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Chebyshev semi-iterative methods, successive overrelaxation iterative methods, and second order Richardson iterative methods

Part I

By

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§ 1. Introduction

One of the major areas of interest in numerical analysis is concerned with the numerically efficient solution of the matrix equation

$$A\vec{x}=\vec{k}, \tag{1.1}$$

where A is a given $N\times N$ real symmetric and positive definite matrix, and $\vec{k}$ is a given column vector. The matrix equation (1.1) can be readily reduced to the analogous matrix equation

$$\vec{x}=B\vec{x}+\vec{g}, \tag{1.2}$$

where B is an $N\times N$ real symmetric matrix which is *convergent*, i.e., if the eigenvalues of the matrix B are μ_j, $1\leqq j\leqq N$, then the *spectral radius* [9] $\varrho(B)$ of B satisfies

$$\varrho(B)=\max_j|\mu_j|<1. \tag{1.3}$$

At this point, one can consider the different convergent systematic iterative methods in the title of this paper, and basically, the literature on the analysis of these methods can be conveniently separated on the following classification of the matrix B. With B symmetric, we say [20] that B is *cyclic* (of index 2) if there exists an $N\times N$ permutation matrix Λ such that

$$\Lambda B\Lambda^T=\begin{pmatrix}0 & F\\ F^T & 0\end{pmatrix}=\tilde{B}, \tag{1.4}$$

where the non-vacuous diagonal blocks of $\tilde{B}$ are square, with zero entries. In the more familiar notation of YOUNG [24, 26], B satisfies property A, and $\tilde{B}$ is consistently ordered with the σ_1 ordering. If no such permutation matrix Λ exists, we say that B is *primitive***.

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* This paper includes work from the doctoral dissertation [7] of the first author, who wishes to thank Professor A. H. TAUB of the University of Illinois for guidance and encouragement in the preparation of that dissertation.

** Usually, the terms primitive and cyclic are reserved (see [23]) for irreducible matrices with non-negative entries. In the case that the matrix B of (1.2) is symmetric and irreducible, with non-negative entries, these definitions agree with the classical terminology.

If B is primitive, then the only systematic iterative methods of the title which have been analysed* and used in large scale digital machine codes are the Chebyshev semi-iterative method [*5', 11, 16, 18, 25*], and the second order Richardson iterative method [*6, 13*]. Actually, one can also define the successive overrelaxation iterative method [*6, 26*] for an expanded matrix equation of the form (1.2), and in §2, our first result is that all three methods, when optimized with respect to acceleration parameters, are from a certain point of view remarkably similar**. In §3, we shall compare these three systematic iterative methods, using the spectral norms of the respective matrix operators as a basis for comparison, and we shall show that the matrix operator for the Chebyshev semi-iterative method possesses the smallest spectral norm. Since the practical application of the Chebyshev semi-iterative method in the primitive case requires effectively no additional arithmetic operations or vector storage over the other iterative methods, it would appear that of these three systematic iterative methods, there is no practical or theoretical reason for not always using the Chebyshev semi-iterative method for the primitive case.

If B is cyclic, then several results already exist in the literature [*13, 18, 25, 27*] comparing the three basic systematic iterative methods of the title. In §4, we shall define a new systematic iterative method, called the *cyclic Chebyshev semi-iterative method* for cyclic matrices B, which again requires effectively no more additional arithmetic operations or vector storage over the other iterative methods. This new systematic iterative method, which has combined the observations of several others, will be shown in §5, using spectral norms of matrix operators as a basic for comparison, to have the smallest spectral norm. Again, of the three systematic iterative methods of the title, it would appear that the modified Chebyshev semi-iterative method is the best choice in the cyclic case.

In §6, we shall show how these results can be used in conjunction with various block methods [*1, 8, 12, 21*] to numerically solve elliptic difference equations, and finally in §7, we shall give some of the results of recent numerical experiments on the comparison between the systematic iterative methods of the title.

§2. Primitive Matrices

We assume in this section that the matrix B of (1.2) is primitive. If $\vec{\alpha}^{(0)}$ is a given vector guess of the unique solution $\vec{x}$ of (1.2), then we iteratively form the vectors $\vec{\alpha}^{(i+1)}$, defined by

$$\vec{\alpha}^{(i+1)} = B\vec{\alpha}^{(i)} + \vec{g}, \qquad i \geqq 0. \tag{2.1}$$

Since $\varrho(B) < 1$, the sequence vectors $\vec{\alpha}^{(i)}$ converges [*9*] to the solution vector $\vec{x}$. If $\vec{\varepsilon}^{(i)} \equiv \vec{x} - \vec{\alpha}^{(i)}$, $i \geqq 0$, is the error vector associated with the iterate $\vec{\alpha}^{(i)}$, then

$$\vec{\varepsilon}^{(i+1)} = B\vec{\varepsilon}^{(i)}, \qquad i \geqq 0, \tag{2.2}$$

and thus, by induction

$$\vec{\varepsilon}^{(i)} = B^i \vec{\varepsilon}^{(0)}, \qquad i \geqq 0. \tag{2.2'}$$

* A notable exception to this is KAHAN's theoretical extension [*10*] of the successive overrelaxation iterative method to the case where B is primitive, and has non-negative entries.

** It has been generally assumed that the successive overrelaxation iterative method could not be applied in as general cases as could the Chebyshev semiiterative method. See [*27*, p. 291].

We now consider forming linear combinations* of the vectors $\vec{\alpha}^{(i)}$ in order to accelerate the convergence of (2.1). Let

$$\vec{\beta}^{(i)} = \sum_{k=0}^{i} a_{i,k} \vec{\alpha}^{(k)}, \qquad i \geqq 0. \tag{2.3}$$

As in [18], we impose the natural condition that $\sum_{k=0}^{i} a_{i,k} = 1$. Thus, if $\vec{\overset{*}{\varepsilon}}^{(i)} = \vec{x} - \vec{\beta}^{(i)}$, $i \geqq 0$, then

$$\vec{\overset{*}{\varepsilon}}^{(i)} = \left(\sum_{k=0}^{i} a_{i,k} B^k \right) \vec{\varepsilon}^{(0)}. \tag{2.4}$$

If $p_i(t) = \sum_{k=0}^{i} a_{i,k} t^k$, then (2.4) becomes formally

$$\vec{\overset{*}{\varepsilon}}^{(i)} = p_i(B) \vec{\varepsilon}^{(0)}, \tag{2.4'}$$

where $p_i(1) = 1$. Let $\{\vec{y}_i\}_{i=1}^N$ be an orthonormal set of eigenvectors of B, where $B\vec{y}_i = \mu_i \vec{y}_i$, $1 \leqq i \leqq N$. If $\vec{\varepsilon}^{(0)} = \sum_{k=1}^{N} c_k \vec{y}_k$, then it follows that

$$\vec{\overset{*}{\varepsilon}}^{(m)} = \sum_{k=1}^{N} c_k p_m(\mu_k) \vec{y}_k. \tag{2.5}$$

If all the eigenvalues μ_i of B were known a priori, then we could determine a polynomial $p_N(x)$ such that $p_N(\mu_k) = 0$ for $1 \leqq k \leqq N$. Since this is seldom the case, let S_m be the set of all polynomials $p_m(x)$ of degree m, normalized so that $p_m(1) = 1$. Since B is symmetric and convergent, all its eigenvalues μ_i satisfy $-1 < -\varrho(B) \leqq \mu_i \leqq \varrho(B) < 1$, and we seek** the polynomial $\tilde{p}_m(x)$ for which

$$\min_{p_m \in S_m} \left\{ \max_{-\varrho \leqq x \leqq \varrho} |p_m(x)| \right\} = \max_{-\varrho \leqq x \leqq \varrho} |\tilde{p}_m(x)|, \tag{2.6}$$

where $\varrho = \varrho(B)$. As is well known [4], the *unique* solution of this problem is given explicitly by

$$\tilde{p}_m(x) = \frac{C_m(x/\varrho(B))}{C_m(1/\varrho(B))}, \quad m \geqq 0, \tag{2.7}$$

where

$$C_m(x) = \begin{cases} \cos(m \cos^{-1} x), & x \leqq 1, \quad m \geqq 0, \\ \cosh(m \cosh^{-1} x), & x \geqq 1, \quad m \geqq 0, \end{cases} \tag{2.8}$$

is the Chebyshev polynomial of degree m. Since the Chebyshev polynomials satisfy the well-known recurrence relation

$$C_{m+1}(x) = 2x C_m(x) - C_{m-1}(x), \qquad m \geqq 1, \tag{2.8'}$$

where $C_0(x) = 1$, $C_1(x) = x$, we can use (2.8') to deduce a recurrence relation for the polynomials $\tilde{p}_m(x)$ which, when inserted into (2.4'), leads to the following

* This is called "linear acceleration" by FORSYTHE [5]. Professor A. H. TAUB has kindly pointed out to us that these results were known much earlier to VON NEUMANN. See [2].

** If B is known to be non-negative, irreducible, and primitive, then the smallest interval $a \leqq x \leqq b$ which contains the eigenvalues of B is such [23] that $|a| < b = \varrho(B)$. While this change in the problem of (2.6) would result in improved convergence rates, it is in general difficult to obtain the lower bound in practical problems.

relationship★ for the vectors $\vec{\beta}^{(i)}$:

$$\vec{\beta}^{(i+1)} = \omega_{i+1}\{B\vec{\beta}^{(i)} + \vec{g} - \vec{\beta}^{(i-1)}\} + \vec{\beta}^{(i-1)}, \qquad i \geqq 1, \tag{2.9}$$

where

$$\omega_{i+1} = \frac{2C_i(1/\varrho(B))}{\varrho(B)\, C_{i+1}(1/\varrho(B))}, \qquad i \geqq 1, \quad \omega_1 = 1. \tag{2.10}$$

With $\omega_1 = 1$, (2.9) reduces to $\vec{\beta}^{(1)} = B\vec{\beta}^{(0)} + \vec{g} = B\vec{\alpha}^{(0)} + \vec{g}$, since $\vec{\beta}^{(0)} = \vec{\alpha}^{(0)}$. Using (2.8′), we can also express the parameters ω_{i+1} as

$$\omega_{i+1} = \frac{1}{1 - \left(\frac{\varrho^2\omega_i}{4}\right)}, \qquad i \geqq 2, \quad \omega_1 = 1, \quad \omega_2 = \frac{2}{2-\varrho^2}, \tag{2.11}$$

which is more convenient for actual computations. From (2.9), we notice that the determination of vector iterates $\vec{\beta}^{(i)}$ does not require the computation or storage of the auxiliary vector iterates $\vec{\alpha}^{(i)}$ of (2.1).

Having described the Chebyshev semi-iterative method, we now consider the successive overrelaxation iterative method of Young and Frankel [*6, 26*], applied to the matrix equation (1.2) where B is primitive. Without making further assumptions on the matrix B, such as B having entries only of one sign [*10*], successive overrelaxation applied directly to (2.1) has not as yet been completely rigorously analysed. We now show that by considering matrix equations with twice as many components, successive overrelaxation can be rigorously applied to a system of equations derived from (1.2). From (1.2), we consider the coupled pair of matrix equations

$$\begin{cases} \vec{x} = B\vec{y} + \vec{g} \\ \vec{y} = B\vec{x} + \vec{g}, \end{cases} \tag{2.12}$$

which in matrix notation becomes

$$\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} + \begin{pmatrix} \vec{g} \\ \vec{g} \end{pmatrix}. \tag{2.12′}$$

If

$$J = \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}, \tag{2.13}$$

then the matrix J is also convergent, and $\varrho(J) = \varrho(B)$. Since $\varrho(B) < 1$, there is a unique solution of (2.12′), and evidently $\vec{x} = \vec{y}$.

The successive overrelaxation iterative method applied to (2.12′) is defined by

$$\begin{cases} \vec{x}^{(m+1)} = \vec{x}^{(m)} + \omega\{B\vec{y}^{(m)} + \vec{g} - \vec{x}^{(m)}\}, \\ \vec{y}^{(m+1)} = \vec{y}^{(m)} + \omega\{B\vec{x}^{(m+1)} + \vec{g} - \vec{y}^{(m)}\}, \qquad m \geqq 0, \end{cases} \tag{2.14}$$

where ω is the *relaxation factor*, and $\vec{x}^{(0)}$, $\vec{y}^{(0)}$ are initial guess vectors. Since the matrix J of (2.13) is evidently real, symmetric, cyclic, and consistently ordered in the sense of Young [*26*], then we can apply the general theory of successive

★ This is a somewhat simpler computational form of the recurrence relationship than is found, say, in [*5′, 16, 18*].

overrelaxation due to YOUNG [26] to (2.12′), and the optimum value of ω is given by

$$\omega_b = \frac{2}{1+\sqrt{1-\varrho^2(J)}} = \frac{2}{1+\sqrt{1-\varrho^2(B)}}. \tag{2.15}$$

To show the similarity of (2.14) to (2.9), we now define a sequence of vectors $\vec{\zeta}^{(j)}$, where

$$\begin{cases} \vec{\zeta}^{(2l)} = \vec{x}^{(l)}, \\ \vec{\zeta}^{(2l+1)} = \vec{y}^{(l)}, \qquad l \geqq 0. \end{cases} \tag{2.16}$$

In terms of the vectors $\vec{\zeta}^{(j)}$, we can write (2.14) in the compact form

$$\vec{\zeta}^{(m+1)} = \omega\{B\vec{\zeta}^{(m)} + \vec{g} - \vec{\zeta}^{(m-1)}\} + \vec{\zeta}^{(m-1)}, \qquad m \geqq 1, \tag{2.17}$$

where $\vec{\zeta}^{(0)}$, and $\vec{\zeta}^{(1)}$ are given vectors guesses. Thus, we conclude that the successive overrelaxation iterative method applied to (2.12′) is in the same form as the Chebyshev semi-iterative method of (2.9), except that in (2.9) the relaxation factors vary with iteration, whereas in (2.17) the relaxation factor ω is fixed. Even more interesting is the fact that the numbers ω_i of (2.11) are strictly decreasing for $i \geqq 2$ $(0 < \varrho(B) < 1)$, and, as can be readily exhibited,

$$\lim_{m\to\infty} \omega_m = \frac{2}{1+\sqrt{1-\varrho^2(B)}} = \omega_b, \tag{2.18}$$

where ω_b is defined in (2.15).

We now consider the second order Richardson iterative method [6, 13], which is defined from (1.2) by

$$\vec{\eta}^{(m+1)} = \vec{\eta}^{(m)} + \alpha\{B\vec{\eta}^{(m)} + \vec{g} - \vec{\eta}^{(m)}\} + \beta\{\vec{\eta}^{(m)} - \vec{\eta}^{(m-1)}\}, \qquad m \geqq 1, \tag{2.19}$$

where $\vec{\eta}^{(0)}$, $\vec{\eta}^{(1)}$ are given initial vector guesses to the unique solution $\vec{x}$ of (1.2), and α and β are fixed acceleration parameters. If $\beta' = \beta - \alpha$, this is equivalent to

$$\vec{\eta}^{(m+1)} = \vec{\eta}^{(m)} + \alpha\{B\vec{\eta}^{(m)} + \vec{g} - \vec{\eta}^{(m-1)}\} + \beta'\{\vec{\eta}^{(m)} - \vec{\eta}^{(m-1)}\}, \qquad m \geqq 1. \tag{2.19′}$$

One can extend the analysis of FRANKEL and RILEY [6, 26], and the best acceleration parameters, those giving the fastest asymptotic convergence, are given* by

$$\alpha = \frac{2}{1+\sqrt{1-\varrho^2(B)}}, \qquad \beta' = -1. \tag{2.20}$$

With this choice of parameters α and β', we see that the second order Richardson iterative method of (2.19′) is identical with the successive overrelaxation iterative method of (2.17), with $\omega = \omega_b$. Of course, RILEY [13] pointed out this correspondence in the case of the numerical solution of the Dirichlet problem.

Having compared three systematic iterative methods for solving (1.2) when B is primitive, we see that each method, when optimized, bears a strong resemblance to the other methods. In the next section, we shall compare these three iterative methods using the spectral norms of the corresponding matrix operators as a basis for comparison.

* See also [22, p. 485].

§ 3. Primitive Matrices: Comparison of Methods

As in the previous section, we assume that B is real, primitive, and symmetric, with $\{\vec{y}_i\}_{i=1}^N$ as an orthonormal set of eigenvectors of B. If, as in § 2, $\vec{\varepsilon}^{(0)} = \sum_{k=1}^N c_k \vec{y}_k$, then Euclidean norm of $\vec{\varepsilon}^{(0)}$ is defined by

$$\|\vec{\varepsilon}^{(0)}\| = \left(\sum_{k=1}^N |c_k|^2\right)^{\frac{1}{2}}. \tag{3.1}$$

To review some facts [9] concerning norms of matrices and vectors, let M be any real $N \times N$ matrix. For any real vector $\vec{x}$ with N components, then from $\vec{y} = M\vec{x}$, we have

$$\|\vec{y}\| = \|M\vec{x}\| \leqq \tau(M)\, \|\vec{x}\|, \tag{3.2}$$

where

$$\tau(M) \equiv [\varrho(M^T M)]^{\frac{1}{2}}. \tag{3.3}$$

The quantity $\tau(M)$ is called the *spectral norm* of the matrix M. Characterized in a different manner, we have that

$$\tau(M) = \max_{\|\vec{x}\| \neq 0} \frac{\|M\vec{x}\|}{\|\vec{x}\|}. \tag{3.4}$$

It is clear that if M is symmetric, then the spectral norm $\tau(M)$ of M coincides with the spectral radius $\varrho(M)$ of M.

For the Chebyshev semi-iterative method of (2.9), we have that $\vec{\overset{*}{\varepsilon}}^{(0)} = \vec{\varepsilon}^{(0)}$, and $\vec{\overset{*}{\varepsilon}}^{(m)} = \tilde{p}_m(B)\, \vec{\varepsilon}^{(0)}$. Thus,

$$\|\vec{\overset{*}{\varepsilon}}^{(m)}\| \leqq \tau(\tilde{p}_m(B))\, \|\vec{\varepsilon}^{(0)}\|, \qquad m \geqq 0. \tag{3.5}$$

Since the matrix B is symmetric, so is the matrix $\tilde{p}_m(B)$, and we can express $\tau(\tilde{p}_m(B))$ as

$$\tau(\tilde{p}_m(B)) = \varrho(\tilde{p}_m(B)) = \max_{1 \leqq i \leqq N} |\tilde{p}_m(\mu_i)|. \tag{3.6}$$

From (2.7) and (2.8), it follows that

$$\tau(\tilde{p}_m(B)) = \frac{\max_{1 \leqq i \leqq N} |C_m(\mu_i/\varrho(B))|}{C_m(1/\varrho(B))}, \qquad m \geqq 0, \tag{3.7}$$

and from (1.3) and (2.8),

$$\tau(\tilde{p}_m(B)) = \frac{1}{C_m(1/\varrho(B))}, \qquad m \geqq 0. \tag{3.7'}$$

To simplify the expression in (3.7′), we recall from (2.8) that $C_m(1/\varrho(B)) = \cosh(m\sigma)$, where $\cosh\sigma = \frac{1}{\varrho(B)}$, and $\varrho(B) < 1$. Thus, $C_m(1/\varrho(B)) = e^{m\sigma}\left(\frac{1 + e^{-2m\sigma}}{2}\right)$, and since $\sigma = \ln\left\{\frac{1}{\varrho(B)} + \sqrt{\frac{1}{\varrho^2(B)} - 1}\right\}$, we have that $e^{-m\sigma} = \left\{\frac{\varrho(B)}{1 + \sqrt{1 - \varrho^2(B)}}\right\}^m$. From Young's basic formula [26] we can relate $e^{-m\sigma}$ to the quantity ω_b in (2.15), and we have that $e^{-m\sigma} = (\omega_b - 1)^{m/2}$. Combining, we now write (3.7′) as

$$\tau(\tilde{p}_m(B)) = (\omega_b - 1)^{m/2} \cdot \left\{\frac{2}{1 + (\omega_b - 1)^m}\right\}, \qquad m \geqq 0. \tag{3.8}$$

Since $\frac{2x}{1 + x^2} < 1$ for $0 \leqq x < 1$, it follows that the right side of (3.8) is less than unity, and is strictly decreasing with increasing m. Thus, we conclude that the matrix operator $\tilde{p}_m(B)$ for the Chebyshev semi-iterative method is *norm reducing* for all $m \geqq 1$.

For the successive overrelaxation iterative method, or equivalently for the second order Richardson iterative methods with $\alpha=\omega$ and $\beta=-1$, we have the following recurrence relation for the error vectors of the iterates of (2.17):

$$\vec{\varepsilon}^{(m+1)}=\omega B\vec{\varepsilon}^{(m)}+(1-\omega)\vec{\varepsilon}^{(m-1)},\qquad m\geqq 1, \tag{3.9}$$

where $\vec{\varepsilon}^{(m)}=\vec{x}-\vec{\zeta}^{(m)}$, $m\geqq 0$, so that $\vec{\varepsilon}^{(0)}$ and $\vec{\varepsilon}^{(1)}$ are dependent on the given vectors $\vec{\zeta}^{(0)}$ and $\vec{\zeta}^{(1)}$. If $\alpha_0(B)=I$, and $\alpha_1(B)=\omega B$, we define now the polynomials $\alpha_m(B)$ from the recurrence relation

$$\alpha_{m+1}(B)=\omega B\alpha_m(B)+(1-\omega)\alpha_{m-1}(B),\qquad m\geqq 1. \tag{3.10}$$

By induction, $\alpha_m(B)$ is, for $\omega\neq 0$, a polynomial of degree m in B, and it is easily verified that

$$\vec{\varepsilon}^{(m)}=\alpha_{m-1}(B)\vec{\varepsilon}^{(1)}+(1-\omega)\alpha_{m-2}(B)\vec{\varepsilon}^{(0)},\qquad m\geqq 2. \tag{3.11}$$

Upon replacing the matrix B by the variable x in (3.10), the linear difference equation of (3.10) can be solved, and $\alpha_m(x)$ can be explicitly represented by

$$\alpha_m(x)=\left\{\begin{array}{ll}\dfrac{\varphi_1^{m+1}(x)-\varphi_2^{m+1}(x)}{\varphi_1(x)-\varphi_2(x)}, & \varphi_1(x)\neq\varphi_2(x)\\ (m+1)\,\varphi_1^m(x), & \varphi_1(x)=\varphi_2(x)\end{array}\right\},\qquad m\geqq 0, \tag{3.12}$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are the roots of the equation

$$\varphi^2(x)-\omega x\varphi(x)+(\omega-1)=0. \tag{3.13}$$

If $\omega=\omega_b$ of (2.15), and $-\varrho(B)\leqq x\leqq+\varrho(B)$, then as shown in [6], all the roots of (3.13) are complex conjugates of the form $(\omega_b-1)^{\frac{1}{2}}e^{\pm i\vartheta}$, where $\cos\vartheta=x/\varrho(B)$. Consequently,

$$\alpha_m(x)=(\omega_b-1)^{m/2}\cdot\left\{\begin{array}{ll}\dfrac{\sin(m+1)\vartheta}{\sin\vartheta}, & \vartheta\neq 0,\pi,\\ m+1, & \vartheta=0,\\ (-1)^m(m+1), & \vartheta=\pi,\end{array}\right\},\qquad m\geqq 0. \tag{3.14}$$

It is clear from (3.11) that $\vec{\varepsilon}^{(m)}$ depends on the relationship between $\vec{\varepsilon}^{(0)}$ and $\vec{\varepsilon}^{(1)}$. For example, if $\vec{\varepsilon}^{(1)}=-\vec{\varepsilon}^{(0)}$ and $\omega=\omega_b$, then

$$\vec{\varepsilon}^{(m)}=q_{m-1}(B)\vec{\varepsilon}^{(0)},\qquad m\geqq 2, \tag{3.15}$$

where

$$q_{m-1}(B)=-[\alpha_{m-1}(B)+(\omega_b-1)\alpha_{m-2}(B)],\qquad m\geqq 2. \tag{3.16}$$

From the symmetry of the matrix B, we have that

$$\tau[q_{m-1}(B)]=\varrho[q_{m-1}(B)]=\max_{1\leqq i\leqq N}|q_{m-1}(\mu)|, \tag{3.17}$$

but from (3.14), we can directly express $\tau[q_{m-1}(B)]$ as

$$\tau[q_{m-1}(B)]=(\omega_b-1)^{\frac{m-1}{2}}\{|m|+|m-1|(\omega_b-1)^{\frac{1}{2}}\},\qquad m\geqq 0. \tag{3.17'}$$

We note that this spectral norm can actually initially increase with m, if ω_b is sufficiently large*.

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* An analogous observation was made by SHELDON [15] in the cyclic case. See also § 5.

We shall now show that the situation of (3.17′) can be considerably improved. Let

$$\vec{\zeta}^{(1)} = B\vec{\zeta}^{(0)} + \vec{g}, \tag{3.18}$$

so that

$$\vec{\varepsilon}^{(1)} = B\vec{\varepsilon}^{(0)}. \tag{3.19}$$

For $\vec{\zeta}^{(1)}$ selected in this way, then

$$\vec{\varepsilon}^{(m)} = r_m(B)\,\vec{\varepsilon}^{(0)}, \qquad m \geqq 2, \tag{3.20}$$

where $r_m(B)$ is a polynomial of degree m in B defined recursively, using (3.11) and (3.19), by

$$r_m(B) = B\,\alpha_{m-1}(B) + (1-\omega)\,\alpha_{m-2}(B), \qquad m \geqq 2, \tag{3.21}$$

with $r_0(B) = I$, and $r_1(B) = B$.

Again,

$$\tau[r_m(B)] = \varrho[r_m(B)] = \max_{1 \leqq i \leqq N} |r_m(\mu_i)|, \qquad m \geqq 0, \tag{3.22}$$

and, for the case $\omega = \omega_b$, a short calculation [7, pp. 22—23] based on (3.14) shows that

$$\tau[r_m(B)] = |r_m\,\varrho(B)| = (\omega_b - 1)^{m/2}\left\{1 + m\sqrt{1-\varrho^2(B)}\right\}, \qquad m \geqq 0. \tag{3.23}$$

It is readily verified [7, pp. 23—24] that the right side of (3.23) is monotone decreasing for all $m \geqq 0$, showing that the matrix operator for the successive overrelaxation iterative method of (2.17) is also norm reducing for $m \geqq 1$ with $\vec{\zeta}^{(1)}$ chosen according to (3.18).

While the Chebyshev semi-iterative method of (2.9) requires but one vector guess $\vec{\beta}^{(0)} = \vec{\zeta}^{(0)}$, (2.9) shows that $\vec{\beta}^{(1)}$ also satisfies (3.18), so that we can directly compare the spectral norm (3.8) of the Chebyshev semi-iterative method with the spectral norm (3.23) of the primitive successive overrelaxation iterative with $\vec{\zeta}^{(1)}$ chosen according to (3.18) method. Now, since $r_0(x) = 1$, and $r_1(x) = x$, it follows easily from (3.21) and (3.10) that $r_m(1) = 1$ for all $m \geqq 0$. But from (2.7), the same is true of the polynomials $\tilde{p}_m(x)$. Moreover, since $\tau[r_m(B)] = |r_m(\varrho(B))|$, and similarly $\tau[\tilde{p}_m(B)] = |\tilde{p}_m(\varrho(B))|$, we can use, as in (2.6), the well known property of the Chebyshev polynomials that among all polynomials $g_m(x)$ of degree m with $g_m(1) = 1$, $\tilde{p}_m(x)$ is the *unique* polynomial whose maximum absolute value on the interval $-\varrho(B) \leqq x \leqq +\varrho(B)$ is minimal. This gives us

Lemma 1. In the primitive case where $\vec{\zeta}^{(0)}$ is arbitrary and $\vec{\zeta}^{(1)} \equiv B\vec{\zeta}^{(0)} + \vec{g}$,

$$\tau(\tilde{p}_m(B)) < \tau(r_m(B)), \qquad m > 1. \tag{3.24}$$

We shall now consider the successive overrelaxation iterative method with another starting procedure*. Let

$$\begin{aligned} \vec{\zeta}^{(1)} &= B\vec{\zeta}^{(0)} + \vec{g} \\ \vec{\zeta}^{(2)} &= B\vec{\zeta}^{(1)} + \vec{g}, \end{aligned} \tag{3.25}$$

* Such a starting procedure is suggested for the primitive case from results in the cyclic case by SHELDON [*12*]. See also § 5.

and $\vec{\zeta}^{(m)}$ $(m \geqq 3)$ we generated by (2.17). Then

$$\vec{\varepsilon}^{(1)} = B\,\vec{\varepsilon}^{(0)}, \qquad \vec{\varepsilon}^{(2)} = B^2\,\vec{\varepsilon}^{(0)}, \tag{3.26}$$

and in general

$$\vec{\varepsilon}^{(m)} = t_m(B)\,\vec{\varepsilon}^{(0)}, \qquad m \geqq 0, \tag{3.27}$$

where the matrix $t_m(B)$ is a polynomial of degree m in the matrix B, and is defined recursively by

$$t_m(B) = B^2\,\alpha_{m-2}(B) + (1-\omega)\,B\,\alpha_{m-3}(B), \qquad m \geqq 3, \tag{3.28}$$

where $t_j(B) = B^j$ for $0 \leqq j \leqq 2$. Again,

$$\tau\,[t_m(B)] = \varrho\,[t_m(B)] = \max_{1 \leqq i \leqq N} |t_m(\mu_i)|, \qquad m \geqq 0, \tag{3.29}$$

and for $\omega = \omega_b$, a short calculation based on (3.14) shows that

$$\tau\,[t_m(B)] = |t_m(\varrho(B))| = (\omega_b - 1)^{\frac{m-1}{2}}\,\varrho(B)\left\{1 + (m-1)\sqrt{1-\varrho^2(B)}\right\}, \quad m \geqq 1. \tag{3.30}$$

With $s_m(t) \equiv t^m$, corresponding to the basic iterative method of (2.1), it is not difficult to show that

$$\tau\left(r_m(B)\right) < \tau\left(t_m(B)\right) < \tau\left(s_m(B)\right) = \varrho^m(B) \quad \text{for} \quad m > 1. \tag{3.31}$$

Consequently, we have

Theorem 1. In the primitive case where $\vec{\zeta}^{(0)}$ is arbitrary and $\vec{\zeta}^{(1)} = B\,\vec{\zeta}^{(0)} + \vec{g}$, then for $0 < \varrho(B) < 1$, and $m > 1$,

$$\tau\left(\tilde{p}_m(B)\right) < \tau\left(r_m(B)\right) < \tau\left(t_m(B)\right) < \tau\left(s_m(B)\right) = \varrho^m(B). \tag{3.32}$$

Thus, the spectral norm of the matrix operator for $m > 1$ iterations of the Chebyshev semi-iterative method is less than the spectral norms of the matrix operators for m iterations of the two variants (3.18) and (3.25) of the successive overrelaxation iterative method, as well as the spectral norms for m iterations of the iterative method of (2.1).

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