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REAL VS. COMPLEX RATIONAL CHEBYSHEV APPROXIMATION ON AN INTERVAL

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Dedicated to Professor A. Sharma on his retirement from the University of Alberta

1. Introduction. Let π_m^r and π_m^c be respectively the sets of polynomials of degree at most m, with real and complex coefficients. For any pair (m,n) of nonnegative integers, $\pi_{m,n}^r$ and $\pi_{m,n}^c$, then respectively denote the sets of rational functions of the form $p(x)/q(x)$, where $p \in \pi_m^r(\pi_m^c)$ and where $q \in \pi_n^r(\pi_n^c)$. Let I denote the real interval $[-1,+1]$ and let $||\cdot||_I$ denote the supremum norm on I, i.e., $||f||_I := \sup_{x\in I}|f(x)|$. If $C^r(I)$ denotes the set of all continuous real-valued functions on I, then for $f \in C^r(I)$, we set

$$(1.1)\qquad \begin{aligned} E_{m,n}^r(f) &:= \inf\{||f-g||_I : g \in \pi_{m,n}^r\},\\ E_{m,n}^c(f) &:= \inf\{||f-g||_I : g \in \pi_{m,n}^c\}. \end{aligned}$$

For $f \in C^r(I)$, it is known (cf. Meinardus [**3**, p. 161]) that there is a *unique* $g \in \pi_{m,n}^r$ such that $E_{m,n}^r(f) = ||f-g||_I$, while in the complex case, there is also a $g \in \pi_{m,n}^c$ for which $E_{m,n}^c(f) = ||f-g||_I$, but g is in general *not unique* (cf. Lungu [**2**], Saff and Varga [**4**], and [**6**].)

Since $\pi_{m,n}^r \subset \pi_{m,n}^c$, then evidently $E_{m,n}^c(f) \le E_{m,n}^r(f)$ for any $f \in C^r(I)$, and it was shown in [**4**] that, for each (m,n) with $n \ge 1$, there is an $f \in C^r(I)$ for which

$$(1.2)\qquad E_{m,n}^c(f)/E_{m,n}^r(f) < 1.$$

Thus, on setting

$$(1.3)\qquad \gamma_{m,n} := \inf\{E_{m,n}^c(f)/E_{m,n}^r(f) : f \in C^r(I)/\pi_{m,n}^r\},$$

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Saff and Varga [4] asked in essence how *small* the ratios of (1.2) could be for each pair (m, n) of nonnegative integers with $n \geq 1$.

Recently, two major results on the precise determination of $\gamma_{m,n}$ have appeared. First, Trefethen and Gutknecht [5] established, by means of a direction construction, the surprising result that

$$\gamma_{m,n} = 0, \text{ for each pair } (m,n) \text{ of nonnegative integers with } n \geq m+3. \tag{1.4}$$

Then, Levin [1] established the complementary result that

$$\gamma_{m,n} = \frac{1}{2}, \text{ for each pair } (m,n) \text{ of nonnegative integers with } m+1 \geq n \geq 1. \tag{1.5}$$

Levin's proof of (1.5) consisted of a direction construction to show that $\gamma_{m,n} \leq 1/2$, and an algebraic method to show that $\gamma_{m,n} < 1/2$ was impossible when $m + 1 \geq n \geq 1$.

Thus, to complete the precise determination of *all* $\gamma_{m,n}$ $(m \geq 0, n \geq 1)$, it remains only to determine the $\gamma_{m,n}$'s on the "missing diagonal", i.e., $\gamma_{m,m+2} (m \geq 0)$. It turns out that Levin's direct construction applies also in this case, so that

$$\gamma_{m,m+2} \leq \frac{1}{2}, \text{ for each integer } m \geq 0. \tag{1.6}$$

(We remark that some mathematicians have privately speculated that $\gamma_{m,m+2} = 0$ for each $m \geq 0$.)

Our object here is to show that

$$\gamma_{m,m+2} \leq \frac{1}{3}, \text{ for each integer } m \geq 0, \tag{1.7}$$

which improves (1.6). What may be of independent interest is that our direct construction to establish (1.7) is quite *different* from the direction constructions of Trefethen and Gutknecht [5] and Levin [1].

2. Main result. We have the

THEOREM. *For each nonnegative integer* m,

$$\gamma_{m,m+2} \leq \frac{1}{3}. \tag{2.1}$$

PROOF. First, suppose that m is an arbitrary (but fixed) *even* nonnegative integer, and suppose that ε is any number satisfying $0 < \varepsilon < 1/(m+1)$. For any complex number z, set

$$\ell_j(z) = \ell_j(z;\varepsilon,m) := \frac{\frac{-2\varepsilon i}{3}(-1)^j}{z-1+\frac{2j}{m+1}-\varepsilon i}, \quad j = 0, 1, \ldots, m+1. \tag{2.2}$$

It is evident from (2.2) that

$$\ell_j\left(1-\frac{2j}{m+1}\right) = \frac{2}{3}(-1)^j, \text{ and } \ell_j\left(1-\frac{2j}{m+1}\pm\varepsilon\right) = \frac{(1\mp i)(-1)^j}{3}, \tag{2.3}$$

for $j = 0, 1, \ldots, m+1$.

Since $\ell_j(z)$ is a linear fractional transformation, it maps the real axis $-\infty < x < +\infty$ onto some (generalized) circle in the complex plane. As $\ell_j(\infty) = 0$, this (generalized) circle necessarily passes through the origin. Moreover, as the pole of $\ell_j(z)$, namely $1 - \frac{2j}{m+1} + \varepsilon i$, when reflected in the real axis, is the point $w_j := 1 - \frac{2j}{m+1} - \varepsilon i$, then from (2.2),

$$\ell_j(w_j) = \frac{1}{3}(-1)^j, \quad j = 0, 1, \ldots, m+1.$$

Thus, the image of the real axis under $\ell_j(z)$ is the circle with center $\frac{1}{3}(-1)^j$ and radius $1/3$ (since this circle passes through the origin). It is then geometrically clear that

$$\|\ell_j\|_{(-\infty,+\infty)} = \frac{2}{3}, \text{ and } \|\operatorname{Im} \ell_j\|_{(-\infty,+\infty)} = \frac{1}{3}, \quad j = 0, 1, \ldots, m+1, \tag{2.4}$$

where, for any subset K of the infinite interval $(-\infty, +\infty)$, we use the notation $\|f\|_K := \sup_{x\in K} |f(x)|$.

To extend the statements of (2.4), consider the real intervals $I_k(m)$, defined by

$$I_k(m) := \left[1-\frac{2k+1}{m+1}, 1-\frac{2k-1}{m+1}\right] \cap I, \quad k = 0, 1, \ldots, m+1, \tag{2.5}$$

so that these intervals cover $I := [-1, +1]$; that is,

$$\cup_{k=1}^{m+1} I_k(m) = I.$$

From the definitions of $\ell_j(x)$ and $I_k(m)$, it follows (as m is fixed) that

$$(2.6) \qquad ||\ell_j||_{I_k(m)} = O(\varepsilon), \text{ as } \varepsilon \to 0 \quad (k \neq j),$$

and from (2.3) that

$$(2.7) \quad ||\ell_j||_{I_j(m)} = \frac{2}{3}, \text{ and } ||\mathrm{Im}\, \ell_j||_{I_j(m)} = \frac{1}{3}, \quad j = 0, 1, \ldots, m+1.$$

Next, consider the complex rational function $g(x)$ defined by

$$(2.8) \qquad g(x) = g(x; \varepsilon, m) := \sum_{j=0}^{m+1} \ell_j(x).$$

On rationalizing $g(x)$,

$$(2.9) \qquad g(x) = \frac{\frac{-2\varepsilon i}{3} \sum_{j=0}^{m+1} (-1)^j \prod_{\substack{k=0\\k\neq j}}^{m+1} \{x - 1 + \frac{2k}{m+1} - \varepsilon i\}}{\prod_{k=0}^{m+1} \{x - 1 + \frac{2k}{m+1} - \varepsilon i\}},$$

so that g is at least an element of $\pi^c_{m+1,m+2}$. However, the numerator of $g(x)$ of (2.9) is

$$\frac{-2\varepsilon i}{3}\left\{x^{m+1} \sum_{j=0}^{m+1} (-1)^j + \text{ lower terms in } x^s (0 \leq s \leq m)\right\}.$$

But, since m is assumed even, it follows that $\sum_{j=0}^{m+1} (-1)^j = 0$, which shows that $g(x)$ is an element in $\pi^c_{m,m+2}$. More precisely, it can be verified from the above definition that the coefficient of X^m in the numerator of $g(x)$ is

$$\frac{2(m+2)\varepsilon i}{3(m+1)} \neq 0,$$

so that $g(x)$ is not an element of $\pi_{s,m+2}$ for any $s < m$. (We remark that the representation of $g(x)$ in (2.8) is just the *partial fraction decomposition* of $g(x)$.)

Consider now the real continuous function $s(u)$ on $(-\infty, +\infty)$ defined by

$$(2.10) \qquad s(u) := \begin{cases} \frac{1-u^2}{1+u^2}, & -1 \leq u \leq +1, \\ 0, & \text{otherwise,} \end{cases}$$

so that $s(0) = 1, s(\pm 1) = 0$, and $0 < s(u) < 1$ for $0 < |u| < 1$. Recalling that $0 < \varepsilon < 1/(m+1)$, set

$$(2.11) \qquad S(x) := \frac{1}{3} \sum_{j=0}^{m+1} (-1)^j s\Big(\frac{x - 1 + \frac{2j}{m+1}}{\varepsilon}\Big), \quad -\infty < x < \infty.$$

It follows from (2.11) that $S(x)$ is a real continuous function on $(-\infty, +\infty)$, with

$$(2.12) \qquad S\Big(1 - \frac{2j}{m+1}\Big) = \frac{1}{3}(-1)^j \text{ and } S\Big(1 - \frac{2j}{m+1} \pm \varepsilon\Big) = 0, \quad j = 0, 1, \ldots, m+1.$$

Geometrically, we note that $S(x)$ has $m + 2$ alternating *spikes* on $I := [-1, +1]$.

With the above definition of $S(x)$ and $g(x)$, set

$$(2.13) \qquad f(x) = f(x; \varepsilon, m) := S(x) + \operatorname{Re} g(x) \quad (x \in I),$$

so that $f(x) \in C^r(I)$. From (2.3), (2.6), (2.8), and (2.12),

$$(2.14) \quad f\Big(1 - \frac{2j}{m+1}\Big) = (-1)^j + O(\varepsilon), \text{ as } \varepsilon \to 0 \quad (j = 0, 1, \ldots, m+1).$$

Now, for $\varepsilon > 0$ small, (2.14) asserts that $f(x)$ has $m + 2$ near "alternants" in the distinct points $\{1 - \frac{2j}{m+1}\}_{j=0}^{m+1}$ of I. On choosing the identically zero function in $\pi^r_{m,m+2}$, an application of the de la Vallée-Poussin Theorem (cf. Meinardus [**3**, p. 161]) gives us that

$$(2.15) \qquad E^r_{m,m+2}(f) = 1 + O(\varepsilon), \text{ as } \varepsilon \to 0.$$

To determine an upper bound for $E^c_{m,m+2}(f)$, note from (2.13) that

$$(2.16) \qquad f(x) - g(x) = S(x) - i \operatorname{Im} g(x) \quad (x \in I).$$

On considering the particular interval $I_k(m)$, it follows from (2.6)-(2.7) that

$$S(x) - i\mathrm{Im}\, g(x) = S(x) - i\mathrm{Im}\, \ell_k(x) + O(\varepsilon), \quad x \in I_k(m). \tag{2.17}$$

Moreover, a short calculation shows that

$$||S(x) - i\mathrm{Im}\, l_k(x)||_{I_k(m)} = \frac{1}{3} + O(\varepsilon), \quad k = 0, 1, \ldots, m+1,$$

so that with (2.16) and (2.6),

$$||f - g||_I = ||S - i\mathrm{Im}\, g||_I = \frac{1}{3} + O(\varepsilon). \tag{2.18}$$

Then, since $g(x)$ is an element of $\pi^c_{m,m+2}$.

$$E^c_{m,m+2}(f) \le ||f - g||_I = \frac{1}{3} + O(\varepsilon), \text{ as } \varepsilon \to 0, \tag{2.19}$$

from (1.1) and (2.18). With (2.15), we see that $E^c_{m,m+2}(f)/E^r_{m,m+2}(f) \le 1/3 + O(\varepsilon)$. Letting $\varepsilon \to 0$ then gives

$$\gamma_{m,m+2} \le \frac{1}{3}, \tag{2.20}$$

which establishes the desired result of (2.7) when m is an even nonnegative integer.

For the case when m is an odd positive integer, the above discussion is easily modified. Set

$$\ell_j(z) = \ell_j(z, \varepsilon, m) := \frac{\frac{-2\varepsilon i}{3}\mu_j(-1)^j}{z - 1 + \frac{2j}{m+1} - \varepsilon\mu_j i}, \; j = 0, 1, \ldots, m+1, \tag{2.21}$$

where $\{\mu_j\}_{j=0}^{m+1}$ are any $m+2$ *fixed* positive numbers satisfying $0 \le \mu_j < 1$ and

$$\sum_{j=0}^{m+1} (-1)^j \mu_j = 0, \text{ and } \sum_{j=0}^{m+1} j(-1)^j_{\mu_j} \neq 0. \tag{2.22}$$

With (2.22), it follows that $\sum_{j=0}^{m+1} \ell_j(z)$ is an element of $\pi_{m,m+2}$, but not an element of $\pi_{s,m+2}$ for any $s < m$. Then exactly the same construction can be carried out to deduce the desired result that $\gamma_{m,m+2} \leq 1/3$ in the case when m is an odd positive integer. □

To conclude, we conjecture that

$$\gamma_{m,m+2} = \frac{1}{3} \text{ for each nonnegative integer } m, \tag{2.23}$$

i.e., we conjecture that the upper bound of (2.1) is *sharp* for each nonnegative integer m. If this conjecture is true, then the "missing diagonal" $\gamma_{m,m+2}$ is, in fact, structurally different from the remaining cases treated in [**5**] and [**1**].

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