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Asymptotics for the Zeros of the Partial Sums of e^z. II

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1. Introduction

With $s_n(z) := \sum_{j=0}^{n} z^j/j!$ $(n = 1, 2, \cdots)$ denoting the familiar partial sums of the exponential function e^z, we continue our investigation here on the location of the zeros of the *normalized* partial sums, $s_n(nz)$, which are known to lie (cf. Anderson, Saff, and Varga [1]) for every $n > 1$ in the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. For notation, let the Szegö curve, D_∞, be defined by

$$D_\infty := \{z \in \mathbb{C} : |ze^{1-z}| = 1 \text{ and } |z| \leq 1\}. \tag{1.1}$$

It is known that D_∞ is a simple closed curve in the closed unit disk $\bar{\Delta}$, and that D_∞ is star-shaped with respect to the origin, $z = 0$.

If $\{z_{k,n}\}_{k=1}^{n}$ denotes the zeros of $s_n(nz)$ (for $n = 1, 2, \cdots$), then it was shown by Szegö [7] in 1924 that each accumulation point of all these zeros, $\{z_{k,n}\}_{k=1,n=1}^{n,\infty}$, must lie on D_∞, and, conversely, that each point of D_∞ is an accumulation point of the zeros $\{z_{k,n}\}_{k=1,n=1}^{n,\infty}$. Subsequently, it was shown by Buckholtz [2] that the zeros $\{z_{k,n}\}_{k=1,n=1}^{n,\infty}$ all lie *outside* the simple closed curve D_∞.

As for a measure of the rate at which the zeros, $\{z_{k,n}\}_{k=1}^{n}$, tend to D_∞, we use the quantity

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$$\text{dist}\ [\{z_{k,n}\}_{k=1}^{n}; D_\infty] := \max_{1\le k\le n}(\text{dist}\ [z_{k,n}; D_\infty]), \tag{1.2}$$

and a result of Buckholtz [2] gives that

$$\text{dist}\ [\{z_{k,n}\}_{k=1}^{n}; D_\infty] \le \frac{2e}{\sqrt{n}} = \frac{5.43656\cdots}{\sqrt{n}} \qquad (n = 1, 2, \cdots) \tag{1.3}$$

It was later shown in Carpenter, Varga, and Waldvogel [3] that the result of (1.3) is *best possible*, as a function of n, since

$$\varliminf_{n\to\infty}\{\sqrt{n}\cdot \text{dist}\ [\{z_{k,n}\}_{k=1}^{n}; D_\infty]\} \ge 0.63665\cdots > 0. \tag{1.4}$$

It was also shown in [3] that there is substantially *faster* convergence of the subset of the zeros $\{z_{k,n}\}_{k=1}^{n}$, to the Szegö curve D_∞, which stay *uniformly* away from the point $z = 1$. More precisely, for the open disk C_δ about the point $z = 1$, defined by

$$C_\delta := \{z \in \mathbb{C} : |z - 1| < \delta\} \qquad (0 < \delta \le 1), \tag{1.5}$$

it was shown in [3] that, for any fixed δ with $0 < \delta \le 1$,

$$\text{dist}\ [\{z_{k,n}\}_{k=1}^{n}\backslash C_\delta; D_\infty] = O\left(\frac{\log n}{n}\right) \qquad (n \to \infty), \tag{1.6}$$

and, the result of (1.6) is also *best possible*, as a function of n, since (cf. [3, eq. (2.27)])

$$\varliminf_{n\to\infty}\left\{\frac{n}{\log n}\cdot \text{dist}\ [\{z_{k,n}\}_{k=1}^{n}\backslash C_\delta; D_\infty]\right\} \ge 0.10890\cdots > 0, \tag{1.7}$$

for any fixed δ with $0 < \delta \le 1$.

In [3], an arc, D_n, was defined for each $n = 1, 2, \cdots$ by

$$D_n := \left\{z \in \mathbb{C} : |ze^{1-z}|^n = \tau_n\sqrt{2\pi n}\left|\frac{1-z}{z}\right|,\ |z| \le 1, \text{ and } |\arg z| \ge \cos^{-1}\left(\frac{n-2}{n}\right)\right\}, \tag{1.8}$$

where from Stirling's formula,

$$\tau_n := \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} \approx 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \cdots, \text{ as } n \to \infty. \tag{1.9}$$

This arc was introduced to provide a much closer approximation to the zeros $\{z_{k,n}\}_{k=1}^{n}$ of $s_n(nz)$, than does the Szegö curve. With the notation of (1.5), it was shown in [3] that, for any fixed δ with $0 < \delta \le 1$,

$$\text{dist}\ [\{z_{k,n}\}_{k=1}^{n}\backslash C_\delta; D_n] = O\left(\frac{1}{n^2}\right) \qquad (n \to \infty), \tag{1.10}$$

and moreover that (1.10) is *best possible*, as a function of n, since (cf. [3, eq. (3.18)])

$$\varliminf_{n\to\infty}\{n^2\cdot \text{dist}\ [\{z_{k,n}\}_{k=1}^{n}\backslash C_\delta; D_n]\} \ge 0.13326\cdots > 0, \tag{1.11}$$

for any fixed δ with $0 < \delta \leq 1$.

It turns out (cf. [3, Prop. 3]) that, for each positive integer n, the arc D_n is *star-shaped* with respect to the origin, $z = 0$, i.e., for each real number θ in $[-\pi, +\pi]$ with $|\theta| \geq \cos^{-1}(\frac{n-2}{n})$, there is a unique positive number $r = r_n(\theta)$ such that $z = re^{i\theta}$ lies on the arc D_n of (1.8). Let $\mathcal{D}_n$ be the closed star-shaped (with respect to $z = 0$) set defined from the arc D_n, i.e.,

$$\mathcal{D}_n := \left\{ z \in \mathbb{C} : |ze^{1-z}|^n \leq \tau_n \sqrt{2\pi n} \left| \frac{1-z}{z} \right|, |z| \leq 1, \text{ and } |\arg z| \geq \cos^{-1}(\frac{n-2}{n}) \right\}, \quad (n = 1, 2, \cdots). \tag{1.12}$$

Recently, R. Barnard and K. Pierce asked if the zeros, $\{z_{k,n}\}_{k=1}^{n}$, of $s_n(nz)$ all lie outside $\mathcal{D}_n$ for *every* $n \geq 1$. (This would be the natural analogue of the result of Buckholtz [2] which established that all the zeros $\{z_{k,n}\}_{k=1,n=1}^{n,\infty}$ lie outside of D_∞.) This is not at all obvious from the graphs of [3], since it appeared that the zeros $\{z_{k,16}\}_{k=1}^{16}$ and $\{z_{k,27}\}_{k=1}^{27}$ of $s_{16}(16z)$ and $s_{27}(27z)$ were, to plotting accuracy, respectively *on* the curves D_{16} and D_{27}.

It turns out that the zeros, $\{z_{k,n}\}_{k=1}^{n}$ of $s_n(nz)$ do *not* all lie outside $\mathcal{D}_n$ for every $n \geq 1$. This follows from our first result below (to be established in §2).

Proposition 1 *If $\{z_{k,n}\}_{k=1}^{n}$ denotes the zeros of $s_n(nz)$ with increasing arguments, i.e.,*

$$0 < \arg z_{1,n} \leq \arg z_{2,n} \leq \cdots \leq \arg z_{n,n} < 2\pi, \tag{1.13}$$

then (cf. (1.12)) *$z_{1,n}$ is an element of $\mathcal{D}_n$ for all positive n sufficiently large.*

As a consequence of Proposition 1, there is a least positive integer, n_0, such that (cf. (1.12))

$$\{z_{k,n}\}_{k=1}^{n} \bigcap \mathcal{D}_n \neq \emptyset \text{ for all positive integers } n > n_0, \tag{1.14}$$

i.e., at least one zero of $s_n(nz)$ lies in $\mathcal{D}_n$ for every $n > n_0$. By direct calculation of the zeros of $s_n(nz)$, it appears that

$$n_0 = 96, \tag{1.15}$$

and also that

$$\{z_{k,n}\}_{k=1}^{n} \bigcap \mathcal{D}_n = \emptyset \quad (n = 1, 2, \cdots, n_0). \tag{1.16}$$

The *size* of $n_0 = 96$ is somewhat surprising. Because n_0 is so large, it was necessary to calculate the zeros of $s_n(nz)$ with great precision, and for this, Richard Brent's MP package was used with 120 significant digits.

As a consequence of Proposition 1, it is natural to ask if there is a simple *modification*, say $\hat{\mathcal{D}}_n$, of the definition of the closed set $\mathcal{D}_n$ of (1.12) which would have all the zeros $\{z_{k,n}\}_{k=1}^{n}$ *outside* $\hat{\mathcal{D}}_n$ for *all* $n \geq 1$. To give an affirmative answer to this question, we define, for each $n = 1, 2, \cdots$, the arc

$$\hat{D}_n := \left\{ z \in \mathbb{C} : |ze^{1-z}|^n = \tau_n \sqrt{2\pi n} \left| \frac{1 - Re\ z}{z} \right|, \quad |z| \leq 1, \text{ and } |\arg z| \geq \cos^{-1}\left(\frac{n-2}{n}\right) \right\}, \tag{1.17}$$

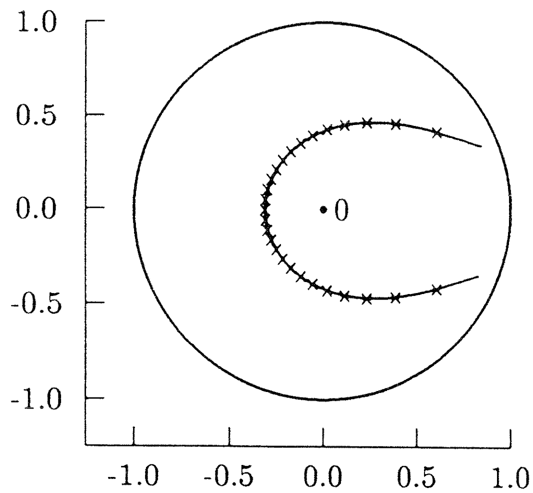

Figure 1. D_{27}

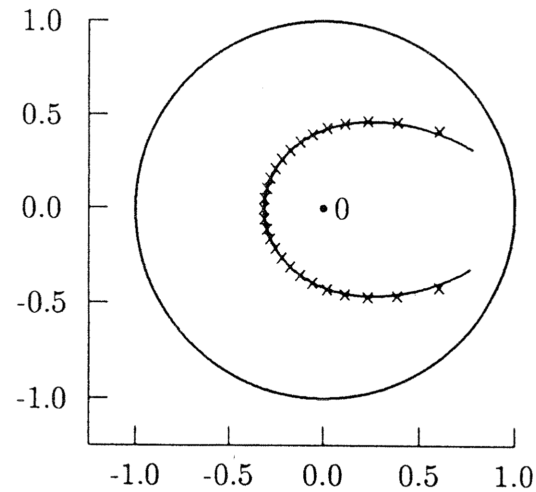

Figure 2. $\hat{D}_{27}$

and its associated closed star-shaped (with respect to $z = 0$) set

$$\hat{\mathcal{D}}_n := \left\{ z \in \mathbb{C} : |ze^{1-z}|^n \le \tau_n \sqrt{2\pi n} \left| \frac{1 - Re\ z}{z} \right|, \quad |z| \le 1, \text{ and } |\arg z| \ge \cos^{-1}\left(\frac{n-2}{n}\right) \right\}. \tag{1.18}$$

Unfortunately, this modification does *not* preserve the accuracy of (1.10). Our ma result (which will be sketched in §2) is

Theorem 2 *With the definition of the set* $\hat{\mathcal{D}}_n$ *of* (1.18), *then*

$$\{z_{k,n}\}_{k=1}^n \bigcap \hat{\mathcal{D}}_n = \emptyset \qquad (n = 1, 2, \cdots), \tag{1.19}$$

and, with the definition of (1.5),

$$\text{dist } [\{z_{k,n}\}_{k=1}^n \backslash C_\delta; \hat{\mathcal{D}}_n] = O\left(\frac{1}{n}\right) \qquad (n \to \infty), \tag{1.20}$$

for any fixed δ *with* $0 < \delta \le 1$.

We remark that the bound of (1.20) is *best possible*, as a function of n. We incluc here Figures 1 and 2 which respectively display the arcs D_{27} and $\hat{D}_{27}$, along with tl zeros, $\{z_{k,27}\}_{k=1}^{27}$, of $s_{27}(27z)$. These zeros are denoted by $\times$'s on Figures 1 and 2. Figur 3 and 4 similarly display the arcs D_{49} and $\hat{D}_{49}$, along with the zeros $\{z_{k,49}\}_{k=1}^{49}$ of $s_{49}(49z$

2. Proof of Proposition 1

It is easy to verify (by differentiation) that

$$e^{-z} s_n(z) = 1 - \frac{1}{n!} \int_0^z \zeta^n e^{-\zeta} d\zeta \qquad (z \in \mathbb{C}, \ n = 0, 1, \cdots), \tag{2.1}$$

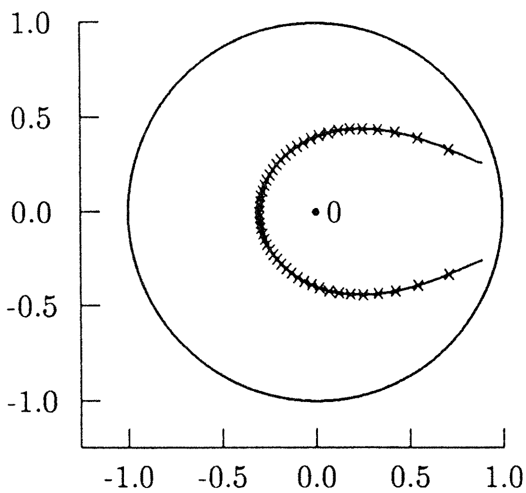

Figure 3. D_{49}

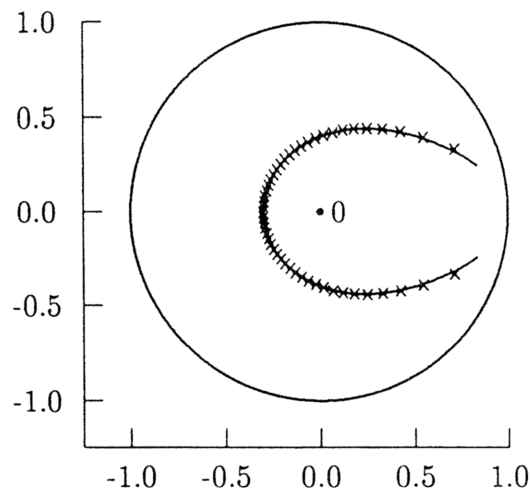

Figure 4. $\hat{D}_{49}$

and replacing ζ and z, respectively, by $n\zeta$ and nz in(2.1) results in

$$e^{-nz}s_n(nz) = 1 - \frac{n^{n+1}}{n!}\int_0^z \zeta^n e^{-n\zeta} d\zeta. \tag{2.2}$$

With the definition of τ_n of (1.9), the above equation becomes

$$e^{-nz}s_n(nz) = 1 - \frac{\sqrt{n}}{\tau_n\sqrt{2\pi}}\int_0^z (\zeta e^{1-\zeta})^n d\zeta \qquad (z \in \mathbb{C},\ n = 0, 1, \cdots). \tag{2.3}$$

Now, in [3, eq. (2.14)], it is shown that

$$e^{-nz}s_n(nz) = 1 - \frac{z(ze^{1-z})^n}{\tau_n\sqrt{2\pi n}(1-z)}\left\{1 - \frac{1}{(n+1)(1-z)^2} + O\left(\frac{1}{n^2}\right)\right\}, \tag{2.4}$$

uniformly on any compact subset Ω of $\bar{\Delta}\backslash\{1\}$. On fixing Ω, then for any zero, $z_{k,n}$, of $s_n(nz)$ in Ω, we evidently have from (2.4) that

$$\frac{z_{k,n}(z_{k,n}e^{1-z_{k,n}})^n}{\tau_n\sqrt{2\pi n}(1-z_{k,n})}\left\{1 - \frac{1}{(n+1)(1-z_{k,n})^2} + O\left(\frac{1}{n^2}\right)\right\} = 1, \tag{2.5}$$

so that

$$\frac{|z_{k,n}(z_{k,n}e^{1-z_{k,n}})^n|}{\tau_n\sqrt{2\pi n}\cdot|1-z_{k,n}|}\left\{\left|1 - \frac{1}{(n+1)(1-z_{k,n})^2} + O\left(\frac{1}{n^2}\right)\right|\right\} = 1. \tag{2.6}$$

It is now clear that the arc D_n of (1.8), is just the approximation of (2.6), with a continuous variable z, when the quantity in braces in (2.6) is replaced by unity, i.e.,

$$\lambda_n(z) := \frac{|z(ze^{1-z})^n|}{\tau_n\sqrt{2\pi n}|1-z|}. \tag{2.7}$$

It is further evident from (1.12) that

$$\text{a zero } z_{k,n} \text{ of } s_n(nz) \text{ lies in } \mathcal{D}_n \text{ iff } \lambda_n(z_{k,n}) \le 1. \tag{2.8}$$

We now examine the particular zero $z_{1,n}$ of $s_n(nz)$ which has smallest argument (1.13)). As discussed in [3, eq. (2.3)], we can write

$$z_{1,n} = 1 + \sqrt{\frac{2}{n}}(t_1 + o(1)) \qquad (n \to \infty), \tag{2.9}$$

where t_1 is the zero of $\operatorname{erfc}(w) := \frac{2}{\sqrt{\pi}} \int_w^\infty e^{-t^2} dt$ in the upper half-plane (i.e., Im $t_1 >$ which is closest to the origin, $w = 0$, and it is known numerically (cf. Fettis, Caslin, Cramer [4]) that

$$t_1 = -1.354810 \cdots + i1.991467 \cdots . \tag{2.10}$$

On evaluating $\lambda_n(z_{1,n})$ from (2.7), (2.9), and (2.10), it can be verified that

$$\lim_{n\to\infty} \lambda_n(z_{1,n}) = 0.985964 \cdots < 1. \tag{2.11}$$

Thus, with (2.8), $z_{1,n}$ is contained in $\mathcal{D}_n$ for all positive n sufficiently large, wh establishes Proposition 1.

We remark that it is because the constant, $0.985964 \cdots$, of (2.11) is so close to un that it is *difficult* to see, graphically, that there are zeros of $s_n(nz)$ which lie interio $\mathcal{D}_n$, for all n sufficiently large.

3. Proof of Theorem 2

We consider the integral (cf. (2.3))

$$I_n(z) := \int_0^z (\zeta e^{1-\zeta})^n d\zeta \qquad (z \in \mathbb{C},\ n = 0, 1, \cdots), \tag{3.1}$$

and, with $z = re^{i\theta}$, we choose the line segment $\zeta = \rho e^{i\theta}(0 \le \rho \le r)$ for the path integration in (3.1). Then,

$$|I_n(z)| \le \int_0^r (\rho e^{1-\rho\cos\theta})^n d\rho =: J_n(r;\theta). \tag{3.2}$$

For $\theta = \pm\pi/2$, we see that $J_n(r; \pm\pi/2)$ can be expressed as

$$J_n(r; \pm\pi/2) = \frac{r(re^{1-r\cos\theta})^n}{n+1} < \frac{r(re^{1-r\cos\theta})^n}{n(1 - r\cos\theta)}. \tag{3.3i}$$

When $\cos\theta < 0$, $J_n(r;\theta)$ can be expressed as

$$J_n(r;\theta) = \frac{1}{|\cos\theta|^{n+1}} \int_0^{r|\cos\theta|e^{1+r|\cos\theta|}} v^{n-1} \frac{u(v)}{1+u(v)} dv \qquad (v := ue^{1+u}).$$

Because $u/(1+u)$ is strictly increasing, it can be verified that

$$J_n(r;\theta) < \frac{r(re^{1-r\cos\theta})^n}{n(1-r\cos\theta)} \qquad (0 < r < 1, \text{ and } \cos\theta < 0). \tag{3.3ii}$$

But this same derivation also shows that the above holds for all $0 < r < 1$ and $\cos\theta > 0$. Thus, with (3.2) and (3.3), we have

$$|I_n(z)| < \frac{|z||ze^{1-z}|^n}{n(1-\text{Re } z)} \quad (0 < |z| < 1,\ n = 1, 2, \cdots), \tag{3.4}$$

and from (2.3), we further have that

$$\frac{\sqrt{n}}{\tau_n\sqrt{2\pi}} I_n(z) = 1 - e^{-nz}s_n(nz). \tag{3.5}$$

Thus, if $\{z_{k,n}\}_{k=1}^n$ denotes the set of zeros of $s_n(nz)$, then $\frac{\sqrt{n}}{\tau_n\sqrt{2\pi}} I_n(z_{k,n}) = 1$, which implies from (3.4) that

$$\frac{|z_{k,n}||z_{k,n}e^{1-z_{k,n}}|^n}{\tau_n\sqrt{2\pi n}(1-\text{Re } z_{k,n})} > 1. \tag{3.6}$$

From (1.18), this means that all zeros $\{z_{k,n}\}_{k=1}^n$ of $s_n(nz)$ lie *outside* the set $\hat{\mathcal{D}}_n$, for all $n \geq 1$, which is the desired result of (1.19) of Theorem 2.

The remainder of Theorem 2, to establish (1.20), now similarly follows, as in the proof given in [3, Theorem 4], by expressing a zero, $z_{k,n}$, of $s_n(nz)$, as $\hat{z} + \delta$, where $\hat{z}$ is a suitable boundary point of $\hat{\mathcal{D}}_n$, and where δ is assumed small. This argument also shows that the result of (1.20) is best possible. ■

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