infty}$ satisfies the Turán inequalities. But then by Proposition 3.8, $\left\{\dfrac{b_m}{(2m)!}\right\}_{m=0}^{\infty} \in \mathscr{P}_2$.

The aforementioned conditions will be established via three preliminary lemmas. The proof of the first lemma is based on an idea of Matiyasevich [15].

LEMMA 3.9 ([15]) *With* (3.1), *set*

$$\left.\begin{aligned} J_{m,n} := J_{m,n}(K) := \int_0^{\infty}\int_0^{\infty} [u^{2m-1}v^{2n+2} - u^{2m+1}v^{2n}]K'(u)K(v)\,du\,dv \\ (m, n = 1, 2, 3, \ldots). \end{aligned}\right\} \tag{3.38}$$

If $\log(K(\sqrt{t}))$ *is strictly concave for* $0 < t < \infty$, *then*

$$J_{m,n} + J_{n,m} > 0 \qquad (m, n = 1, 2, 3, \ldots). \tag{3.39}$$

Proof If $\log(K(\sqrt{t}))$ is strictly concave for $0 < t < \infty$, then an elementary calculation shows that

$$E(t) := \frac{d}{dt}\left(-\frac{K'(t)}{tK(t)}\right) = -4t\left\{\left[\frac{d^2}{ds^2}\log(K(\sqrt{s}))\right]_{s=t^2}\right\} > 0 \qquad (t > 0). \tag{3.40}$$

But then

$$\left.\begin{aligned} J_{m,n} + J_{n,m} = \frac{1}{2}\int_0^\infty \int_0^\infty u^{2m} v^{2n} K(u) K(v)(v^2 - u^2)\left(\int_u^v E(t)\,dt\right) du\,dv \\ (m, n = 1, 2, 3, \ldots). \end{aligned}\right\} \tag{3.41}$$

Therefore, by (3.40) and (3.41) the desired conclusion (3.39) follows. ■

The next lemma is of independent interest and it may be used to give an alternative proof of Theorem 3.12. This lemma is a refinement of a result of Barlow, Marshall and Proschan [2] (cf. Csordas and Varga [9, Proposition 2.3] or Marshall and Olkin [16, Proposition E.4]).

LEMMA 3.10 ([2], [9], [16]) *With* (3.1) *set*

$$\mu(x) := \frac{1}{\Gamma(x+1)}\int_0^\infty t^x K(\sqrt{t})\,dt \qquad (x > -1), \tag{3.42}$$

where $\Gamma(x)$ *denotes the gamma function. If* $\log(K(\sqrt{t}))$ *is strictly concave for* $0 < t < \infty$, *then* $\mu(x)$ *is strictly concave for* $-1 < x < \infty$.

Remark The connection between Lemma 3.10 and Theorem 2.2 is particularly interesting. By (3.1(iii)) we know that if

$$K_1(t) := K(\sqrt{t}) \qquad (t \geqslant 0), \tag{3.43}$$

then there is a positive number $R(=(\tau(K))^{1/2})$ such that K is a real analytic function in $|t| < R$, i.e.,

$$K_1(t) = \sum_{j=0}^\infty c_j t^j \qquad (t \in (-R, R),\ c_j \in \mathbb{R}); \tag{3.44}$$

so that assumption (2.4) of Theorem 2.2 is fulfilled. Now for $0 < r < R$, we set

$$H_1(z) := \int_0^r \sum_{j=0}^\infty c_j t^{j+z}\,dt + \int_r^\infty t^z K_1(t)\,dt \qquad (\operatorname{Re} z > -1), \tag{3.45}$$

and therefore by the absolute and uniform convergence of the power series (3.44), we have that for $0 < r < R$,

$$H_1(z) = r^{z+1}\sum_{j=0}^\infty \frac{c_j r^j}{j+z+1} + \int_{\log r}^\infty K_1(e^u) e^{(z+1)u}\,du \qquad (\operatorname{Re} z > -1). \tag{3.46}$$

Since the integral in (3.46) represents an entire function and since $z=-n$ $(n=1,2,3,\ldots)$, is a simple pole of the sum in (3.46), and hence also for $H_1(z)$, it follows that

$$\mu(z):=\frac{1}{\Gamma(z+1)}H_1(z) \qquad (z\in\mathbb{C}), \tag{3.47}$$

where $\Gamma(z)$ denotes the gamma function, is a real entire function (cf. [23, p. 15]). Therefore, if $\mu(z)$ has only real negative zeros, then by (3.43) and Theorem 2.2

$$F_2(z):=\int_{-\infty}^{\infty}K_1(t^2)e^{izt}\,dt=\int_{-\infty}^{\infty}K(t)e^{izt}\,dt\in\mathscr{L}-\mathscr{P}. \tag{3.48}$$

Another type of condition on $K(t)$, which is different from those assumed in Lemma 3.9 and Lemma 3.10, is used in the following lemma. It is a generalization of a result of [8, Proposition 2.1].

Lemma 3.11 ([8]) *With* (3.1), *set*

$$I(t):=\int_t^{\infty}K(\sqrt{s})\,ds \qquad (t>0), \tag{3.49}$$

and

$$\mu(x):=\frac{1}{\Gamma(x+1)}\int_0^{\infty}t^xI(u)\,du \qquad (x>-1), \tag{3.50}$$

where $\Gamma(x)$ *denotes the gamma function. If* $\log(I(t))$ *is strictly concave for* $0<t<\infty$, *then* $\log(\mu(x)))$ *is strictly concave for* $-1<x<\infty$.

Remark If we set

$$g(t):=g(t;K):=t[(K'(t))^2-K(t)K''(t)]+K(t)K'(t) \qquad (t>0), \tag{3.51}$$

then it is easy to verify that $\log(K(\sqrt{t}))$ is strictly concave for $0<t<\infty$ if and only if $g(t)>0$ for $t>0$. Thus, by (3.51) and (3.1(i), (ii)), the strict concavity of $\log(K(\sqrt{t}))$ for $0<t<\infty$ implies the strict concavity of $\log(K(t))$ for $0<t<\infty$. But the condition that $\log(K(t))$ is strictly concave for $0<t<\infty$, is not strong enough for our purposes to deduce the Turán inequalities (3.20). Similarly, in reference to Lemma 3.11, we note that $\log(I(t))$ (cf. (3.49)) is strictly concave for $0<t<\infty$ if and only if

$$h(t):=h(t;K):=\left(\int_t^{\infty}sK(s)\,ds\right)K'(t)+t(K(t))^2>0 \qquad (t>0). \tag{3.52}$$

Setting

$$I_1(t):=\int_t^{\infty}K(s)\,ds \qquad (t>0), \tag{3.53}$$

then $\log(I_1(t))$ is strictly concave for $0<t<\infty$ if and only if

$$h_1(t):=h_1(t;K):=I_1(t)K'(t)+(K(t))^2>0 \qquad (t>0). \tag{3.54}$$

Now by (3.1(ii)), $K'(t) < 0$ for $t > 0$ and thus using (3.52) and (3.54),

$$th_1(t) > h(t) \qquad (t > 0) \tag{3.55}$$

if and only if

$$tI_1(t) < \int_t^\infty sK(s)\,ds \qquad (t > 0). \tag{3.56}$$

Therefore, upon integrating by parts, we have

$$\int_t^\infty sK(s)\,ds = tI_1(t) + \int_t^\infty I_1(s)\,ds \qquad (t > 0). \tag{3.57}$$

So by (3.54)–(3.57), if $\log(I(t))$ is strictly concave for $0 < t < \infty$, then $\log(I_1(t))$ is also strictly concave for $0 < t < \infty$. Once again, this latter condition does *not* imply, by our methods, the Turán inequalities (3.20). Indeed, the new and essential idea in the foregoing lemmas is that the hypotheses involve the modified function $K(\sqrt{t})$, rather than the kernel $K(t)$ itself.

Preliminaries aside, we now prove

THEOREM 3.12 *With* (3.1), *set*

$$b_m := 2\int_0^\infty t^{2m}K(t)\,dt \qquad (m = 0, 1, 2, \ldots), \tag{3.58}$$

and

$$\gamma_m := \frac{m!}{(2m)!}\,b_m \qquad (m = 0, 1, 2, \ldots). \tag{3.59}$$

If $\log(K(\sqrt{t}))$ *is strictly concave for* $0 < t < \infty$, *then*

$$\gamma_m^2 > \gamma_{m-1}\gamma_{m+1} \qquad (m = 1, 2, 3, \ldots). \tag{3.60}$$

Moreover, $\left\{\dfrac{\gamma_m}{m!}\right\}_{m=0}^\infty \in \mathscr{P}_2$.

Proof By Lemma 3.9, $J_{m,m} > 0$ (cf. (3.38) and (3.39)) for $m = 1, 2, 3, \ldots$. Now, if we integrate (3.58) by parts, we obtain

$$b_m = \frac{-2}{(2m+1)}\int_0^\infty t^{2m+1}K'(t)\,dt \qquad (m = 1, 2, 3, \ldots). \tag{3.61}$$

Therefore, with $n = m$ and using (3.61), we have

$$\begin{aligned} 4J_{m,m} &= 4\int_0^\infty\int_0^\infty [u^{2m-1}v^{2m+2} - u^{2m+1}v^{2m}]K'(u)K(v)\,du\,dv \\ &= -(2m-1)b_{m-1}b_{m+1} + (2m+1)b_m^2 > 0 \qquad (m = 1, 2, 3, \ldots). \end{aligned} \tag{3.62}$$

Thus, (3.62) implies (3.60) (cf. (3.20)). Furthermore, by Proposition 3.8, $\left\{\dfrac{\gamma_m}{m!}\right\}_{m=0}^\infty \in \mathscr{P}_2$. ■

In view of Theorem 3.12, it is natural to raise the following question.

Open Problem 3.4 Is it true that the extended Turán inequalities (3.10) are also a consequence of the assumption that $\log(K(\sqrt{t}))$ is strictly concave for $0 < t < \infty$, where $K(t)$ satisfies (3.1)?

If $\log(K(\sqrt{t}))$ is concave for $0 < t < \infty$, then Lemma 3.9 can be used to generate various types of inequalities involving the moments (3.58) (see also [15]). However, apart from (3.60), we do not know whether or not such inequalities yield, *ipso facto*, necessary conditions for the entire function

$$F(x) := F(x; K) = \int_{-\infty}^{\infty} K(t) e^{izt}\, dt, \tag{3.63}$$

to have only real zeros. In contrast, by means of Pólya's universal factors (see Definition 3.5 below), it is possible to generate a plethora of the required necessary conditions. To motivate Pólya's notion of a universal factor, we first recall (cf. Section 2) that multiplier sequences may be regarded as generalized forms of differentiation. But since the Laguerre–Pólya class is closed under differentiation (that is, if $f(x) \in \mathscr{L}-\mathscr{P}$, then $f'(x) \in \mathscr{L}-\mathscr{P}$; see, for example, [25]), it is natural to investigate the actions of more general differential operators on the class $\mathscr{L}-\mathscr{P}$. By extending the classical Hermite–Poulain Theorem (see, for example, Obreschkoff [17, p. 4]), to transcendental entire functions, Pólya (cf. [20, p. 242]) has shown, in particular, that if $f(x), F(x) \in \mathscr{L}-\mathscr{P}$ and if the series

$$H(x) := H(x; f, F) := \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} F^{(k)}(x), \tag{3.64}$$

converges, where

$$f(x) := \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} x^k, \tag{3.65}$$

then $H(x) \in \mathscr{L}-\mathscr{P}$.

Definition 3.5 ([23], [3]) With (3.1), suppose that

$$F(x) := F(x; K) := \int_{0}^{\infty} K(t) e^{izt}\, dt \in \mathscr{L}-\mathscr{P}. \tag{3.66}$$

Then a function $U(t)\colon \mathbb{R} \to \mathbb{R}$ is called a *universal factor*, if

$$\int_{-\infty}^{\infty} U(t) K(t) e^{izt}\, dt \in \mathscr{L}-\mathscr{P}. \tag{3.67}$$

The following theorem of Pólya [23] provides a complete characterization of universal factors.

THEOREM 3.13 ([23]) *Set*

$$U(t) := \sum_{k=0}^{\infty} \frac{\alpha_k}{k!} t^k \qquad (t \in (-\infty, \infty),\ \alpha_k \in \mathbb{R}). \tag{3.68}$$

Then $U(t)$ is a universal factor if and only if $U(t) = f(it)$, where $f(t) \in \mathscr{L} - \mathscr{P}$.

Therefore, if $U(t) = f(it)$, $f \in \mathscr{L} - \mathscr{P}$ and if $F(x) \in \mathscr{L} - \mathscr{P}$, where $F(x)$ is given by (3.63), then

$$\begin{aligned} H(x) := f(D)F(x) &= \int_{-\infty}^{\infty} f(it)K(t)e^{izt}\,dt \qquad \left(D := \frac{d}{dx}\right) \\ &= \int_{-\infty}^{\infty} U(t)K(t)e^{izt}\,dt \in \mathscr{L} - \mathscr{P}. \end{aligned} \tag{3.69}$$

To generalize Theorem 3.12, we need the following proposition (cf. Csordas and Varga [9]).

PROPOSITION 3.14 ([9]) *Let $f(z) \in \mathscr{L} - \mathscr{P}$, where $f(z)$ is even, $f(z) \not\equiv 0$, and normalized so that its first non-zero Taylor coefficient is positive. Then $\log(f(i\sqrt{t}))$ is concave for $0 < t < \infty$.*

Proof Using the representation (2.16) and the assumptions of the proposition, we deduce that

$$f(it) = ct^{2n} \prod_{j=1}^{\omega} (1 + t^2/z_j^2) \qquad (\omega \leqslant \infty), \tag{3.70}$$

where $z_j \in \mathbb{R} - \{0\}$, $\sum_{j=1}^{\omega} z_j^{-2} < \infty$ and $c > 0$. Then a computation shows that

$$\frac{d^2}{dt^2} \log(f(i\sqrt{t})) = -\left\{\frac{n}{t^2} + \sum_{j=1}^{\omega} \frac{1}{(z_j^2 + t)^2}\right\} < 0 \qquad (t > 0). \quad \blacksquare \tag{3.71}$$

THEOREM 3.15 ([9]) *With (3.1) suppose that $\log(K(\sqrt{t}))$ is strictly concave for $0 < t < \infty$. Let $f(z) \in \mathscr{L} - \mathscr{P}$, $f(z) \not\equiv 0$, be an even entire function, normalized so that its first non-zero Taylor coefficient is positive. Set*

$$c_m(K; f) := \int_0^{\infty} t^{2m} f(it)K(t)\,dt \qquad (m = 0, 1, 2, \ldots). \tag{3.72}$$

Then

$$(c_m(K; f))^2 > \left(\frac{2m-1}{2m+1}\right) c_{m-1}(K; f)\,c_{m+1}(K; f) \qquad (m = 1, 2, 3, \ldots). \tag{3.73}$$

Proof Set $V(t) := f(it)K(t)$, $t \geqslant 0$. Then by (3.1) and the assumption that $f \in \mathscr{L} - \mathscr{P}$, we have $V(t)$ satisfies (2.1). Thus, in particular, the moments (3.72) all exist (cf. Proposition 2.1). Also, by Proposition 3.14, $\log(V(\sqrt{t}))$ is strictly concave for $t>0$. Set

$$\gamma_m := \gamma_m(K,f) := \frac{m!}{(2m)!}\, c_m(K,f) \qquad (m=0,1,2,\ldots). \tag{3.74}$$

Then by Theorem 3.12

$$\gamma_m^2 > \gamma_{m-1}\gamma_{m+1} \qquad (m=1,2,3,\ldots). \tag{3.75}$$

Since (3.75) is equivalent to (3.73), the proof of the theorem is complete. ■

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