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On a Particular 2-Periodic Lacunary Trigonometric Interpolation Problem on Equidistant Nodes

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Dedicated to Professor Karl Zeller on his sixty-fifth birthday.

§1. Introduction.

For a given positive integer n, define

$$x_k = x_k(n) := \frac{k\pi}{n} \qquad (k = 0, 1, \cdots, 2n-1), \tag{1.1}$$

so that $\{x_k\}_{k=0}^{2n-1}$ is a set of $2n$ equidistant points, called *nodes*, in $[0, 2\pi)$. Next, assume that

$$\{m_j\}_{j=1}^{p} \text{ are any } p \text{ distinct positive integers } (p \geq 1). \tag{1.2}$$

We consider here the following 2-*periodic lacunary trigonometric interpolation problem*, denoted by

$$(0 =: m_0, m_1, \cdots, m_p; m_1, m_2, \cdots, m_p), \tag{1.3}$$

on the $2n$ equidistant nodes $\{x_k\}_{k=0}^{2n-1}$. For arbitrary data consisting of complex numbers $\{\alpha_{j,\nu}\}_{j=0,\nu=0}^{n-1,m_p}$ and $\{\beta_{j,\nu}\}_{j=0,\nu=1}^{n-1,m_p}$, we ask if there exists a *unique* trigonometric polynomial of the form

$$t_M(x) = a_0 + \sum_{k=1}^{M} (a_k \cos kx + b_k \sin kx) \tag{1.4}$$

or of the form

$$t_M(x) = a_0 + \sum_{k=1}^{M-1} (a_k \cos kx + b_k \sin kx) + a_M \cos\left(Mx + \frac{\varepsilon\pi}{2}\right) \tag{1.4'}$$

(where $\varepsilon = 0$ or where $\varepsilon = 1$), such that

$$\begin{cases} t_M^{(m_\nu)}(x_{2j}) &= \alpha_{j,\nu} \quad (j = 0, 1, \cdots, n-1;\ \nu = 0, 1, \cdots, p), \text{ and} \\ t_M^{(m_\nu)}(x_{2j+1}) &= \beta_{j,\nu} \quad (j = 0, 1, \cdots, n-1;\ \nu = 1, 2, \cdots, p). \end{cases} \tag{1.5}$$

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Note that as the number of nodes in (1.1) is *even* (namely, $2n$), we see that interpolation conditions of (1.5) are broken into interpolation conditions on *two* disjoint sets of n nodes, from which the term, namely, 2-*periodic* lacunary trigonometric interpolation, is derived.

By way of background, the problem of lacunary trigonometric interpolation on equidistant nodes in $[0, 2\pi)$ was initiated in 1960 by O. Kis in [Ki], and the question concerning the regularity of the more general k-*periodic* lacunary trigonometric interpolation was proposed and solved in 1981 by Sharma, Smith, and Tzimbalario in [S]. But, these necessary and sufficient conditions given in [S] involve the *nonvanishing* of a number of *determinants* of large order; in this form, these necessary and sufficient conditions are generally impractical to apply.

But, there are known results in this area, such as those of Cavaretta, Sharma, and Varga [C] and Sharma, Szabados, and Varga [SSV], which give necessary and sufficient conditions for the regularity of lacunary trigonometric and 2-periodic lacunary trigonometric interpolation respectively. These results depend only on the *number* of even and odd integers in (1.2) and on specific *choices* of ε in (1.4′). Such results seem to be more practical, as they can be easily checked and as they avoid the evaluation of determinants.

Our goal in this paper is to derive, as in [C] and [SSV], necessary and sufficient conditions for the regularity of the 2-periodic lacunary trigonometric interpolation problem (1.5), which similarly depend only on the number of even and odd integers in (1.2) and on a specific choice of ε in (1.4′)

Continuing, the total number of interpolation conditions in (1.5) is evidently

$$N := n(2p+1). \tag{1.6}$$

Thus, N is odd iff n is odd, and the sought unique trigonometric interpolant $t_M(x)$ is necessarily of the form (1.4) (which has an odd number of parameters), and in this case $M = (N-1)/2$. Similarly, N is even iff n is even, and the sought unique trigonometric interpolant $t_M(x)$ is necessarily of the form (1.4′) with $M = N/2$, where $\varepsilon(= 0 \text{ or } 1)$ is to be appropriately determined. To summarize,

$$\begin{cases} M := (N-1)/2 & \text{when} \quad N \text{ is odd, and} \\ M := N/2 & \text{when} \quad N \text{ is even.} \end{cases} \tag{1.7}$$

We say that this $(0, m_1, \cdots, m_p; m_1, \cdots, m_p)$ 2-periodic lacunary trigonometric interpolation problem is *regular* if, for *arbitrary* data, (1.5) admits a *unique* solution $t_M(x)$, where

$t_M(x)$ is of the form (1.4) when N is odd, or of the form (1.4′) when N is even.

Next, we use the notation of

$$\begin{cases} e_p := \text{number of even integers in the set } \{m_1, m_2, \cdots, m_p\}, \text{ and} \\ o_p := \text{number of odd integers in the set } \{m_1, m_2, \cdots, m_p\}, \end{cases} \tag{1.8}$$

so that

$$e_p + o_p = p. \tag{1.9}$$

Our result can be stated as

Theorem 1. *Let $\{m_j\}_{j=1}^p$ be p distinct positive integers and let $\{x_k(n)\}_{k=0}^{2n-1}$ be the $2n$ equidistant nodes in $[0, 2\pi)$ of* (1.1).

1. *If $N := n(2p+1)$ is odd, so that n is also odd, then the 2-periodic lacunary trigonometric interpolation problem* (1.5), *with $t_M(x)$ of the form* (1.4), *is regular if and only if p is even and* (cf. (1.8))

$$e_p = o_p = p/2. \tag{1.10}$$

2. *If $N := n(2p+1)$ is even, so that n is also even, then the 2-periodic lacunary trigonometric interpolation problem* (1.5), *with $t_M(x)$ of the form* (1.4′), *is regular if and only if p is even,* (1.10) *is satisfied, and $\varepsilon = 0$ in* (1.4′).

In particular, the interpolation problem (1.5) *is never regular when p is odd.*

For N odd, necessary and sufficient determinantal conditions are given, in Proposition 1 of §2, for the regularity of the associated 2-periodic interpolation problem (1.5). From Proposition 1, the necessary conditions that p is even and that (1.10) holds for regularity are derived in Corollary 1 in §2 for this case. These necessary conditions are then shown in §4 to be sufficient when N is odd.

For N even, necessary and sufficient determinantal conditions are derived in Proposition 2 of §3. From Proposition 2, the necessary conditions that p is even and that (1.10) holds for regularity are given in Corollary 2 of §3 for this case. Then, these necessary conditions are used in §5 to complete that part of the proof of Theorem 1 when N is even; this hinges on the proper choice of ε (namely, 0) in (1.4′).

Next, we give two simple examples (one for n odd and one for n even) to illustrate that the trigonometric interpolation problem (1.5) is *never* regular when p is odd. Consider first the case when n is odd and when p is odd. One such problem is $n = p = 1$ with, say, $m_1 = 2$. In this case, $N = 3$ from (1.6), and (1.3) reduces to

$$(0, 2; 2). \tag{1.11}$$

It is easy to verify that the specific trigonometric polynomial $\sin x$ and the identically zero trigonometric polynomial both satisfy (1.5) in this case of null data (i.e., $\alpha_{j,\nu} = \beta_{j,\nu} = 0$ in (1.5)), so that interpolation is clearly not unique in this case. Similarly, for the case when n is even, and p is odd, suppose that $n = 2$ and $p = 1$, so that $N = 6$ from (1.6), and (1.3) reduces in this case, for $m_1 = 1$, to

$$(0, 1; 1). \tag{1.12}$$

In this case, the specific trigonometric polynomial $1 - \cos 2x$ and the identically zero trigonometric polynomial both satisfy (1.5) in this case for null data, so that interpolation is again not unique here. Note also in this case that $M = N/2 = 3$ from (1.7), so that $1 - \cos 2x$ is of the form (1.4′) with $M = 3$, *independent* of the choice of $\varepsilon = 0$ or $\varepsilon = 1$.

To conclude this section, we remark that the determination, as in [SSV], of the associated *fundamental polynomials* for the interpolation problem of (1.5), will be carried out in a subsequent paper.

§2. Necessary and Sufficient Determinental Conditions for Regularity: N odd.

We assume here that $N := n(2p+1)$ is odd, so that n is necessarily odd, and we write $n = 2r+1$. In this case,, we have (cf. (1.7))

$$M = \frac{N-1}{2} = np + r, \text{ where } n = 2r+1, \tag{2.1}$$

and the sought trigonometric polynomials must be of the form (1.4).

The derivation of necessary and sufficient determinental conditions for a *unique* trigonometric interpolant of (1.5) is based on the elementary notion that the interpolation of null data is unique if and only if certain associated determinants are *nonzero*. To begin, we use the familiar device of identifying (in a 1-1 way) trigonometric polynomials with algebraic polynomials, through the transformation $z = e^{ix}$. Then, any trigonometric polynomial $t_M(x)$ of the form (1.4) can be expressed as

$$t_M(x) = z^{-M} q_{2M}(z), \qquad (q_{2M}(z) \in \pi_{2M}), \tag{2.2}$$

where π_s denotes the set of all complex algebraic polynomials of degree at most s. The case $\nu = 0$ of (1.5), in the case of null data, shows that $z^n - 1$ is a factor of $q_{2M}(z)$ in (2.2). Thus, with (2.1), we can express (2.2) as

$$t_M(x) = z^{-M}(z^n - 1)\sum_{\lambda=0}^{2p-1} z^{\lambda n} Q_\lambda(z) \qquad (Q_\lambda(z) \in \pi_{n-1};\ \lambda = 0, 1, \cdots, 2p-1). \tag{2.3}$$

Writing $Q_\lambda(z) := \sum_{j=0}^{n-1} a_{\lambda,j} z^j$, then $t_M(x)$ takes the form

$$t_M(x) = \sum_{j=0}^{n-1} \sum_{\lambda=0}^{2p-1} a_{\lambda,j} \left\{ z^{j+(\lambda+1)n-M} - z^{j+\lambda n - M} \right\}. \tag{2.3$'$}$$

Then since $z = e^{ix}$ implies $d/dx = izd/dz$, the remaining conditions of (1.5) for null data reduce respectively, with $z_{2k} := e^{ix_{2k}}$ and $z_{2k+1} := e^{ix_{2k+1}}$, to

$$\begin{cases} \sum_{j=0}^{n-1} z_{2k}^j \left[\sum_{\lambda=0}^{2p-1} a_{\lambda,j} \left\{ (j + (\lambda+1)n - M)^{m_\nu} - (j + \lambda n - M)^{m_\nu} \right\} \right] = 0, \\ \qquad\qquad (\nu = 1, 2, \cdots, p;\ k = 0, 1, \cdots, n-1), \\ \sum_{j=0}^{n-1} z_{2k+1}^j \left[\sum_{\lambda=0}^{2p-1} (-1)^\lambda a_{\lambda,j} \left\{ (j + (\lambda+1)n - M)^{m_\nu} + (j + \lambda n - M)^{m_\nu} \right\} \right] = 0, \\ \qquad\qquad (\nu = 1, 2, \cdots, p;\ k = 0, 1, \cdots, n-1), \end{cases} \tag{2.4}$$

where we have used the facts (cf. (1.1)) that $z_{2k}^n = 1$ and $z_{2k+1}^n = -1$. Next, if the quantity in square brackets in the first display of (2.4) is defined to be B_j, and if we set $p_{n-1}(z) := \sum_{j=0}^{n-1} B_j z^j$, then $p_{n-1}(z)$ is an element of π_{n-1} and (2.4) implies that $p_{n-1}(z_{2k}) = 0$ for $k = 0, 1, \cdots, n-1$. But as the n associated nodes $\{z_{2k}\}_{k=0}^{n-1}$ are all *distinct*, then $p_{n-1}(z) \equiv 0$; whence, $B_j = 0$ for all $j = 0, 1, \cdots, n-1$. As the same argument applies to the second display of (2.4),we thus have

$$\begin{cases} \sum_{\lambda=0}^{2p-1} a_{\lambda,j} \left\{ (j + (\lambda+1)n - M)^{m_\nu} - (j + \lambda n - M)^{m_\nu} \right\} = 0 \\ \qquad\qquad (j = 0, 1, \cdots, n-1;\ \nu = 1, 2, \cdots p), \\ \sum_{\lambda=0}^{2p-1} (-1)^\lambda a_{\lambda,j} \left\{ (j + (\lambda+1)n - M)^{m_\nu} + (j + \lambda n - M)^{m_\nu} \right\} = 0 \\ \qquad\qquad (j = 0, 1, \cdots, n-1;\ \nu = 1, 2, \cdots p). \end{cases} \tag{2.5}$$

Recalling from (2.1) that $M = np + r$ and that $n = 2r + 1$ in the case when N is odd, then on dividing out a factor of n^{m_ν} in each equation above and on setting

$$\alpha_j := \frac{j - r}{n} \qquad (j = 0, 1, \cdots, n-1), \tag{2.6}$$

we have, for each $j = 0, 1, \cdots, n-1$, that

$$\left\{\begin{array}{ll} \sum_{\lambda=0}^{2p-1} a_{\lambda,j}\left\{(\alpha_j + \lambda + 1 - p)^{m_\nu} - (\alpha_j + \lambda - p)^{m_\nu}\right\} = 0 & (\nu = 1, 2, \cdots, p), \\ \\ \sum_{\lambda=0}^{2p-1} (-1)^\lambda a_{\lambda,j}\left\{(\alpha_j + \lambda + 1 - p)^{m_\nu} + (\alpha_j + \lambda - p)^{m_\nu}\right\} = 0 & (\nu = 1, 2, \cdots, p). \end{array}\right. \tag{2.7}$$

But, for each choice of $j = 0, 1, \cdots, n-1$, the equation (2.7) represents $2p$ homogeneous linear equations in the $2p$ unknowns $\{a_{\lambda,j}\}_{\lambda=0}^{2p-1}$. If $\Delta(\alpha_j)$ denotes the determinant of order $2p$ of the coefficients of $a_{\lambda,j}$ in (2.7), then $\Delta(\alpha_j) \neq 0$ for *all* $j = 0, 1, \cdots, n-1$ if and only if $t_M(x) \equiv 0$ in (2.2). Consequently, we immediately have the result of

Proposition 1. *Let $N = n(2p+1)$ be odd, so that $n(= 2r+1)$ is also odd. Then, in order for the 2-periodic lacunary interpolation problem (1.5) to be regular on the $2n$ equidistant nodes $\{x_k\}_{k=0}^{2n-1}$ in $[0, 2\pi)$, it is necessary and sufficient that*

$$\Delta(\alpha_j) \neq 0 \qquad \left(\alpha_j := \frac{j-r}{n};\ j = 0, 1, \cdots, n-1\right), \tag{2.8}$$

where $\Delta(\alpha_j)$ denotes the determinant of order $2p$ of the coefficients of $a_{\lambda,j}$ in (2.7).

We now use the result of Proposition 1 to derive (cf. Corollary 1) particularly *necessary conditions* for regularity of the interpolation problem (1.5) when N is odd.

For convenience, first let $\Delta(\alpha)$ denote the determinant (of order $2p$) of the coefficients of $a_{\lambda,j}$ in (2.7) when α_j is replaced by the variable α. Then, border the determinant $\Delta(\alpha)$ on the *left* by the column vector (of length $2p$), defined by

$$\left((\alpha-p)^{m_1}, (\alpha-p)^{m_2}, \cdots, (\alpha-p)^{m_p}, -(\alpha-p)^{m_1}, -(\alpha-p)^{m_2} \cdots, -(\alpha-p)^{m_p}\right)^T,$$

and then add the row vector (of length $2p+1$), defined by

$$(1, 0, 0, \cdots, 0),$$

to the *top* of $\Delta(\alpha)$, thereby forming a determinant $\tilde{\Delta}(\alpha)$ of order $2p+1$. It is evident from this construction that $\Delta(\alpha) = \tilde{\Delta}(\alpha)$ for arbitrary α, so that the conditions

$$\tilde{\Delta}(\alpha_j) \neq 0 \qquad \left(\alpha_j := \frac{j-r}{2r+1};\ j = 0, 1, \cdots, 2r\right) \tag{2.8'}$$

are necessary and sufficient for the regularity of the interpolation problem (1.5) when N is odd.

To make the following determinantal operations clearer, we examine in detail the specific case $p = 2$. In this case, the determinant $\tilde{\Delta}(\alpha)$, of order 5, is, from (2.7), given by

$$\tilde{\Delta}(\alpha) = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ (\alpha-2)^{m_1} & (\alpha-1)^{m_1}-(\alpha-2)^{m_1} & \alpha^{m_1}-(\alpha-1)^{m_1} & (\alpha+1)^{m_1}-\alpha^{m_1} & (\alpha+2)^{m_1}-(\alpha+1)^{m_1} \\ (\alpha-2)^{m_2} & (\alpha-1)^{m_2}-(\alpha-2)^{m_2} & \alpha^{m_2}-(\alpha-1)^{m_2} & (\alpha+1)^{m_2}-\alpha^{m_2} & (\alpha+2)^{m_2}-(\alpha+1)^{m_2} \\ -(\alpha-2)^{m_1} & (\alpha-1)^{m_1}+(\alpha-2)^{m_1} & -\alpha^{m_1}-(\alpha-1)^{m_1} & (\alpha+1)^{m_1}+\alpha^{m_1} & -(\alpha+2)^{m_1}-(\alpha+1)^{m_1} \\ -(\alpha-2)^{m_2} & (\alpha-1)^{m_2}+(\alpha-2)^{m_2} & -\alpha^{m_2}-(\alpha-1)^{m_2} & (\alpha+1)^{m_2}+\alpha^{m_2} & -(\alpha+2)^{m_2}-(\alpha+1)^{m_2} \end{vmatrix} \tag{2.9}$$

On adding the first column of $\tilde{\Delta}(\alpha)$ to its second column, on adding the second column of the resulting determinant to the third column, etc. (which clearly leaves $\tilde{\Delta}(\alpha)$ invariant), we obtain from (2.9)

$$\tilde{\Delta}(\alpha) = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ (\alpha-2)^{m_1} & (\alpha-1)^{m_1} & \alpha^{m_1} & (\alpha+1)^{m_1} & (\alpha+2)^{m_1} \\ (\alpha-2)^{m_2} & (\alpha-1)^{m_2} & \alpha^{m_2} & (\alpha+1)^{m_2} & (\alpha+2)^{m_2} \\ -(\alpha-2)^{m_1} & (\alpha-1)^{m_1} & -\alpha^{m_1} & (\alpha+1)^{m_1} & -(\alpha+2)^{m_1} \\ -(\alpha-2)^{m_2} & (\alpha-1)^{m_2} & -\alpha^{m_2} & (\alpha+1)^{m_2} & -(\alpha+2)^{m_2} \end{vmatrix} . \tag{2.10}$$

On adding the fourth row of the above determinant to the second row and the fifth row to the third row, on taking out a factor of 2 from the resulting rows 2 and 3, on subtracting the second row from the fourth and the third row from the fifth, and on changing the signs of the last two rows, we obtain, up to a nonzero scalar which is independent of α, that

$$\tilde{\Delta}(\alpha) = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & (\alpha-1)^{m_1} & 0 & (\alpha+1)^{m_1} & 0 \\ 0 & (\alpha-1)^{m_2} & 0 & (\alpha+1)^{m_2} & 0 \\ (\alpha-2)^{m_1} & 0 & \alpha^{m_1} & 0 & (\alpha+2)^{m_1} \\ (\alpha-2)^{m_2} & 0 & \alpha^{m_2} & 0 & (\alpha+2)^{m_2} \end{vmatrix} \tag{2.11}$$

Then, on rearranging the rows and columns of $\tilde{\Delta}(\alpha)$ above, the determinant of (2.11) becomes, up to a sign change,

$$\tilde{\Delta}(\alpha) = \left|\begin{array}{ccc|cc} 1 & 1 & 1 & 1 & 1 \\ (\alpha-2)^{m_1} & \alpha^{m_1} & (\alpha+2)^{m_1} & 0 & 0 \\ (\alpha-2)^{m_2} & \alpha^{m_2} & (\alpha+2)^{m_2} & 0 & 0 \\ \hline 0 & 0 & 0 & (\alpha-1)^{m_1} & (\alpha+1)^{m_1} \\ 0 & 0 & 0 & (\alpha-1)^{m_2} & (\alpha+1)^{m_2} \end{array}\right| \tag{2.12}$$

Because of the upper block-triangular structure of the partitioned determinant in (2.12), it immediately follows that $\tilde{\Delta}(\alpha)$ can be represented as the following product of determinants:

$$\tilde{\Delta}(\alpha) = \begin{vmatrix} 1 & 1 & 1 \\ (\alpha-2)^{m_1} & \alpha^{m_1} & (\alpha+2)^{m_1} \\ (\alpha-2)^{m_2} & \alpha^{m_2} & (\alpha+2)^{m_2} \end{vmatrix} \cdot \begin{vmatrix} (\alpha-1)^{m_1} & (\alpha+1)^{m_1} \\ (\alpha-1)^{m_2} & (\alpha+1)^{m_2} \end{vmatrix}, \tag{2.13}$$

again up to a nonzero multiplicative factor which is independent of α.

For the regularity of the 2-periodic interpolation problem (1.5), we know from (2.8′) that it is necessary and sufficient that $\tilde{\Delta}(\alpha_j) \neq 0$ for *each* $\alpha_j = \frac{j-r}{2r+1}$ $(j = 0, 1, \cdots, 2r)$. Because $\alpha_r = 0$ is one such value, we consider the special case $\alpha = 0$ of (2.13). This gives us in this case $p = 2$ that

$$\tilde{\Delta}(0) = \begin{vmatrix} 1 & 1 & 1 \\ (-2)^{m_1} & 0 & 2^{m_1} \\ (-2)^{m_2} & 0 & 2^{m_2} \end{vmatrix} \cdot \begin{vmatrix} (-1)^{m_1} & 1 \\ (-1)^{m_2} & 1 \end{vmatrix}. \tag{2.14}$$

The second determinant on the right in (2.14) is just $(-1)^{m_1} - (-1)^{m_2}$, which is evidently nonzero only if the positive integers m_1 and m_2 are of *different parity*, i.e., in this case of $p = 2$, that (cf. (1.8)) $e_2 = o_2 = 1$.

Having carefully worked through the specific case $p = 2$ of (2.9), the general case follows similarly, the general analogue of (2.12) being given by the following determinant of order $2p + 1$:

$$\tilde{\Delta}(\alpha) = \left|\begin{array}{cccc|cccc} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ (\alpha-p)^{m_1} & (\alpha-p+2)^{m_1} & & (\alpha+p)^{m_1} & 0 & 0 & & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ (\alpha-p)^{m_p} & (\alpha-p+2)^{m_p} & \cdots & (\alpha+p)^{m_p} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & (\alpha-p+1)^{m_1} & (\alpha-p+3)^{m_1} & \cdots & (\alpha+p-1)^{m_1} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 & (\alpha-p+1)^{m_p} & (\alpha-p+3)^{m_p} & \cdots & (\alpha+p-1)^{m_p} \end{array}\right|, \tag{2.15}$$

up to a nonzero multiplicative factor which is independent of α, where the upper-left diagonal subdeterminant of (2.15) is of order $p+1$. As in (2.12), the upper block-triangular structure of the partitioned determinant $\tilde{\Delta}(\alpha)$ in (2.15) allows us to express $\tilde{\Delta}(\alpha)$ as the following product of two determinants:

$$\tilde{\Delta}(\alpha) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ (\alpha-p)^{m_1} & (\alpha-p+2)^{m_1} & & (\alpha+p)^{m_1} \\ \vdots & & & \vdots \\ (\alpha-p)^{m_p} & (\alpha-p+2)^{m_p} & \cdots & (\alpha+p)^{m_p} \end{vmatrix} \cdot \begin{vmatrix} (\alpha-p+1)^{m_1} & (\alpha-p+3)^{m_1} & \cdots & (\alpha+p-1)^{m_1} \\ (\alpha-p+1)^{m_2} & (\alpha-p+3)^{m_2} & & (\alpha+p-1)^{m_2} \\ \vdots & & & \vdots \\ (\alpha-p+1)^{m_p} & (\alpha-p+3)^{m_p} & \cdots & (\alpha+p-1)^{m_p} \end{vmatrix}. \tag{2.16}$$

Again, as in the special case $p = 2$ of (2.13), we consider the special case $\alpha = 0$ in (2.16), for which we know, from (2.8′), that $\tilde{\Delta}(0) \neq 0$ is a *necessary* condition for the regularity of the interpolation problem (1.5). But from the factorization in (2.16), it is then clear that a *necessary* condition for the regularity of the interpolation problem (1.5) is, for this case $\alpha = 0$, that the last determinant (of order p) of (2.16) is not zero, i.e.,

$$\begin{vmatrix} (-p+1)^{m_1} & (-p+3)^{m_1} & \cdots & (p-1)^{m_1} \\ (-p+1)^{m_2} & (-p+3)^{m_2} & & (p-1)^{m_2} \\ \vdots & & & \vdots \\ (-p+1)^{m_p} & (-p+3)^{m_p} & \cdots & (p-1)^{m_p} \end{vmatrix} \neq 0. \tag{2.17}$$

We consider two cases:

Case 1: p is odd. If p is any odd positive integer, set $p = 2s+1$. But as the middle column (i.e., the $s+1$-st column) of the determinant of (2.17) has only zero entries, the determinant of (2.17) is necessarily zero; whence, $\tilde{\Delta}(0) = 0$ from (2.16). Thus from Proposition 1 (with $j = r$), we see, in this case when $N = n(2p+1)$ is odd, that the trigonometric interpolation problem (1.5) is *never* regular if p is odd.

Case 2: p is even. If p is an even positive integer, set $p = 2s$ and let (cf. (1.8)) $e := e_{2s}$ and $o := o_{2s}$ be respectively the number of even and odd integers in $\{m_j\}_{j=1}^{2s}$. Since the integers $\{m_j\}_{j=1}^{2s}$ of (1.2) can be applied in (1.5) in *any* order, we may assume, without loss of generality, that $m_1, m_2, \cdots, m_e$ are all even positive integers, and that $m_{e+1}, \cdots, m_{2s}$ are all odd positive integers. Assume first that $e > s$. Because $m_1, m_2, \cdots, m_e$ are all even, then on subtracting the first column in (2.17) from the last column, the second column from the second last column, etc., the determinant of (2.17) takes the form (up to a multiplicative factor of a power of 2)

$$\left|\begin{array}{cccc|cccc} (2s-1)^{m_1} & (2s-3)^{m_1} & \cdots & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ (2s-1)^{m_e} & (2s-3)^{m_e} & \cdots & 1 & 0 & \cdots & 0 & 0 \\ \hline -(2s-1)^{m_{e+1}} & -(2s-3)^{m_{e+1}} & \cdots & -1 & 1 & \cdots & (2s-3)^{m_{e+1}} & (2s-1)^{m_{e+1}} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ -(2s-1)^{m_{2s}} & -(2s-3)^{m_{2s}} & \cdots & -1 & 1 & \cdots & (2s-3)^{m_{2s}} & (2s-1)^{m_{2s}} \end{array}\right| . \tag{2.18}$$

What is important here is that there is a $e \times s$ block of zeros in the upper right corner of the partitioned determinant above. But because $e > s$ by assumption, each term of the Laplace expansion (cf. Karlin [K, p. 6]) of the above determinant, in terms of its first s columns, is *necessarily zero*; whence, $\tilde{\Delta}(0) = 0$. Similarly, if $e < s$, we again have (now *adding* the first column to last column, the second column to the second last column, etc.) that the determinant of (2.18) is again zero; whence, $\tilde{\Delta}(0) = 0$. Thus, from Proposition 1 (with $j = r$), we have established, in this case when $p = 2s$, that $e_{2s} = o_{2s} = s$ is a *necessary* condition for the regularity of the interpolation problem of (1.5). Combined with the previous case, we have established

Corollary 1. *Let $N = n(2p+1)$ be odd, so that n is also odd. Then, necessary conditions for the 2-periodic lacunary interpolation problem* (1.5) *to be regular on the $2n$ equidistant nodes $\{x_k\}_{k=0}^{2n-1}$ in $[0, 2\pi)$, are that p is even and that* (cf. (1.8))

$$e_p = o_p = p/2 . \tag{2.19}$$

§3. Necessary and Sufficient Determinental Conditions for Regularity: N Even.

We assume here that $N = n(2p+1)$ is even, so that n is necessarily even, and we write

$n = 2r$. In this case, we have (cf. (1.7))

$$M = \frac{N}{2} = np + r, \text{ where } n = 2r, \tag{3.1}$$

and the sought trigonometric polynomials must be of the form (1.4′), (which depend on an even number of parameters) with an appropriate choice of $\varepsilon = 0$ or $\varepsilon = 1$. Because of the special form of the last term of the trigonometric polynomials of (1.4′), the previous steps used in §2 must be altered.

As in §2, a straight-forward calculation shows, on setting $z = e^{ix}$, that we can express the trigonometric polynomial of (1.4′), which interpolates null data for the case $\nu = 0$ of (1.5), as (cf. (2.3))

$$t_M(x) = z^{-M}(z^n - 1)\left[\sum_{\lambda=0}^{2p-1} z^{\lambda n} Q_\lambda(z) + cz^{2pn}\right], \tag{3.2}$$

where $Q_\lambda(z) := \sum_{j=0}^{n-1} a_{\lambda,j} z^j$ (for $\lambda = 0, 1, \cdots, 2p-1$), with the added condition that

$$a_{0,0} + c(-1)^\varepsilon = 0. \tag{3.3}$$

Next, on expressing $t_M(x)$ of (3.2) as

$$t_M(x) = \sum_{j=0}^{n-1} \sum_{\lambda=0}^{2p-1} a_{\lambda,j}\left\{z^{j+(\lambda+1)n-M} - z^{j+\lambda n-M}\right\} + c\left\{z^{(2p+1)n-M} - z^{2pn-M}\right\},$$

the remaining interpolation conditions of (1.5) for null data similarly require, as in (2.4), that

$$\begin{cases} \displaystyle\sum_{j=0}^{n-1} z_{2k}^j \sum_{\lambda=0}^{2p-1} a_{\lambda,j}\left\{(\alpha_j+\lambda+1-p)^{m_\nu} - (\alpha_j+\lambda-p)^{m_\nu}\right\} + c\left\{\left(p+1-\frac{r}{n}\right)^{m_\nu} - \left(p-\frac{r}{n}\right)^{m_\nu}\right\} = 0, \\ \qquad (\nu = 1, 2, \cdots, p;\ k = 0, 1, \cdots, n-1), \\ \displaystyle\sum_{j=0}^{n-1} z_{2k+1}^j \sum_{\lambda=0}^{2p-1} (-1)^\lambda a_{\lambda,j}\left\{(\alpha_j+\lambda+1-p)^{m_\nu} + (\alpha_j+\lambda-p)^{m_\nu}\right\} + c\left\{\left(p+1-\frac{r}{n}\right)^{m_\nu} + \left(p-\frac{r}{n}\right)^{m_\nu}\right\} = 0, \\ \qquad (\nu = 1, 2, \cdots, p;\ k = 0, 1, \cdots, n-1), \end{cases} \tag{3.4}$$

where, as in (2.6),

$$\alpha_j := (j-r)/n \qquad (j = 0, 1, \cdots, n-1). \tag{3.5}$$

With the reasoning used in §2 to deduce (2.5) from (2.4), we similarly obtain the following set of linear equations for each $j = 1, 2, \cdots n-1$:

$$\begin{cases} \sum_{\lambda=0}^{2p-1} a_{\lambda,j} \{(\alpha_j + \lambda + 1 - p)^{m_\nu} - (\alpha_j + \lambda - p)^{m_\nu}\} = 0 & (\nu = 1, 2, \cdots, p), \\ \sum_{\lambda=0}^{2p-1} (-1)^\lambda a_{\lambda,j} \{(\alpha_j + \lambda + 1 - p)^{m_\nu} + (\alpha_j + \lambda - p)^{m_\nu}\} = 0 & (\nu = 1, 2, \cdots, p), \end{cases} \tag{3.6}$$

and for the remaining case $j = 0$ (noting from (3.5) that $\alpha_0 = -1/2$ in this case since $n = 2r$):

$$\begin{cases} \sum_{\lambda=0}^{2p-1} a_{\lambda,0} \left\{ \left(+\frac{1}{2} + \lambda - p\right)^{m_\nu} - \left(-\frac{1}{2} + \lambda - p\right)^{m_\nu} \right\} + c \left\{ \left(p + 1 - \frac{1}{2}\right)^{m_\nu} - \left(p - \frac{1}{2}\right)^{m_\nu} \right\} = 0 \\ \qquad (\nu = 1, 2, \cdots, p), \\ \sum_{\lambda=0}^{2p-1} (-1)^\lambda a_{\lambda,0} \left\{ \left(+\frac{1}{2} + \lambda - p\right)^{m_\nu} + \left(-\frac{1}{2} + \lambda - p\right)^{m_\nu} \right\} + c \left\{ \left(p + 1 - \frac{1}{2}\right)^{m_\nu} + \left(p - \frac{1}{2}\right)^{m_\nu} \right\} = 0 \\ \qquad (\nu = 1, 2, \cdots, p). \end{cases} \tag{3.7}$$

For each choice of $j = 1, \cdots n-1$, (3.6) gives $2p$ homogeneous linear equations in the unknowns $\{a_{\lambda,j}\}_{\lambda=0}^{2p-1}$, while (3.7) and (3.3) give $2p+1$ homogeneous linear equations in the remaining unknowns c and $\{a_{\lambda,0}\}_{\lambda=0}^{2p-1}$. Thus, if $\Delta(\alpha_j)$ denotes the determinant, of order $2p$, of the coefficients $\{a_{\lambda,j}\}_{\lambda=0}^{2p-1}$ in (3.6) for $j = 1, 2, \cdots, n-1$, and if Δ_0 denotes the determinant, of order $2p+1$, of the coefficients c and $\{a_{\lambda,0}\}_{\lambda=0}^{2p-1}$ from (3.7) and (3.3), we have, in analogy with Proposition 1 of §2, the result of

Proposition 2. *Let $N = n(2p+1)$ be even, so that $n(= 2r)$ is also even. Then, in order for the 2-periodic lacunary interpolation problem (1.5) to be regular on the $2n$ equidistant nodes $\{x_k\}_{k=0}^{2n-1}$ in $[0, 2\pi)$, it is necessary and sufficient that*

$$\Delta(\alpha_j) \neq 0 \qquad \left(\alpha_j := \frac{j-r}{2r}; \quad j = 1, 2, \cdots, 2r-1\right), \tag{3.8}$$

and that

$$\Delta_0 \neq 0 \qquad \left(\alpha_0 := \frac{-r}{2r} = -\frac{1}{2}; \quad j = 0\right), \tag{3.9}$$

where $\Delta(\alpha_j)$ denotes the determinant of order $2p$ of the coefficients of $a_{\lambda,j}$ of (3.6) for $j = 1, 2, \cdots, 2r-1$, and where Δ_0 denotes the determinant of order $2p+1$ of the coefficients of c and $a_{\lambda,0}$ of (3.7) and (3.3) for $j = 0$.

Noting that the equations (3.6) are *exactly* those of (2.7) (except that $n = 2r$ in this present case as opposed to $n = 2r+1$ of §2), we can use the bordering technique and the column operations of §2 to form the determinant $\tilde{\Delta}(\alpha)$ of order $2p+1$ of (2.9). Now, α is a real variable with $\frac{1-r}{2r} \leq \alpha \leq \frac{r-1}{2r}$, which corresponds to the cases $j = 1, 2, \cdots, 2r-1$ of

(3.6). As $\alpha = 0$ falls in this range and corresponds also to the case $j = r$, the conclusions of §2, culminating in Corollary 1, can be applied *without change*. We state this as

Corollary 2. *Let $N = n(2p+1)$ be even, so that n is even. Then, necessary conditions for the 2-periodic lacunary interpolation problem* (1.5) *to be regular on the $2n$ equidistant nodes $\{x_k\}_{k=0}^{2n-1}$ in $[0, 2\pi)$, are that p is even and that* (cf. (1.8))

$$e_p = o_p = p/2. \tag{3.10}$$

§4. Proof of Theorem 1: N odd.

As in §2, we assume that $N = n(2p+1)$ is odd, so that n is necessarily odd, and we again write $n = 2r+1$. For the regularity of the interpolation problem (1.5) in this case, Corollary 1 gives us the necessary conditions that p is even (so we write $p = 2s$), as well as $e_{2s} = o_{2s} = s$. The object of this section is to show that these necessary conditions are also *sufficient* for the determinants $\tilde{\Delta}(\alpha)$ of (2.8′) to satisfy

$$\tilde{\Delta}(\alpha_j) \neq 0 \qquad (j = 0, 1, \cdots, 2r). \tag{4.1}$$

From (2.8′), this will establish Theorem 1 for the case when N is odd. Note that since $n = 2r+1$, then from (2.6), $\alpha_j := \frac{j-r}{2r+1}$ for $j = 0, 1, \cdots, 2r$, so that

$$|\alpha_j| \leq \frac{r}{2r+1} < \frac{1}{2} \qquad (j = 0, 1, \cdots, 2r). \tag{4.2}$$

To begin, since $e_{2s} = o_{2s} = s$ and since the integers $\{m_j\}_{j=1}^{2s}$ can be applied in any order in (1.5), we assume (without loss of generality) that

$$\begin{cases} \{m_j\}_{j=1}^{s} \text{ are even positive integers with} & 0 < m_1 < m_2 < \cdots < m_s; \\ \{m_j\}_{j=s+1}^{2s} \text{ are odd positive integers with} & 0 < m_{s+1} < m_{s+2} < \cdots < m_{2s}. \end{cases} \tag{4.3}$$

Next, for ease of description again, consider the determinant $\tilde{\Delta}(\alpha)$, of order 5, of (2.10). In this case, $p = 2$ and $s = 1$, and, by our convention in (4.3), m_1 is even and m_2 is odd. On replacing α by $-\alpha$ in (2.10), we have

$$\tilde{\Delta}(-\alpha) = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ (2+\alpha)^{m_1} & (1+\alpha)^{m_1} & \alpha^{m_1} & (1-\alpha)^{m_1} & (2-\alpha)^{m_1} \\ -(2+\alpha)^{m_2} & -(1+\alpha)^{m_2} & -\alpha^{m_2} & (1-\alpha)^{m_2} & (2-\alpha)^{m_2} \\ -(2+\alpha)^{m_1} & (1+\alpha)^{m_1} & -\alpha^{m_1} & (1-\alpha)^{m_1} & -(2-\alpha)^{m_1} \\ (2+\alpha)^{m_2} & -(1+\alpha)^{m_2} & \alpha^{m_2} & (1-\alpha)^{m_2} & -(2-\alpha)^{m_2} \end{vmatrix}. \tag{4.4}$$

On interchanging columns 1 and 5 and columns 2 and 4, and then changing the sign of all entrees in rows 3 and 5, (which leaves the sign of $\tilde{\Delta}(-\alpha)$ unchanged), we obtain $\tilde{\Delta}(\alpha)$ of (2.10), i.e.,

$$\tilde{\Delta}(\alpha) = \tilde{\Delta}(-\alpha) \quad \text{for all} \quad \alpha. \tag{4.5}$$

Actually, the relation of (4.5), using similar elementary determinantal operations, can be verified to be valid *in the general case* as well. With this observation, it then suffices, from (4.2), to show that $\tilde{\Delta}(\alpha) \neq 0$ for all $0 \leq \alpha < 1/2$. In fact, we shall prove more; namely, that $\tilde{\Delta}(\alpha) \neq 0$ for all $0 \leq \alpha < 1$.

With $p = 2s$, consider the factored form (2.16), and let $F_1(\alpha)$ be the first determinant (of order $2s+1$) on the right side of (2.16), i.e.,

$$F_1(\alpha) := \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ (\alpha-2s)^{m_1} & (\alpha-2s+2)^{m_1} & & (\alpha+2s-2)^{m_1} & (\alpha+2s)^{m_1} \\ (\alpha-2s)^{m_2} & (\alpha-2s+2)^{m_2} & & (\alpha+2s-2)^{m_2} & (\alpha+2s)^{m_2} \\ \vdots & & & & \vdots \\ (\alpha-2s)^{m_{2s}} & (\alpha-2s+2)^{m_{2s}} & \cdots & (\alpha+2s-2)^{m_{2s}} & (\alpha+2s)^{m_{2s}} \end{vmatrix}. \tag{4.6}$$

We wish to show now that $F_1(\alpha) \neq 0$ for all $0 \leq \alpha < 1$. On interchanging columns of $F_1(\alpha)$, we have, up to a sign change, that

$$F_1(\alpha) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \alpha^{m_1} & (\alpha-2)^{m_1} & (\alpha+2)^{m_1} & & (\alpha-2s)^{m_1} & (\alpha+2s)^{m_1} \\ \vdots & & & & & \vdots \\ \alpha^{m_{2s}} & (\alpha-2)^{m_{2s}} & (\alpha+2)^{m_{2s}} & \cdots & (\alpha-2s)^{m_{2s}} & (\alpha+2s)^{m_{2s}} \end{vmatrix},$$

which can be equivalently written as

$$F_1(\alpha) = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ \alpha^{m_1} & [-(2-\alpha)]^{m_1} & (2+\alpha)^{m_1} & & [-(2s-\alpha)]^{m_1} & (2s+\alpha)^{m_1} \\ \vdots & & & & & \vdots \\ \alpha^{m_{2s}} & [-(2-\alpha)]^{m_{2s}} & (2+\alpha)^{m_{2s}} & \cdots & [-(2s-\alpha)]^{m_{2s}} & (2s+\alpha)^{m_{2s}} \end{vmatrix}. \tag{4.7}$$

We next quote the following result from Cavaretta, Sharma, and Varga [C, Lemma 2, last part (in transposed form)]:

Lemma 1. *Let $m'_1 < m'_2 < \cdots < m'_j$ be distinct positive even integers, and let $m'_{j+1} < m'_{j+2} < \cdots < m'_q$ be distinct positive odd integers where $0 \leq j < q$. For any $q+1$*

positive numbers $t_1 < t_2 < \cdots < t_{q+1}$, *the following determinant is nonzero*:

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ t_1^{m_1'} & (-t_2)^{m_1'} & t_3^{m_1'} & & [(-1)^q t_{q+1}]^{m_1'} \\ \vdots & & & & \vdots \\ t_1^{m_q'} & (-t_2)^{m_q'} & t_3^{m_q'} & \cdots & [(-1)^q t_{q+1}]^{m_q'} \end{vmatrix}. \tag{4.8}$$

To apply Lemma 1 to $F_1(\alpha)$ in (4.7), set

$$\begin{cases} j = s \text{ and } q = 2s, \\ t_1 := \alpha;\ t_2 := 2-\alpha;\ t_3 := 2+\alpha; \cdots;\ t_{2s} := 2s-\alpha;\ t_{2s+1} := 2s+\alpha, \\ m_k' := m_k \qquad (k = 1, 2, \cdots, 2s). \end{cases} \tag{4.9}$$

Then, we see that $0 < t_1 < t_2 < \cdots < t_{2s+1}$, provided that $0 < \alpha < 1$. Thus, we obtain from Lemma 1 that $F_1(\alpha) \neq 0$ for any α with $0 < \alpha < 1$.

For the case $F_1(0)$, which is not covered by the above application of Lemma 1, we have from (4.6) that $F_1(0) =$

$$\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ (-2s)^{m_1} & (-2s+2)^{m_1} & & (-2)^{m_1} & 0 & 2^{m_1} & & (2s)^{m_1} \\ (-2s)^{m_2} & (-2s+2)^{m_2} & & (-2)^{m_2} & 0 & 2^{m_2} & & (2s)^{m_2} \\ \vdots & & & & & & & \vdots \\ (-2s)^{m_{2s}} & (-2s+2)^{m_{2s}} & \cdots & (-2)^{m_{2s}} & 0 & 2^{m_{2s}} & \cdots & (2s)^{m_{2s}} \end{vmatrix}. \tag{4.10}$$

On expanding about the central column (with the zero entries), then $F_1(0)$ is given by the following determinant of order $2s$:

$$F_1(0) = \begin{vmatrix} (-2s)^{m_1} & \cdots & (-2)^{m_1} & 2^{m_1} & \cdots & (2s)^{m_1} \\ \vdots & & & & & \vdots \\ (-2s)^{m_{2s}} & \cdots & (-2)^{m_{2s}} & 2^{m_{2s}} & \cdots & (2s)^{m_{2s}} \end{vmatrix}, \tag{4.11}$$

up to a possible sign change. Since $\{m_j\}_{j=1}^{s}$ and $\{m_j\}_{j=s+1}^{2s}$ are respectively even and odd positive integers from (4.3), (4.11) becomes

$$F_1(0) = \left|\begin{array}{ccc|ccc} (2s)^{m_1} & \cdots & 2^{m_1} & 2^{m_1} & \cdots & (2s)^{m_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ (2s)^{m_s} & \cdots & 2^{m_s} & 2^{m_s} & \cdots & (2s)^{m_s} \\ \hline -(2s)^{m_{s+1}} & \cdots & -2^{m_{s+1}} & 2^{m_{s+1}} & \cdots & (2s)^{m_{s+1}} \\ \vdots & & \vdots & \vdots & & \vdots \\ -(2s)^{m_{2s}} & \cdots & -2^{m_{2s}} & 2^{m_{2s}} & \cdots & (2s)^{m_{2s}} \end{array}\right| . \tag{4.12}$$

On subtracting the first column from the last column, the second column from the second last column, etc., and then permuting the first s columns, we have

$$F_1(0) = \left|\begin{array}{cccc|cccc} 2^{m_1} & \cdots & (2s-2)^{m_1} & (2s)^{m_1} & 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 2^{m_s} & \cdots & (2s-2)^{m_s} & (2s)^{m_s} & 0 & \cdots & 0 & 0 \\ \hline -2^{m_{s+1}} & \cdots & -(2s-2)^{m_{s+1}} & -(2s)^{m_{s+1}} & 2^{m_{s+1}} & \cdots & (2s-2)^{m_{s+1}} & (2s)^{m_{s+1}} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ -2^{m_{2s}} & \cdots & -(2s-2)^{m_{2s}} & -(2s)^{m_{2s}} & 2^{m_{2s}} & \cdots & (2s-2)^{m_{2s}} & (2s)^{m_{2s}} \end{array}\right| , \tag{4.13}$$

up to a factor of $\pm 2^s$. As each diagonal subdeterminant, of order s, in (4.13) is a non-vanishing generalized Vandermonde determinant (cf. Gantmacher [G, p. 99]), it follows that $F_1(0) \neq 0$.Thus, we have shown that the first factor $F_1(\alpha)$ of (2.16) is nonzero for any $0 \leq \alpha < 1$.

Next, let $F_2(\alpha)$ be the second determinant (of order $p = 2s$) on the right side of (2.16), i.e.,

$$F_2(\alpha) := \begin{vmatrix} (\alpha - 2s + 1)^{m_1} & (\alpha - 2s + 3)^{m_1} & \cdots & (\alpha + 2s - 1)^{m_1} \\ (\alpha - 2s + 1)^{m_2} & (\alpha - 2s + 3)^{m_2} & & (\alpha + 2s - 1)^{m_2} \\ \vdots & & & \vdots \\ (\alpha - 2s + 1)^{m_{2s}} & (\alpha - 2s + 3)^{m_{2s}} & \cdots & (\alpha + 2s - 1)^{m_{2s}} \end{vmatrix} , \tag{4.14}$$

where $0 \le \alpha < 1$. Since $\{m_j\}_{j=1}^{s}$ and $\{m_j\}_{s+1}^{2s}$ satisfy (4.3), $F_2(\alpha)$ can be written as

$$F_2(\alpha) = \left| \begin{array}{ccc|ccc} (2s-1-\alpha)^{m_1} & \cdots & (1-\alpha)^{m_1} & (1+\alpha)^{m_1} & \cdots & (2s-1+\alpha)^{m_1} \\ \vdots & & \vdots & \vdots & & \vdots \\ (2s-1-\alpha)^{m_s} & \cdots & (1-\alpha)^{m_s} & (1+\alpha)^{m_s} & \cdots & (2s-1+\alpha)^{m_s} \\ \hline -(2s-1-\alpha)^{m_{s+1}} & \cdots & -(1-\alpha)^{m_{s+1}} & (1+\alpha)^{m_{s+1}} & \cdots & (2s-1+\alpha)^{m_{s+1}} \\ \vdots & & \vdots & \vdots & & \vdots \\ -(2s-1-\alpha)^{m_{2s}} & \cdots & -(1-\alpha)^{m_{2s}} & (1+\alpha)^{m_{2s}} & \cdots & (2s-1+\alpha)^{m_{2s}} \end{array} \right| . \tag{4.15}$$

Next, divide each entry in the j column of the above determinant by the *first* entry in that column, and then rearrange the columns of the resulting determinant so that the first column contains powers of $(1-\alpha)$, the second column contains powers of $(1+\alpha)$, the third column powers of $(3-\alpha)$, etc. Then, changing the sign of all entries in the last s rows, we obtain

$$F_2(\alpha) = \left| \begin{array}{ccccc} 1 & 1 & 1 & \cdots & 1 \\ (1-\alpha)^{m_2-m_1} & [-(1+\alpha)]^{m_2-m_1} & (3-\alpha)^{m_2-m_1} & & [-(2s-1+\alpha)]^{m_2-m_1} \\ \vdots & & & & \vdots \\ (1-\alpha)^{m_{2s}-m_1} & [-(1+\alpha)]^{m_{2s}-m_1} & (3-\alpha)^{m_{2s}-m_1} & \cdots & [-(2s-1+\alpha)]^{m_{2s}-m_1} \end{array} \right| , \tag{4.16}$$

up to a multiplicative factor which is nonzero for any $0 \le \alpha < 1$. Similar to (4.9), we now set

$$\begin{cases} j = s-1 \text{ and } q = 2s-1, \\ t_1 := 1-\alpha;\ t_2 := 1+\alpha; \cdots;\ t_{2s} := 2s-1+\alpha, \\ m'_k := m_{k+1} - m_1 \qquad (k = 1, 2, \cdots, 2s-1). \end{cases} \tag{4.17}$$

and we see that the determinant (of order $2s$) of (4.16) is of the form (4.8) of Lemma 1. Moreover, since $0 < t_1 < t_2 < \cdots < t_{2s}$ in (4.17) for any $0 < \alpha < 1$, and since, from (4.3) and (4.17), $m'_1 < m'_2 < \cdots < m'_{s-1}$ are distinct positive even integers and $m'_s < m'_{s+1} < \cdots < m'_{2s-1}$ are distinct positive odd integers, we can apply Lemma 1 to deduce that the determinant $F_2(\alpha)$ is nonzero for all $0 < \alpha < 1$.

As in the treatment of $F_1(\alpha)$, we similarly separately consider the case $F_2(0)$, which was not covered by the previous application of Lemma 1. From Case 2, in §2, we can write from (2.18), after a permutation of the first s columns of (2.18), that

$$F_2(0) = \left|\begin{array}{cccc|cccc} 1 & 3^{m_1} & \cdots & (2s-1)^{m_1} & 0 & \cdots & & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ 1 & 3^{m_s} & \cdots & (2s-1)^{m_s} & 0 & \cdots & & 0 \\ \hline -1 & -3^{m_{s+1}} & \cdots & -(2s-1)^{m_{s+1}} & 1 & 3^{m_{s+1}} & \cdots & (2s-1)^{m_{s+1}} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ -1 & -3^{m_{2s}} & \cdots & -(2s-1)^{m_{2s}} & 1 & 3^{m_{2s}} & \cdots & (2s-1)^{m_{2s}} \end{array}\right|, \tag{4.18}$$

up to a multiplicative factor of $\pm 2^s$. Again because of (4.3), each diagonal subdeterminant, of order s, in (4.18) is a nonvanishing generalized Vandermonde determinant; whence, $F_2(0) \neq 0$. Thus, with the previous paragraph, $F_2(\alpha) \neq 0$ for all $0 \leq \alpha < 1$. Combined with the same result for the first determinantal factor $F_1(\alpha)$ of (2.16), we deduce that the determinant $\tilde{\Delta}(\alpha)$ of (2.16) is also nonzero for all $0 \leq \alpha < 1$. From (4.5), (4.2), and (4.1), this establishes Theorem 1 when N is odd. □

§5. Proof of Theorem 1: N even.

As in §3, we assume that $N = n(2p+1)$ is even, so that n is necessarily even, and we write $n = 2r$. For the regularity of the interpolation problem of (1.5) in this case, Corollary 2 gives us the necessary condition that p is even (so we write $p = 2s$), as well as $e_{2s} = o_{2s} = s$. The object of this section is to show that, with the above necessary conditions, the choice $\varepsilon = 0$ in (1.4′) is both necessary and sufficient for the regularity of the interpolation problem (1.5). For our use below, the necessary and sufficient conditions of Proposition 2 can be equivalently formulated as (cf. (3.8))

$$\tilde{\Delta}(\alpha_j) \neq 0 \qquad \left(\alpha_j := \frac{j-r}{2r}; \quad j = 1, 2, \cdots, 2r-1\right), \tag{5.1}$$

and (cf. (3.9))

$$\Delta_0 \neq 0 \qquad \left(\alpha_0 := -\frac{1}{2}; \quad j = 0\right), \tag{5.2}$$

where $\tilde{\Delta}(\alpha)$ denotes the bordered determinant (cf. (2.8′)), derived from $\Delta(\alpha)$ of (3.8). Note also that since $n = 2r$, the α_j's under consideration in (5.1) satisfy

$$|\alpha_j| \leq \frac{r-1}{2r} < \frac{1}{2} \qquad (j = 1, 2, \cdots, 2r-1). \tag{5.3}$$

To begin, since $e_{2s} = o_{2s} = s = p/2$, we assume (without loss of generality) as in (4.3) that

$$\begin{cases} \{m_j\}_{j=1}^{s} \text{ are even positive integers with } 0 < m_1 < m_2 < \cdots < m_s; \\ \{m_j\}_{j=s+1}^{2s} \text{ are odd positive integers with } 0 < m_{s+1} < m_{s+2} < \cdots < m_{2s}. \end{cases} \tag{5.4}$$

Next, because the only change from §4 is that $n = 2r$ here (as opposed to $n = 2r + 1$ in §4), it follows, as in (4.5), that the associated bordered determinants $\tilde{\Delta}(\alpha)$ of (2.8′) satisfy

$$\tilde{\Delta}(\alpha) = \tilde{\Delta}(-\alpha) \text{ for all } \alpha, \tag{5.5}$$

so that it suffices to show, as in §4, that $\tilde{\Delta}(\alpha) \neq 0$ for all $0 \leq \alpha < 1/2$. But as the entire argument of §4 can be applied without change, then

$$\tilde{\Delta}(\alpha) \neq 0 \text{ for all } 0 \leq \alpha < 1. \tag{5.6}$$

It remains to show that the determinant Δ_0 (of order $2p+1$) of (3.9) is not zero, and the dependence on $\varepsilon(= 0$ or $= 1)$ now comes into play. For clarity, assume as in §2, that $p = 2$. Then, from (3.3) and (3.7), we have the following determinant Δ_0 (of order $5 = 2p+1$): $\Delta_0 =$

$$\begin{vmatrix} (-\frac{3}{2})^{m_1} - (-\frac{5}{2})^{m_1} & (-\frac{1}{2})^{m_1} - (-\frac{3}{2})^{m_1} & (\frac{1}{2})^{m_1} - (-\frac{1}{2})^{m_1} & (\frac{3}{2})^{m_1} - (\frac{1}{2})^{m_1} & (\frac{5}{2})^{m_1} - (\frac{3}{2})^{m_1} \\ (-\frac{3}{2})^{m_2} - (-\frac{5}{2})^{m_2} & (-\frac{1}{2})^{m_2} - (-\frac{3}{2})^{m_2} & (\frac{1}{2})^{m_2} - (-\frac{1}{2})^{m_2} & (\frac{3}{2})^{m_2} - (\frac{1}{2})^{m_2} & (\frac{5}{2})^{m_2} - (\frac{3}{2})^{m_2} \\ (-\frac{3}{2})^{m_1} + (-\frac{5}{2})^{m_1} & -(-\frac{1}{2})^{m_1} - (-\frac{3}{2})^{m_1} & (\frac{1}{2})^{m_1} + (-\frac{1}{2})^{m_1} & -(\frac{3}{2})^{m_1} - (\frac{1}{2})^{m_1} & (\frac{5}{2})^{m_1} + (\frac{3}{2})^{m_1} \\ (-\frac{3}{2})^{m_2} + (-\frac{5}{2})^{m_2} & -(-\frac{1}{2})^{m_2} - (-\frac{3}{2})^{m_2} & (\frac{1}{2})^{m_2} + (-\frac{1}{2})^{m_2} & -(\frac{3}{2})^{m_2} - (\frac{1}{2})^{m_2} & (\frac{5}{2})^{m_2} + (\frac{3}{2})^{m_2} \\ 1 & 0 & 0 & 0 & (-1)^{\varepsilon} \end{vmatrix}. \tag{5.7}$$

Then, border Δ_0 on the left by the column vector (of length $5 = 2p+1$), defined by

$$\left(\left(-\frac{5}{2}\right)^{m_1}, \left(-\frac{5}{2}\right)^{m_2}, -\left(-\frac{5}{2}\right)^{m_1}, -\left(-\frac{5}{2}\right)^{m_2}, -1\right)^T,$$

and then add the row vector (of length $6 = 2p+2$), defined by

$$(1, 0, 0, 0, 0, 0)$$

to the top to $\Delta(\alpha)$, thereby forming a determinant $\tilde{\Delta}_0$ of order 6 (of order $2p+2$ in the general case). By construction, we note that

$$\tilde{\Delta}_0 = \Delta_0. \tag{5.8}$$

By performing elementary determinantal operations on $\tilde{\Delta}_0$ (exactly as in the transition of the determinant in (2.9) to the determinant in (2.12)), these operations leaving the determinant of $\tilde{\Delta}_0$ invariant (up to nonzero scalar factors), we obtain (in the case $p = 2$)

$$\tilde{\Delta}_0 = \left|\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 3^{m_1} & 5^{m_1} \\ 0 & 0 & 0 & 1 & -3^{m_2} & 5^{m_2} \\ \hline 1 & 3^{m_1} & 5^{m_1} & 0 & 0 & 0 \\ 1 & -3^{m_2} & 5^{m_2} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & (-1)^{\varepsilon} \end{array}\right|, \tag{5.9}$$

where we have made use of the fact (cf. (5.4) for $p = 2$) that m_1 is an even positive integer and m_2 is an odd positive integer. We now apply the Laplace expansion of $\tilde{\Delta}_0$ above in terms of the first three columns of $\tilde{\Delta}_0$, i.e., (cf. [K, p. 6])

$$\tilde{\Delta}_0 = \sum (-1)^{i_1+i_2+i_3+\frac{3\cdot 4}{2}} \Delta \begin{pmatrix} i_1, & i_2, & i_3 \\ 1, & 2, & 3 \end{pmatrix} \cdot \Delta \begin{pmatrix} i'_1, & i'_2, & i'_3 \\ 4, & 5, & 6 \end{pmatrix}; \tag{5.10}$$

the above summation is taken over all integers $\{i_j\}_{j=1}^3$ with $1 \le i_1 < i_2 < i_3 \le 6$, and where i'_1, i'_2, and i'_3 denote the complementary rows to i_1, i_2, i_3 in $(1,2,3,4,5,6)$ with $1 \le i'_1 < i'_2 < i'_3 \le 6$. (For notation, $\Delta \begin{pmatrix} i_1, & i_2, & \cdots, & i_j \\ k_1, & k_2, & \cdots, & k_j \end{pmatrix}$ is the determinant of the entries of Δ_0 from rows $i_1, i_2, \cdots, i_j$ and columns $k_1, k_2, \cdots, k_j$.)

Because of the two zero rows in the first three and last three columns of $\tilde{\Delta}_0$ in (5.9), there are only *two* nontrivial terms in the summation of (5.10), corresponding to the choices $(1,4,5)$ and $(4,5,6)$ of $\{i_j\}_{j=1}^3$. Thus, (5.10) reduces (up to a possible sign change) to

$$\tilde{\Delta}_0 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3^{m_1} & 5^{m_1} \\ 1 & -3^{m_2} & 5^{m_2} \end{vmatrix} \cdot \begin{vmatrix} 1 & 3^{m_1} & 5^{m_1} \\ 1 & -3^{m_2} & 5^{m_2} \\ 0 & 0 & (-1)^{\varepsilon} \end{vmatrix} - \begin{vmatrix} 1 & 3^{m_1} & 5^{m_1} \\ 1 & -3^{m_2} & 5^{m_2} \\ 0 & 0 & -1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3^{m_1} & 5^{m_1} \\ 1 & -3^{m_2} & 5^{m_2} \end{vmatrix},$$

which can be factored as

$$\tilde{\Delta}_0 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3^{m_1} & 5^{m_1} \\ 1 & -3^{m_2} & 5^{m_2} \end{vmatrix} \cdot \left\{ \begin{vmatrix} 1 & 3^{m_1} & 5^{m_1} \\ 1 & -3^{m_2} & 5^{m_2} \\ 0 & 0 & (-1)^{\varepsilon} \end{vmatrix} - \begin{vmatrix} 1 & 3^{m_1} & 5^{m_1} \\ 1 & -3^{m_2} & 5^{m_2} \\ 0 & 0 & -1 \end{vmatrix} \right\}. \tag{5.11}$$

It is evident from (5.11) that if $\varepsilon = 1$, the two determinants in braces in (5.11) are *identical*, so that $\tilde{\Delta}_0 = 0$. But this would imply from (5.8) that $\Delta_0 = 0$, and the corresponding 2-periodic interpolation problem, from (3.9) of Proposition 2, could not be regular.

If, on the other hand, $\varepsilon = 0$ in (5.11), then on dividing, by 3^{m_1}, each entry in the center column in the two determinants in braces and on expanding the determinants in braces in terms of their last rows, it follows, up to a nonzero multiplicative factor, that

$$\tilde{\Delta}_0 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3^{m_1} & 5^{m_1} \\ 1 & -3^{m_2} & 5^{m_2} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 \\ 1 & -3^{m_2-m_1} \end{vmatrix} . \tag{5.12}$$

But, Lemma 1 can be applied to show that *each* of the two determinants of (5.12) is nonzero, and so $\varepsilon = 0$ implies $\tilde{\Delta}_0 \neq 0$. Thus, for the regularity of the interpolation problem (1.5) when N is even, it is necessary and sufficient in this case $p = 2$, that $e_2 = o_2 = 1$, and that $\varepsilon = 0$ in (1.4′).

Having carefully treated the special case $p = 2$, assume that $p = 2s$ and that (5.4) is valid. The bordered determinant $\tilde{\Delta}_0$ is of order $2p+2$, and (5.9) then takes the form $\tilde{\Delta}_0 =$

$$\left|\begin{array}{cccccc|cccccc} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & & 0 & 0 & 1 & (-3)^{m_1} & 5^{m_1} & & [-(2p-1)]^{m_1} & (2p+1)^{m_1} \\ \vdots & & & & & \vdots & \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & (-3)^{m_p} & 5^{m_p} & \cdots & [-(2p-1)]^{m_p} & (2p+1)^{m_p} \\ \hline 1 & (-3)^{m_1} & 5^{m_1} & \cdots & [-(2p-1)]^{m_1} & (2p+1)^{m_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots & \vdots & & & & & \vdots \\ 1 & (-3)^{m_p} & 5^{m_p} & & [-(2p-1)]^{m_p} & (2p+1)^{m_p} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & (-1)^{\varepsilon} \end{array}\right|, \tag{5.13}$$

up to a nonzero multiplicative factor. As in the case $p = 3$, the Laplace expansion of $\tilde{\Delta}_0$ of (5.13), in terms of the first $p+1$ columns of $\tilde{\Delta}_0$, contains only *two* nontrivial terms in its summation, and, up to a sign change, we similarly have (cf. (5.11))

$$\tilde{\Delta}_0 = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & (-3)^{m_1} & 5^{m_1} & & (2p+1)^{m_1} \\ \vdots & & & & \vdots \\ 1 & (-3)^{m_p} & 5^{m_p} & \cdots & (2p+1)^{m_p} \end{vmatrix} \cdot \left\{ \begin{vmatrix} 1 & (-3)^{m_1} & 5^{m_1} & \cdots & (2p+1)^{m_1} \\ \vdots & & & & \vdots \\ 1 & (-3)^{m_p} & 5^{m_p} & & (2p+1)^{m_p} \\ 0 & 0 & 0 & \cdots & (-1)^{\varepsilon} \end{vmatrix} \right. \tag{5.14}$$

$$
\left.- \begin{vmatrix} 1 & (-3)^{m_1} & 5^{m_1} & \cdots & (2p+1)^{m_1} \\ \vdots & & & & \vdots \\ 1 & (-3)^{m_p} & 5^{m_p} & & (2p+1)^{m_p} \\ 0 & 0 & 0 & \cdots & -1 \end{vmatrix} \right\} .
$$

Again, since the two determinants in braces are identical if $\varepsilon = 1$, then $\tilde{\Delta}_0 = 0$ if $\varepsilon = 1$. If $\varepsilon = 0$, then we similarly have, up to a nonzero multiplicative factor, that

$$
\tilde{\Delta}_0 = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & (-3)^{m_1} & 5^{m_1} & & (2p+1)^{m_1} \\ \vdots & & & & \vdots \\ 1 & (-3)^{m_p} & 5^{m_p} & \cdots & (2p+1)^{m_p} \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & (-3)^{m_2-m_1} & & [-(2p-1)]^{m_2-m_1} \\ \vdots & & & \vdots \\ 1 & (-3)^{m_p-m_1} & \cdots & [-(2p-1)]^{m_p-m_1} \end{vmatrix}, \tag{5.15}
$$

and an application of Lemma 1 shows that *each* of the factors of Δ_0 above is nonzero; whence, $\tilde{\Delta}_0$ and Δ_0 are not zero. This completes the proof of Theorem 1 that, if N is even, then the interpolation problem (1.5) is regular iff p is even (say $p = 2s$), that $e_{2s} = o_{2s} = p/2$, and that $\varepsilon = 0$ in (1.4′). □

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