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Necessary and Sufficient Conditions and the Riemann Hypothesis*

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1. INTRODUCTION

The purpose of this paper is (1) to investigate several necessary and sufficient conditions for a real entire function to have only real zeros and to apply these conditions to the Riemann ξ-function (cf. Section 2), and (2) to prove results concerning the distribution of zeros of entire functions related to the Riemann ξ-function (cf. Section 3).

The interest in this area of research stems, in part, from the well-known fact (cf. Pólya [21] or Henrici [12, p. 305]) that the Riemann hypothesis is equivalent to the statement that all the zeros of the Riemann ξ-function

$$\xi\left(\frac{x}{2}\right) := 8\int_0^\infty \Phi(t)\cos(xt)\,dt \tag{1.1}$$

are *real*, where

$$\Phi(t) := \sum_{n=1}^{\infty} a_n(t) \tag{1.2}$$

and where

$$a_n(t) := \pi n^2(2\pi n^2 e^{4t} - 3)\exp(5t - \pi n^2 e^{4t}) \qquad (t \in \mathbb{R}; n = 1,2,3,\ldots). \tag{1.3}$$

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Since $\xi(x/2)$ is a real entire function (cf. Pólya [21] or Titchmarsh [28]), the Riemann hypothesis is valid if and only if the function $\xi(x)$ belongs to the Laguerre–Pólya class (written $\xi \in \mathscr{L}-\mathscr{P}$). This class is defined as the collection of all real entire functions $f(x)$ of the form

$$f(x) = Ce^{-\alpha x^2+\beta x}x^n \prod_{j=1}^{\omega} (1 - x/x_j)e^{x/x_j} \qquad (\omega \leq \infty), \tag{1.4}$$

where $\alpha \geq 0$, β and C are real numbers, n is a nonnegative integer, and the x_j's are real and nonzero with $\sum_{j=1}^{\omega} 1/x_j^2 < \infty$.

In order to outline here the background and motivation of the present work, we first note that the Taylor series of $\frac{1}{8}\xi(x/2)$ about the origin can be written in the form

$$\frac{1}{8}\xi\left(\frac{x}{2}\right) = \sum_{m=0}^{\infty} \frac{(-1)^m \hat{b}_m}{(2m)!} x^{2m}, \tag{1.5}$$

where

$$\hat{b}_m := \int_0^{\infty} t^{2m}\Phi(t)\,dt \qquad (m = 0, 1, 2, \ldots). \tag{1.6}$$

On setting $z := -x^2$ in (1.5), the function $\xi_1(z)$, defined by

$$\xi_1(z) := \sum_{m=0}^{\infty} \gamma_m \frac{z^m}{m!} \tag{1.7}$$

where

$$\gamma_m := \frac{m!}{(2m)!}\hat{b}_m \qquad (m = 0, 1, 2, \ldots), \tag{1.8}$$

is a real entire function of order $\frac{1}{2}$, and the Riemann hypothesis is equivalent to the statement that $\xi_1 \in \mathscr{L}-\mathscr{P}$. Now, it is known (cf. Boas [1, p. 24] or Pólya-Schur [23]) that *a necessary condition* that $\xi_1(z)$ have only real zeros is that the *Turán inequalities* hold,

$$(\hat{b}_m)^2 - \left(\frac{2m-1}{2m+1}\right)\hat{b}_{m-1}\hat{b}_{m+1} > 0 \qquad (m = 1, 2, 3, \ldots), \tag{1.9}$$

or, equivalently in terms of the γ_m's (defined by (1.8)), that

$$\gamma_m^2 - \gamma_{m-1}\gamma_{m+1} > 0 \qquad (m = 1, 2, 3, \ldots). \tag{1.10}$$

In [6] (see also [8, 9] for related results), we established (1.9), and in this paper we use (1.9) to establish analogous results (cf. Theorem 2.7) for the Jensen polynomials (defined in Section 2) associated with $F_c(x) := 2\xi_1(x)$, where $F_c(x)$ has the integral representation (cf. [9])

$$F_c(x) := \int_{-\infty}^{\infty} \cosh(t\sqrt{x})\Phi(t)\, dt. \tag{1.11}$$

We begin Section 2 with a review of the properties of Jensen and Appell polynomials associated with real entire functions (Proposition 2.1). These polynomials are then used to characterize functions in the Laguerre–Pólya class (Theorem 2.4). In addition to these real-variable results, we establish some refinements of known complex-variable characterizations of functions in the Laguerre–Pólya class (cf. Theorems 2.9, 2.10, and 2.12), and we then apply these theorems (cf. Corollaries 2.11 and 2.13) to the function

$$F(x) := 2 \cdot \frac{1}{8}\xi\left(\frac{x}{2}\right) = \int_{-\infty}^{\infty} e^{ixt}\Phi(t)\, dt. \tag{1.12}$$

In Section 3, we examine the distribution of zeros of real entire functions related to $F(x)$ (cf. (1.12)). In particular, we prove certain convexity results (cf. Theorem 3.4) when the kernel $\Phi(t)$ in (1.12) is replaced by

$$\Phi_j(t) := \sum_{n=j+1}^{\infty} a_n(t) \qquad (j = 1, 2, 3, \ldots), \tag{1.13}$$

where $a_n(t)$ $(n = 1, 2, 3, \ldots)$ is defined by (1.3). These results enable us to give a simple, new geometric interpretation of the question of when $F(x)$ has only real zeros. We also show that the Fourier cosine integral of $\Phi(t)$ *on the interval* [0, 0.11] has only real zeros (cf. Theorem 3.6). Finally, in Section 3 we state three open problems concerning the distribution of the zeros of the Fourier cosine transforms of $a_1(t)$ and of $\Phi(t)$.

In the subsequent sections, we repeatedly make use of several known properties of the kernel $\Phi(t)$, defined by (1.2). For the reader's convenience, we state the following theorem which summarizes some of the known properties of $\Phi(t)$ (cf. Theorem A in [6, 8]).

THEOREM A. *The functions* $\Phi(t)$ *and* $a_n(t)$ $(n = 1, 2, 3, \dots)$, *defined by* (1.2) *and* (1.3), *respectively, satisfy the following properties*:

(i) *for each* $n \geq 1$, $a_n(t) > 0$ *for all* $t \geq 0$, *so that* $\Phi(t) > 0$ *for all* $t \geq 0$;

(ii) $\Phi(z)$ *is analytic in the strip* $-\pi/8 < \operatorname{Im} z < \pi/8$;

(iii) $\Phi(t)$ *is an even function, so that* $\Phi^{(2m+1)}(0) = 0$ $(m = 0, 1, 2, \dots)$;

(iv) *for any* $\varepsilon > 0$, $\lim_{t\to\infty} \Phi^{(n)}(t)\exp[(\pi - \varepsilon)e^{4t}] = 0$ $(n = 0, 1, 2, \dots)$;

(v) $\Phi'(t) < 0$ *for all* $t > 0$;

(vi) $a_n'(t) < 0$ *for all* $t \geq 0$, *for each* $n = 2, 3, 4, \dots$;

(vii) *the function* $\log \Phi(\sqrt{t})$ *is strictly concave for* $0 < t < \infty$.

2. JENSEN POLYNOMIALS: NECESSARY AND SUFFICIENT CONDITIONS

There are basically two types of characterizations of functions in the Laguerre–Pólya class: real-variable and complex-variable characterizations. The real-variable characterizations usually depend on the behavior, on the real axis, of real polynomials (such as the Jensen polynomials of (2.2) below), which are used to approximate the functions in the Laguerre–Pólya class. Since some of the necessary and sufficient conditions for $F(x)$, defined by (1.12), to belong to the Laguerre–Pólya class, will be expressed in terms of Jensen polynomials, we will first state and establish in this section several properties of these polynomials. In contrast, we will see in the sequel that the complex-variable characterizations of functions $f(x)$ in the Laguerre–Pólya class require information concerning the behavior of $f(x)$ in the *entire* complex plane.

If

$$f(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k \tag{2.1}$$

is a real entire function, so that $\gamma_k \in \mathbb{R}$ for $k = 0, 1, 2, \dots,$ then we define the nth *Jensen polynomial associated with* $f(x)$ by

$$g_n(t) := g_n(t; f) := \sum_{k=0}^{n} \binom{n}{k} \gamma_k t^k \qquad (n = 0, 1, 2, \dots). \tag{2.2}$$

The nth Jensen polynomial associated with the derivative $f^{(p)}(x)$ for

$p = 0, 1, 2, \ldots,$ will be denoted by

$$g_{n,p}(t) := g_{n,p}(t; f) := \sum_{k=0}^{n} \binom{n}{k} \gamma_{k+p} t^k \qquad (n, p = 0, 1, 2, \ldots),$$

$$g_{n,0}(t) := g_n(t). \tag{2.3}$$

The nth *Appell polynomial* associated with $f(x)$, if $\gamma_0 \neq 0$, is defined by

$$P_n(t) := P_n(t; f) := \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \gamma_k t^{n-k} \qquad (n = 0, 1, 2, \ldots). \tag{2.4}$$

In particular, if $\gamma_0 \neq 0$, and if

$$g_n^*(t) := t^n g_n(t^{-1}) \qquad (t \neq 0; n = 1, 2, 3, \ldots), \tag{2.5}$$

then $P_n(t) = (1/n!) g_n^*(t)$.

Some of the properties of these polynomials are summarized in the following proposition. (The reader will observe that the properties listed in Proposition 2.1 do *not* depend on $f(z)$ having only real coefficients γ_k in (2.1).)

PROPOSITION 2.1. *With* (2.1)–(2.5), *the following properties hold*:

(i) *if* $\gamma_0 \neq 0$, $P_n'(t) = P_{n-1}(t)$ $(t \in \mathbb{R}; n = 1, 2, \ldots)$;

(ii) *the sequence* $\{g_n(t)\}_{n=0}^{\infty}$ *is generated by* $e^x f(xt)$, *i.e.*,

$$e^x f(xt) = \sum_{n=0}^{\infty} g_n(t) \frac{x^n}{n!} \qquad (x, t \in \mathbb{R}), \tag{2.6}$$

while, if $\gamma_0 \neq 0$, *the sequence* $\{n! P_n(t)\}_{n=0}^{\infty}$ *is generated by* $e^{xt} f(x)$, *i.e.*,

$$e^{xt} f(x) = \sum_{n=0}^{\infty} P_n(t) x^n = \sum_{n=0}^{\infty} g_n^*(t) \frac{x^n}{n!} \qquad (x, t \in \mathbb{R}); \tag{2.7}$$

(iii) *the polynomials* $\{g_n(t)\}_{n=0}^{\infty}$ *satisfy*

$$n g_n(t) = n g_{n-1}(t) + t g_n'(t) \qquad (t \in \mathbb{R}; n = 1, 2, 3, \ldots); \tag{2.8}$$

(iv) *the polynomials* $\{g_{n,p}(t)\}$ $(n, p = 0, 1, 2, \ldots)$ *satisfy*

$$g_{n+1,p}(t) = g_{n,p}(t) + t g_{n,p+1}(t) \qquad (t \in \mathbb{R}; n, p = 0, 1, 2, \ldots); \tag{2.9}$$

(v) *if*

$$\Delta_{n,p}(t) := \Delta_{n,p}(t;f) := g_{n,p}^2(t) - g_{n-1,p}(t)g_{n+1,p}(t)$$
$$(n = 1,2,3,\dots;\ p = 0,1,2,\dots), \quad (2.10)$$

then

$$\Delta_{n,p}(t) = t^2\left[g_{n-1,p+1}^2(t) - g_{n-1,p}(t)g_{n-1,p+2}(t)\right]. \quad (2.11)$$

Proof. Direct verification yields (i)–(iv) (for (ii) and (iii) see, for example, Rainville [25, p. 133] and Rota [26]). To prove (v), we use (2.9) in the form

$$g_{n,p}(t) = g_{n-1,p}(t) + tg_{n-1,p+1}(t). \quad (2.12)$$

Substituting (2.9) and (2.12) into (2.10) and then using (2.9) in the form $g_{n,p}(t) = g_{n-1,p}(t) + tg_{n-1,p+1}(t)$, yields

$$\Delta_{n,p}(t) = t^2g_{n-1,p+1}^2(t) + tg_{n-1,p}(t)g_{n-1,p+1}(t) - tg_{n-1,p}(t)g_{n,p+1}(t)$$
$$= t^2g_{n-1,p+1}^2(t) + tg_{n-1,p}(t)\left[g_{n-1,p+1}(t) - g_{n,p+1}(t)\right].$$

Since by (2.9), $g_{n-1,p+1}(t) - g_{n,p+1}(t) = -tg_{n-1,p+2}(t)$, we obtain

$$\Delta_{n,p}(t) = t^2\left[g_{n-1,p+1}^2(t) - g_{n-1,p}(t)g_{n-1,p+2}(t)\right],$$

the desired result of (2.11). □

The following two known propositions provide a characterization of the functions in the Laguerre–Pólya class in terms of the *Turán differences* $\Delta_{n,p}(t)$, defined by (2.10). For simplicity, in the sequel we will adopt also the following notation:

$$\Delta_n(t) := \Delta_{n,0}(t) := \Delta_{n,0}(t;f)$$
$$= g_n^2(t) - g_{n-1}(t)g_{n+1}(t) \qquad (n = 1,2,3,\dots). \quad (2.13)$$

PROPOSITION 2.2 [4, 10, 23]. *Suppose that the real entire function $f(x)$, defined by* (2.1) *with $\gamma_0 \neq 0$, is in the Laguerre–Pólya class. Let $g_n(t)$, $P_n(t)$, and $\Delta_n(t)$ denote the associated Jensen polynomials, Appell polynomials, and Turán differences (cf.* (2.2), (2.4), (2.10), *and* (2.13)). *Then*

$$g_n(t), P_n(t) \in \mathcal{L}\text{-}\mathcal{P} \qquad (n = 0,1,2,\dots), \quad (2.14)$$

and for each real t, either

$$\Delta_n(t) > 0\ (n = 1,2,3,\dots) \qquad or \qquad \Delta_n(t) = 0\ (n = 1,2,3,\dots). \quad (2.15)$$

PROPOSITION 2.3 [5, 10, 21]. *Let $f(x)$ be a real entire function defined by* (2.1) *with* $\gamma_0 \neq 0$. *If*

$$\gamma_{k-1}\gamma_{k+1} < 0 \qquad \text{whenever } \gamma_k = 0 \; (k = 1, 2, 3, \ldots), \tag{2.16}$$

and if (2.15) *holds, then* $f(x) \in \mathcal{L}-\mathcal{P}$. *Moreover, if $f(x)$ has infinitely many zeros, then a necessary and sufficient condition for $f(x)$ to be in the Laguerre–Pólya class is that*

$$\Delta_n(t) = g_n^2(t) - g_{n-1}(t)g_{n+1}(t) > 0 \qquad (t \in \mathbb{R} - \{0\}; n = 1, 2, 3, \ldots). \tag{2.17}$$

Remarks. Consider first the particular even polynomial $\tilde{f}(x) := 1 - 10x^2 + x^6$. A straightforward (but lengthy) calculation shows that, for *all* $n = 1, 2, 3, \ldots,$

$$\Delta_n(t; \tilde{f}) > 0 \; (t \in \mathbb{R} - \{0\}) \qquad \text{and} \qquad \Delta_n(0; \tilde{f}) = 0,$$

so that (2.15) is valid for $\tilde{f}(x)$. However, because $\tilde{f}(x)$ has two nonreal zeros, then $\tilde{f}(x) \notin \mathcal{L}-\mathcal{P}$, and, from the first part of Proposition 2.3, it is evident that (2.16) must fail. Indeed, (2.16) fails for $\tilde{f}(x)$, since $\gamma_k = 0$ and $\gamma_{k-1}\gamma_{k+1} = 0$ for $k = 3$, 4, and 5. This shows that the first part of Proposition 2.3 is false if condition (2.16) is omitted. Continuing, in terms of the polynomials $g_n^*(t)$ and $P_n(t)$ (cf. (2.4) and (2.5)), inequality (2.17) becomes

$$\Delta_n^*(t) := t^{2n}\Delta_n(t^{-1}) = (g_n^*(t))^2 - g_{n-1}^*(t)g_{n+1}^*(t) \tag{2.18}$$

$$= (n+1)!(n-1)!\left[\frac{n}{n+1}P_n^2(t) - P_{n-1}(t)P_{n+1}(t)\right] > 0$$
$$(t \in \mathbb{R}; n = 1, 2, 3, \ldots). \tag{2.19}$$

Pólya's result [21, p. 24] asserts, in particular, that if the real entire function $f(x)$ is *not* of the form $e^{\beta x}Q(x)$, where $\beta \in \mathbb{R}$ and $Q(x)$ is a real polynomial, then $f(x) \in \mathcal{L}-\mathcal{P}$ if and only if (2.19) holds.

Preliminaries aside, we now proceed to relate the foregoing results to the entire functions

$$F(x) := \int_{-\infty}^{\infty} e^{ixt}\Phi(t)\, dt \tag{2.20}$$

and

$$F_c(x) := \int_{-\infty}^{\infty} \cosh(t\sqrt{x})\Phi(t)\, dt, \tag{2.21}$$

where $\Phi(t)$ is defined by (1.2). (For results pertaining to the Taylor coefficients of these real entire functions, see [6, 8, 9].)

THEOREM 2.4. *Consider the real entire function $F(x)$ defined by (2.20), where the kernel $\Phi(t)$ is given by (1.2). Set*

$$c_{m,n}(\alpha) := \int_{-\infty}^{\infty} (it)^m(\alpha + it)^n \Phi(t)\, dt \qquad (\alpha \in \mathbb{R};\ m, n = 0, 1, 2, \ldots). \tag{2.22}$$

Then, $F(x) \in \mathcal{L}-\mathcal{P}$ if and only if the moments $c_{m,n}(\alpha)$ in (2.22) satisfy the Turán inequalities

$$I_n(\alpha) := c_{1,n-1}^2(\alpha) - c_{0,n-1}(\alpha)c_{2,n-1}(\alpha) > 0 \qquad (\alpha \in \mathbb{R};\ n = 1, 2, 3, \ldots). \tag{2.23}$$

Proof. Since, by Theorem A, $\Phi(t)$ is an even function, an easy verification shows that $c_{m,n}(\alpha)$ is real for all $\alpha \in \mathbb{R}$ and for all $m, n = 0, 1, 2, \ldots$. Now by (2.7) of Proposition 2.1, the sequence $\{g_n^*(\alpha)\}_{n=0}^{\infty}$ is generated by

$$e^{x\alpha}F(x) = \sum_{n=0}^{\infty} \frac{g_n^*(\alpha)}{n!}x^n \qquad (\alpha \in \mathbb{R})$$

$$= \int_{-\infty}^{\infty} e^{(\alpha+it)x}\Phi(t)\, dt.$$

Consequently, from this last integral we infer that $g_n^*(\alpha)$ has the representation

$$g_n^*(\alpha) = \int_{-\infty}^{\infty} (\alpha + it)^n \Phi(t)\, dt, \tag{2.24}$$

so that from (2.22),

$$g_n^*(\alpha) = c_{0,n}(\alpha) \qquad (\alpha \in \mathbb{R};\ n = 0, 1, 2, \ldots). \tag{2.25}$$

Next, with (2.22) we can directly represent $I_n(\alpha)$ of (2.23) as the double integral

$$I_n(\alpha) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Phi(s)\Phi(t)(\alpha + is)^{n-1}(\alpha + it)^{n-1}[t^2 - st]\, dt\, ds.$$

As interchanging the roles of s and t in the above integral clearly leaves $I_n(\alpha)$ unchanged, then

$$I_n(\alpha) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Phi(t)\Phi(s)(\alpha + it)^{n-1}(\alpha + is)^{n-1}[s^2 - st]\, ds\, dt,$$

and averaging the above two expressions yields

$$I_n(\alpha) = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\Phi(s)\Phi(t)(\alpha + is)^{n-1}(\alpha + it)^{n-1}(s - t)^2\,dt\,ds$$
$$(\alpha \in \mathbb{R};\ n = 1,2,3,\dots). \quad (2.26)$$

In a completely similar fashion, it is easily seen from (2.25) and (2.22) that $((g_n^*(\alpha))^2 - g_{n-1}^*(\alpha)g_{n+1}^*(\alpha)$ has the *same* integral representation as $I_n(\alpha)$ in (2.26), whence

$$I_n(\alpha) = (g_n^*(\alpha))^2 - g_{n-1}^*(\alpha)g_{n+1}^*(\alpha) \qquad (\alpha \in \mathbb{R};\ n = 1,2,3,\dots). \quad (2.27)$$

Since it is known (cf. Hardy [11] or Pólya [19]) that $F(x)$ has an infinite number of real zeros, it follows from (2.17) of Proposition 2.3, (2.18), and (2.27) that $I_n(\alpha) > 0$ $(\alpha \in \mathbb{R};\ n = 1,2,3,\dots)$ if and only if $F(x) \in \mathcal{L}-\mathcal{P}$.

□

Remarks. The significance of Theorem 2.4 is further underscored when it is expressed in terms of Pólya's universal factors (cf. [20 or 22]). We recall that an entire function $f(it)$ is a *universal factor* if the entire function

$$\int_{-\infty}^{\infty} e^{izt}f(it)\varphi(t)\,dt$$

has only real zeros, whenever the zeros of the Fourier transform

$$\int_{-\infty}^{\infty} e^{ixt}\varphi(t)\,dt$$

are all real, where $\varphi(t)\colon \mathbb{R} \to \mathbb{R}$ is integrable over $\mathbb{R}$, $\varphi(t) = \varphi(-t)$ for all $t \in \mathbb{R}$ and $\varphi(t) = O(e^{-|t|^{2+\varepsilon}})$ for some $\varepsilon > 0$, as $t \mapsto \pm\infty$. Now, Pólya [20] has shown that $f(it)$ is a universal factor if and only if $f(t) \in \mathcal{L}-\mathcal{P}$. Therefore, since

$$h_n(x) := h_n(x;\alpha) := (x + \alpha)^n \qquad (\alpha \in \mathbb{R};\ n = 0,1,2,\dots), \quad (2.28)$$

is in $\mathcal{L}-\mathcal{P}$, then $h_n(it) = h_n(it;\alpha)$ is a universal factor for each $\alpha \in \mathbb{R}$ and each $n = 0,1,2,\dots$. Moreover, if we apply the differential operator $h_n(D) := (D + \alpha)^n$, $D := d/dx$, to $F(x)$ defined by (2.20), then

$$H_n(x) := H_n(x;\alpha) := h_n(D)F(x) = \int_{-\infty}^{\infty} e^{ixt}(\alpha + it)^n\Phi(t)\,dt, \quad (2.29)$$

where the differentiation under the integral sign can be readily justified by virtue of the properties of $\Phi(t)$ (cf. Theorem A). But then $H_n(0) = g_n^*(\alpha)$ (cf. (2.24)) and, consequently, Theorem 2.4 states that the moments in (2.22), corresponding to the universal factors $h_n(it;\alpha)$ $(n = 0,1,2,\dots)$, satisfy the Turán inequalities (2.23).

The interest in the applications of universal factors stems, in part, from the fact that it is known (cf. Bruijn [3] or Newman [16]) that if we apply the universal factors $f_\lambda(it) := e^{\lambda t^2}$ $(\lambda \geq 0)$ and $h_\mu(it) = \cosh(\mu t)$ $(\mu \in \mathbb{R})$ to $F(x)$ of (2.20), then the functions

$$F_\lambda(x) := \int_{-\infty}^{\infty} e^{ixt} e^{\lambda t^2}\Phi(t)\,dt \qquad (\lambda \geq \tfrac{1}{2}) \tag{2.30}$$

and

$$H_\mu(x) := \int_{-\infty}^{\infty} e^{ixt}\cosh(\mu t)\Phi(t)\,dt \qquad (\mu \geq 1) \tag{2.31}$$

belong to the Laguerre–Pólya class for the indicated values of λ and μ. In the case when λ is negative, it was proved in [7] that $F_\lambda(x)$ has nonreal zeros for $\lambda \leq -50$, so that $F_\lambda(x) \notin \mathscr{L}-\mathscr{P}$ for $\lambda \leq -50$.

Now, the above remarks, together with (2.14) of Proposition 2.2, imply that the Jensen polynomials associated with $F_\lambda(x)$, for $\lambda \geq \frac{1}{2}$, and with $H_\mu(x)$, for $\mu \geq 1$, all have only real zeros. Therefore, using the properties of $\Phi(t)$, the generating relation (2.6), and the fact that $\mathrm{Re}(1 + ixt)^n = (1 + x^2t^2)^{n/2}\cos(n\tan^{-1}(xt))$ $(n = 0,1,2,\dots)$, we directly obtain the following proposition.

PROPOSITION 2.5. *The Jensen polynomials $g_n(x;F_\lambda)$ and $g_n(x;H_\mu)$, associated with $F_\lambda(x)$ (cf. (2.30)) and $H_\mu(x)$ (cf. (2.31)), respectively, have the following representations*:

$$g_n(x;F_\lambda) := 2\int_0^{\infty} (1 + x^2t^2)^{n/2} e^{\lambda t^2}\Phi(t)\cos(n\tan^{-1}(xt))\,dt \qquad (\lambda \in \mathbb{R};\ n = 0,1,2,\dots), \tag{2.32}$$

and

$$g_n(x;H_\mu) := 2\int_0^{\infty} (1 + x^2t^2)^{n/2}\cosh(\mu t)\Phi(t)\cos(n\tan^{-1}(xt))\,dt \qquad (\mu \in \mathbb{R};\ n = 0,1,2,\dots). \tag{2.33}$$

Moreover, $g_n(x;F_\lambda)$ and $g_n(x;H_\mu)$ $(n = 0,1,2,\dots)$ have only real zeros for $\lambda \geq \frac{1}{2}$ and $\mu \geq 1$, respectively.

Since, for fixed real numbers λ and μ, the entire functions $F_\lambda(x)$ and $H_\mu(x)$ are even entire functions of order one, it follows from the Hadamard factorization theorem that $F_\lambda(x)$ and $H_\mu(x)$ each have infinitely many zeros. Therefore, since $g_n(x; F_\lambda)$ of (2.32) is *even* in x, then by Proposition 2.3, for each fixed λ, $F_\lambda(x) \in \mathscr{L}-\mathscr{P}$ if and only if the Jensen polynomials, $g_n(x; F_\lambda)$ (cf. (2.32)), associated with $F_\lambda(x)$, satisfy the Turán inequality (2.17) for all $x > 0$ and all $n = 1, 2, 3, \ldots$. Similarly, for each fixed μ, $H_\mu(x) \in \mathscr{L}-\mathscr{P}$ if and only if the Jensen polynomials $g_n(x; H_\mu)$ (cf. (2.33)), associated with $H_\mu(x)$, satisfy the Turán inequality (2.17) for all $x > 0$ and all $n = 1, 2, 3, \ldots$.

We next consider the function $F_c(x)$ (cf. (2.21)) and observe, in contrast with the previous paragraph, that its associated Jensen polynomials $g_n(x; F_c)$, are *not* even polynomials. However, we will prove that the polynomials $g_n(x; F_c)$ satisfy the Turán inequality (2.17) for all $x > 0$ and all $n = 1, 2, 3, \ldots$. To this end, we consider

$$F_c(x) := 2 \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k, \tag{2.34}$$

where

$$\gamma_k := \frac{k!}{(2k)!} \hat{b}_k \quad \text{and} \quad \hat{b}_k := \int_0^\infty t^{2k} \Phi(t)\, dt \quad (k = 0, 1, 2, \ldots), \tag{2.35}$$

and we first establish a relationship between the Turán differences

$$T_k := \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \quad (k = 1, 2, 3, \ldots) \tag{2.36}$$

and

$$\Delta_{n,p}(t) := g_{n,p}^2(t) - g_{n-1,p}(t) g_{n+1,p}(t) \quad (t \in \mathbb{R};\ n = 1, 2, 3, \ldots), \tag{2.37}$$

where

$$g_{n,p}(t) := \sum_{k=0}^{n} \binom{n}{k} \gamma_{k+p} t^k \quad (t \in \mathbb{R};\ n = 0, 1, 2, \ldots). \tag{2.38}$$

The fact that $T_k > 0$ $(k = 1, 2, 3, \ldots)$ is a necessary condition for $\Delta_n(t) := \Delta_{n,0}(t) > 0$, for all $t \in \mathbb{R} - \{0\}$ and $n = 1, 2, 3, \ldots$ to hold, is a consequence of the following direct calculations: If

$$\Delta_n(t) := \sum_{k=2}^{2n} d_k(n) t^k \quad (t \in \mathbb{R};\ n = 1, 2, 3, \ldots),$$

then

$$\begin{aligned} d_{2n}(n) &:= \gamma_n^2 - \gamma_{n-1}\gamma_{n+1}, \\ d_{2n-1}(n) &:= (n-1)(\gamma_{n-1}\gamma_n - \gamma_{n-2}\gamma_{n+1}), \\ d_2(n) &:= \gamma_1^2 - \gamma_0\gamma_2. \end{aligned} \tag{2.39}$$

In addition, if $T_k > 0$ $(k = 1, 2, 3, \dots)$, then, since (cf. (2.39) and (2.36))

$$\Delta_n(t) = T_1 t^2 + \sum_{k=3}^{2n-1} d_k(n) t^k + T_n t^{2n} \qquad (n = 2, 3, 4, \dots),$$

it is evident that, for *each* $n \geq 1$, there are positive constants $M_n := M(n, \gamma_0, \gamma_1, \dots, \gamma_n)$ and $\mu_n := \mu(n, \gamma_0, \gamma_1, \dots, \gamma_n)$ with $0 < \mu_n \leq M_n$ such that

$$\Delta_n(t) > 0 \qquad \text{for all } t \in S_n, \tag{2.40}$$

where

$$S_n := (-\mu_n, 0) \cup (0, \mu_n) \cup (-\infty, -M_n) \cup (M_n, \infty). \tag{2.41}$$

LEMMA 2.6. *Let* $f(x) := \sum_{k=0}^{\infty}(\gamma_k/k!)x^k$ *denote a real entire function and let* $g_{n,p}(t)$ $(n, p = 0, 1, 2, \dots)$ *denote the Jensen polynomials associated with* $f^{(p)}(x)$ $(p = 0, 1, 2, \dots)$. *If*

$$T_k := \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} > 0 \; (k = 1, 2, 3, \dots) \qquad \textit{and if}$$
$$\gamma_k > 0 \; (k = 0, 1, 2, \dots), \tag{2.42}$$

then the Turán differences, evaluated at $t = 1$, *satisfy*

$$\Delta_{n,p}(1) := g_{n,p}^2(1) - g_{n-1,p}(1) g_{n+1,p}(1) > 0$$
$$(n = 1, 2, 3, \dots; p = 0, 1, 2, \dots). \tag{2.43}$$

Proof. We will prove (2.43) by induction on n. For $n = 1$ and $p = 0, 1, 2, \dots$, we have from (2.38) that

$$\Delta_{1,p}(1) = g_{1,p}^2(1) - g_{0,p}(1) g_{2,p}(1) = \gamma_{p+1}^2 - \gamma_p \gamma_{p+2} = T_{p+1} > 0. \tag{2.44}$$

For $n = 2$ and $p = 0, 1, 2, \dots$,

$$\begin{aligned} \Delta_{2,p}(1) &= g_{2,p}^2(1) - g_{1,p}(1) g_{3,p}(1) \\ &= T_{p+1} + T_{p+2} + (\gamma_{p+1}\gamma_{p+2} - \gamma_p \gamma_{p+3}) > 0, \end{aligned} \tag{2.45}$$

where the hypotheses of (2.42) imply that $T_{p+1} > 0$ and $T_{p+2} > 0$, as well as

$$\frac{\gamma_{p+1}}{\gamma_p} > \frac{\gamma_{p+2}}{\gamma_{p+1}} > \frac{\gamma_{p+3}}{\gamma_{p+2}}, \tag{2.46}$$

whence, on cross-multiplying, $\gamma_{p+1}\gamma_{p+2} - \gamma_p\gamma_{p+3} > 0$. Next, we assume that

$$\Delta_{k,p}(1) > 0 \qquad \text{for } k = 1, 2, \ldots, n \text{ and } p = 0, 1, 2, \ldots. \tag{2.47}$$

Then by (2.11) of Proposition 2.1 and the induction assumption (2.47),

$$\Delta_{n,p}(1) = g_{n-1,p+1}^2(1) - g_{n-1,p}(1)g_{n-1,p+2}(1) > 0 \qquad (p = 0, 1, 2, \ldots), \tag{2.48}$$

and, since (from (2.38)) the positivity of the γ_k's implies $g_{n,p}(1) > 0$ for all $n, p = 0, 1, 2, \ldots,$ then (2.48) yields

$$\frac{g_{n-1,p+1}(1)}{g_{n-1,p}(1)} > \frac{g_{n-1,p+2}(1)}{g_{n-1,p+1}(1)} > \frac{g_{n-1,p+3}(1)}{g_{n-1,p+2}(1)} \qquad (p = 0, 1, 2 \ldots). \tag{2.49}$$

Using the induction assumption (2.47), we will show that

$$\Delta_{n+1,p}(1) > 0 \qquad (p = 0, 1, 2, \ldots). \tag{2.50}$$

Let p be a fixed, but arbitrary, nonnegative integer. Then by (2.11),

$$\Delta_{n+1,p}(1) = g_{n,p+1}^2(1) - g_{n,p}(1)g_{n,p+2}(1), \tag{2.51}$$

and by (2.9),

$$g_{n,p}(1) = g_{n-1,p}(1) + g_{n-1,p+1}(1). \tag{2.52}$$

Thus, if we apply (2.52) to each of the three terms on the right of (2.51), then after some simplifications, (2.51) becomes

$$\Delta_{n+1,p}(1) = \Delta_{n,p}(1) + \Delta_{n,p+1}(1) + g_{n-1,p+1}(1)g_{n-1,p+2}(1) - g_{n-1,p}(1)g_{n-1,p+3}(1). \tag{2.53}$$

Therefore, it follows from (2.47) and (2.49) that $\Delta_{n+1,p}(1) > 0$ for any $p = 0, 1, 2, \ldots,$ which completes the induction. □

We remark that if $t_0 > 0$, if $\alpha_k > 0$ $(k = 0, 1, 2, \ldots)$, and if $\alpha_k^2 - \alpha_{k-1}\alpha_{k+1} > 0$ $(k = 1, 2, 3, \ldots)$, then with $\gamma_k := t_0^k \alpha_k$, it follows that

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} = t_0^{2k}\left(\alpha_k^2 - \alpha_{k-1}\alpha_{k+1}\right) > 0 \qquad (k = 1, 2, 3, \ldots). \quad (2.54)$$

Therefore, using Lemma 2.6 and (2.54), we obtain the following theorem.

THEOREM 2.7. *Let* $f(x) := \sum_{k=0}^{\infty}(\gamma_k/k!)x^k$ *denote a real entire function and let* $g_{n,p}(t)$ $(n, p = 0, 1, 2, \ldots)$ *denote the Jensen polynomials associated with* $f^{(p)}(x)$ $(p = 0, 1, 2, \ldots)$. *If* (2.42) *holds, then the Turán differences satisfy*

$$\Delta_{n,p}(t) := g_{n,p}^2(t) - g_{n-1,p}(t)g_{n+1,p}(t) > 0$$
$$(t > 0;\ n = 1, 2, 3, \ldots;\ p = 0, 1, 2, \ldots). \quad (2.55)$$

As an application of Theorem 2.7 and Theorem A, we have

THEOREM 2.8. *Set*

$$F_c(x) := 2\sum_{k=0}^{\infty} \frac{\gamma_k}{k!}x^k, \quad (2.56)$$

where

$$\gamma_k := \frac{k!}{(2k)!}\hat{b}_k \quad \textit{and} \quad \hat{b}_k := \int_0^{\infty} t^{2k}\Phi(t)\,dt \qquad (k = 0, 1, 2, \ldots). \quad (2.57)$$

Then,

$$\Delta_{n,p}(t) := \Delta_{n,p}(t; F_c) = g_{n,p}^2(t) - g_{n-1,p}(t)g_{n+1,p}(t) > 0$$
$$(t > 0;\ n = 1, 2, 3, \ldots;\ p = 0, 1, 2, \ldots), \quad (2.58)$$

where

$$g_{n,p}(t) := g_{n,p}(t; F_c) := 2\sum_{k=0}^{n}\binom{n}{k}\gamma_{k+p}t^k$$
$$(n = 0, 1, 2, \ldots;\ p = 0, 1, 2, \ldots). \quad (2.59)$$

Proof. By Theorem A, $\log(\Phi(\sqrt{t}))$ is strictly concave for $0 < t < \infty$. Hence, it follows from a known result (cf. [8]) that

$$\left(\hat{b}_k\right)^2 - \left(\frac{2k-1}{2k+1}\right)\hat{b}_{k-1}\hat{b}_{k+1} > 0 \qquad (k = 1, 2, 3, \ldots).$$

But from (2.57), the above inequalities are equivalent to $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} > 0$ $(k = 1, 2, 3, \dots)$. In addition, since $\Phi(t) > 0$ for all $t \in \mathbb{R}$, then $\gamma_k > 0$ $(k = 0, 1, 2, \dots)$, and so (2.42) holds. Therefore, (2.58) is an immediate consequence of Theorem 2.7. □

Remarks. Since $F(x)$ is an entire function of order 1 (cf. Section 1), it follows that $F_c(x)$ is of order $\frac{1}{2}$ and thus has an infinite number of zeros. Hence by Proposition 2.3, a necessary and sufficient condition for $F_c(x)$ to be in the Laguerre–Pólya class is that the Turán inequalities hold:

$$\left(g_{n,p}(t; F_c)\right)^2 - g_{n-1,p}(t; F_c) g_{n+1,p}(t; F_c) > 0$$
$$(t \in \mathbb{R} - \{0\}; n = 1, 2, 3, \dots; p = 0, 1, 2, \dots), \tag{2.60}$$

where $g_{n,p}(t; F_c)$ is defined by (2.59). In light of this, the importance of Theorem 2.8 stems from the fact that it establishes that the Turán inequalities of (2.60) hold for the Jensen polynomials associated with $F_c(x)$, for *all* $t > 0$. (We note that if (2.60) holds for *all* real $t \neq 0$, then by Proposition 2.3, $F_c(x) \in \mathscr{L}-\mathscr{P}$ and, as noted in Section 1, this is *equivalent* to the truth of the Riemann hypothesis!)

We also remark that by (2.57) we have the following integral representation for the Jensen polynomials $g_n(t; F_c)$:

$$g_n(t; F_c) = 2\int_0^\infty G_n(ts^2)\Phi(s)\, ds \qquad (t \in \mathbb{R}; n = 0, 1, 2, \dots), \tag{2.61}$$

where

$$G_n(s) := \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(2k)!} s^k \qquad (s \in \mathbb{R}; n = 0, 1, 2, \dots). \tag{2.62}$$

Moreover, the polynomials $G_n(s)$ of (2.62) are the Jensen polynomials associated with the particular Mittag–Leffler function $E_2(x)$, defined by

$$E_2(x) := \sum_{k=0}^{\infty} \frac{x^k}{(2k)!}. \tag{2.63}$$

Since it is known that $E_2(x)$ is in $\mathscr{L}-\mathscr{P}$ and that $E_2(x)$ has infinitely many zeros (cf. Pólya [18]), $G_n(s)$ has only real (negative) zeros (cf. Proposition 2.2), and by Proposition 2.3, the Turán inequalities, namely

$$G_n^2(s) - G_{n-1}(s)G_{n+1}(s) > 0 \qquad (s \in \mathbb{R} - \{0\}; n = 1, 2, 3, \dots), \tag{2.64}$$

are satisfied. Combining the foregoing results, we have by Proposition 2.3, (2.61), and Theorem 2.8, that $F_c(x) \in \mathscr{L}-\mathscr{P}$ if and only if

$$\int_0^{\infty}\int_0^{\infty}\Phi(s)\Phi(t)\left[G_n(xt^2)G_n(xs^2) - G_{n-1}(xt^2)G_{n+1}(xs^2)\right] dt\, ds > 0$$
$$(x < 0;\ n = 1,2,3,\dots). \quad (2.65)$$

(For related results, see also [9].)

We next turn to the extensions and applications of some known results pertaining to the complex-variable characterizations of functions in the Laguerre–Pólya class.

THEOREM 2.9 [14, 15, 21]. *Let*

$$f(z) := e^{-\alpha z^2} f_1(z) \qquad (\alpha \geq 0,\ f(z) \not\equiv 0), \qquad (2.66)$$

where $f_1(z)$ is a real entire function of genus 0 *or* 1. *Set*

$$L_n(f(x)) := \sum_{k=0}^{2n} \frac{(-1)^{k+n}}{(2n)!}\binom{2n}{k} f^{(k)}(x) f^{(2n-k)}(x)$$
$$(x \in \mathbb{R};\ n = 0,1,2,\dots). \quad (2.67)$$

Then, $f(z) \in \mathscr{L}-\mathscr{P}$ *if and only if*

$$L_n(f(x)) \geq 0 \qquad (x \in \mathbb{R};\ n = 0,1,2,\dots). \qquad (2.68)$$

Proof. Since $f(z)$ is a real entire function, the Taylor series expansion of $|f(z)|^2$ ($z := x + iy$; $x, y \in \mathbb{R}$), with respect to y, about the origin, has the form

$$|f(z)|^2 = f(x+iy) f(x-iy) =: \sum_{n=0}^{\infty} A_n(x) y^{2n} \qquad (x, y \in \mathbb{R}), \quad (2.69)$$

where, by direct verification, $A_n(x) = L_n(f(x))$ of (2.67) for all $x \in \mathbb{R}$ and $n = 0, 1, 2, \dots$. In [15], Patrick proved that if $f(z) \in \mathscr{L}-\mathscr{P}$, then (2.68) holds.

Conversely, assume that (2.68) holds, i.e., $A_n(x) \geq 0$ for all $x \in \mathbb{R}$ and $n = 0, 1, 2, \dots$. Suppose that $z_0 := x_0 + iy_0$ is a nonreal zero of $f(z)$, so that by (2.69),

$$0 = |f(z_0)|^2 = \sum_{n=0}^{\infty} A_n(x_0) y_0^{2n}. \qquad (2.70)$$

As $y_0 \neq 0$ and as $A_n(x_0) \geq 0$, it follows from (2.70) that $A_n(x_0) = 0$ for all $n = 0, 1, 2, \ldots,$ and thus, for *any* $y \in \mathbb{R}$,

$$0 = |f(x_0 + iy)|^2 = \sum_{n=0}^{\infty} A_n(x_0) y^{2n}. \qquad (2.71)$$

But, as $f(z)$ is a real entire function, (2.71) implies that $f(z) \equiv 0$, which contradicts (2.66). Thus if (2.68) holds, then all zeros of $f(z)$ are real, and hence, $f(z) \in \mathscr{L}-\mathscr{P}$. □

Remarks. Since $\mathscr{L}-\mathscr{P}$ is closed under differentiation (cf. Obreschkoff [17]), then as a consequence of Theorem 2.9, the function $f(z)$ defined by (2.66) is in $\mathscr{L}-\mathscr{P}$ if and only if (cf. (2.67))

$$L_n(f^{(p)}(x)) \geq 0 \qquad (x \in \mathbb{R};\, n = 0, 1, 2, \ldots;\, p = 0, 1, 2, \ldots). \quad (2.72)$$

In particular, if we apply the result (2.72) to the function $F(x)$ of (2.20), we obtain, after some calculations, that $F(x) \in \mathscr{L}-\mathscr{P}$ if and only if

$$L_n(F^{(p)}(x)) = \frac{(-1)^p}{(2n)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (ts)^p \Phi(t) \Phi(s) \cdot \cos(x(t+s))(s-t)^{2n}\, dt\, ds \geq 0$$
$$(x \in \mathbb{R};\, n, p = 0, 1, 2, \ldots). \quad (2.73)$$

An elegant characterization of functions in the Laguerre–Pólya class is contained in the following theorem essentially due to Jensen [14].

THEOREM 2.10 [14]. *Let $f(z)$ be the real entire function defined by* (2.66). *Then, $f(z) \in \mathscr{L}-\mathscr{P}$ if and only if*

$$|f'(z)|^2 \geq \operatorname{Re}\{f(z)\overline{f''(z)}\} \qquad \textit{for all } z \in \mathbb{C}. \quad (2.74)$$

Proof. Suppose $f(z) \in \mathscr{L}-\mathscr{P}$. Then as in the proof of Theorem 2.8 (cf. (2.69)) with $A_n(x) := L_n(f(x))$, we have

$$|f(x + iy)|^2 = \sum_{n=0}^{\infty} L_n(f(x)) y^{2n} \qquad (x, y \in \mathbb{R}). \quad (2.75)$$

Moreover, from Theorem 2.9, $L_n(f(x)) \geq 0$ for all $x \in \mathbb{R}$ and $n = 0, 1, 2, \ldots$. Hence,

$$\frac{\partial^2}{\partial y^2}|f(x + iy)|^2 = \sum_{n=0}^{\infty} (2n+2)(2n+1) L_{n+1}(f(x)) y^{2n} \geq 0$$
$$(x, y \in \mathbb{R}). \quad (2.76)$$

But a computation shows that

$$\frac{\partial^2}{\partial y^2}|f(x+iy)|^2 = 2|f'(z)|^2 - 2\,\mathrm{Re}\{f(z)\overline{f''(z)}\}. \tag{2.77}$$

Thus, (2.76) and (2.77) establish (2.74).

Conversely, suppose (2.74) holds. Let $z_0 := x_0 + iy_0$ be a zero of $f(z)$, so that $f(x_0 + iy_0) = 0$. Let

$$M(y; x_0) := |f(x_0 + iy)|^2 \qquad (y \in \mathbb{R}). \tag{2.78}$$

Clearly, $M(y; x_0) \geq 0$ for all $y \in \mathbb{R}$, and $M(y; x_0)$ is an even function of y. Furthermore, by (2.74) and (2.77), $(\partial^2/\partial y^2)M(y; x_0) \geq 0$ for all $y \in \mathbb{R}$, so that $M(y; x_0)$ is a convex even function of y. Consequently, since $f(z) \not\equiv 0$ from (2.66), then $M(y; x_0)$ has a *unique* minimum, which must occur for $y = 0$. Since $f(x_0 + iy_0) = 0$, we conclude that $y_0 = 0$. Therefore, we have proved that any zero of $f(z)$ of the form (2.66) must be real, and hence, $f(z) \in \mathscr{L}-\mathscr{P}$. □

If we apply condition (2.74) of Theorem 2.10 to the function $F(z)$ of (2.20), then using the double integral method of the proof of Theorem 2.4 gives a double integral condition (2.79′) which is an even as a function of x and even as a function of y (where $z = x + iy$). Thus, for this application of Theorem 2.10, it suffices to restrict $z = x + iy$ to the first quadrant, and we have the following corollary (cf. Pólya [21]).

COROLLARY 2.11. *The function $F(x)$ defined by* (2.20) *is in the Laguerre–Pólya class if and only if*

$$I(x, y; \Phi) \geq 0 \qquad \textit{for all } x, y > 0, \tag{2.79}$$

where

$$I(x, y; \Phi) := \int_0^\infty \int_0^\infty \Phi(s)\Phi(t)\{(s+t)^2 \cos(x(s-t))\cosh(y(s+t))$$

$$+(s-t)^2\cos(x(s+t))\cosh(y(s-t))\}\,dt\,ds. \tag{2.79′}$$

Remark. Note that if we replace $\Phi(t)$ by $\Psi(t) := \Phi(t)\cosh t$, then as in the discussion of (2.31),

$$I(x, y; \Psi) \geq 0 \qquad \text{for all } x, y > 0. \tag{2.80}$$

The next theorem provides a new and particularly simple characterization of the functions in the Laguerre–Pólya class.

THEOREM 2.12. *Let $f(z)$ be an entire function of the form*

$$f(z) := Ce^{-\alpha z^2+\beta z}z^n \prod_{k=1}^{\omega}(1-z/z_k)e^{z/z_k} \qquad (\omega \le \infty), \tag{2.81}$$

where $\alpha \ge 0, C$ and β are real numbers, n is a nonnegative integer, and the z_k's are nonzero with $\sum_{k=1}^{\omega}|z_k|^{-2} < \infty$, and the zeros $\{z_k\}_{k=1}^{\infty}$ of $f(z)$ are counted according to multiplicity and are arranged so that $0 < |z_1| \le |z_2| \le \cdots$. Then, $f(z) \in \mathscr{L}-\mathscr{P}$ if and only if

$$\frac{1}{y}\operatorname{Im}\{-f'(z)\overline{f(z)}\} \ge 0 \qquad \textit{for all } z := x+iy \in \mathbb{C}, y \ne 0. \tag{2.82}$$

Proof. Setting $z_k := x_k + iy_k$ $(k = 1,2,3,\dots)$ and $z = x + iy$, then for $y \ne 0$ and $z \ne z_k$, a straightforward calculation yields that

$$R(z) := \frac{1}{y}\operatorname{Im}\left\{-\frac{f'(z)}{f(z)}\right\} = \frac{1}{y|f(z)|^2}\operatorname{Im}\{-f'(z)\overline{f(z)}\}$$
$$= \frac{n}{x^2+y^2} + \sum_{k=1}^{\omega}\left\{\frac{1-y_k/y}{(x-x_k)^2+(y-y_k)^2} + \frac{y_k/y}{x_k^2+y_k^2}\right\}, \tag{2.83}$$

where, in case $\omega = \infty$, the uniform and absolute convergence of the series, on compact subsets S of $\mathbb{C}$ with $0 \notin S$ and $z_k \notin S$ $(k = 1,2,3,\dots)$, follows from (2.81). Now, if $f(z) \in \mathscr{L}-\mathscr{P}$, then $y_k = 0$ $(k = 1,2,3,\dots)$ and (2.82) then holds.

Conversely, suppose that (2.82) holds. We will show that the assumption that $\operatorname{Im} z_k = y_k \ne 0$ for some (positive integer) k, leads to a contradiction. Without loss of generality, we may assume that $y_1 \ne 0$ and that the zero $z_1 = x_1 + iy_1$ is simple, since the argument in the case when the multiplicity of z_1 is greater than one is, *mutatis mutandis*, the same as the following argument. Indeed, with $y_1 \ne 0$, set $z(\varepsilon) = x_1 + iy_1(1-\varepsilon)$, where $0 < \varepsilon < 1$ and ε is sufficiently small. Then, by (2.83), we have

$$R(z(\varepsilon)) = \frac{n}{x_1^2+y_1^2(1-\varepsilon)^2} - \frac{1}{\varepsilon y_1^2(1-\varepsilon)}$$
$$+ \frac{1}{(1-\varepsilon)(x_1^2+y_1^2)} + S_2(z(\varepsilon)), \tag{2.84}$$

where

$$S_2(z(\varepsilon)) := \sum_{k=2}^{\omega}\left[\frac{1-y_k/[y_1(1-\varepsilon)]}{(x_1-x_k)^2+[y_1(1-\varepsilon)-y_k]^2} + \frac{y_k/[y_1(1-\varepsilon)]}{x_k^2+y_k^2}\right]. \tag{2.85}$$

Thus, for $\varepsilon > 0$ sufficiently small, the sum in (2.84) will be negative, which contradicts (2.82). Hence, Im $z_k = y_k = 0$, and therefore, $f(z) \in \mathscr{L}-\mathscr{P}$. □

Remark. It is also possible to give a geometric interpretation of condition (2.82). To see this, we first note that (2.82) can be written as

$$m(y;x) := \frac{1}{2y}\frac{\partial}{\partial y}|f(z)|^2 = \frac{1}{y}\mathrm{Im}\{-f'(z)\overline{f(z)}\} \geq 0$$
$$(z = x + iy \in \mathbb{C};\ y \neq 0). \quad (2.86)$$

Thus, we see that for each fixed $x \in \mathbb{R}$, $|f(x+iy)|^2$, as a function of y, is nondecreasing for $y > 0$ and is nonincreasing for $y < 0$, and so $|f(x+iy)|^2$ attains its minimum only for $y = 0$, unless $f(z) \equiv C$ in (2.81).

Applying condition (2.82) of Theorem 2.12 to the function $F(z)$ of (2.20) similarly yields the following corollary (by means of the double integral method previously employed).

COROLLARY 2.13. *The function $F(x)$ defined by (2.20) is in the Laguerre–Pólya class if and only if*

$$\int_0^\infty\int_0^\infty \Phi(s)\Phi(t)\{(t-s)\cos(x(t+s))\sinh(y(t-s))$$
$$+(t+s)\cos(x(t-s))\sinh(y(t+s))\}\,dt\,ds \geq 0 \quad (2.87)$$

for all $y > 0$ and $x \geq 0$.

Remark. Since the Jacobian of the transformation $(t,s) \to (u,v)$ defined by $2u := t + s$ and $2v := t - s$, is nonzero, inequality (2.87) can be cast in the following equivalent form:

$$\iint_{0\leq|v|\leq u} \Phi(u+v)\Phi(u-v)$$
$$\cdot\{v\cos(2xu)\sinh(2yv) + u\cos(2xv)\sinh(2yu)\}\,du\,dv \geq 0 \quad (2.88)$$

for all $y > 0$ and $x \geq 0$.

3. NEW PROPERTIES OF $\Phi_j(t)$: SCHOLIA AND OPEN PROBLEMS

The purpose of this section is twofold. First, we establish some results concerning the distribution of zeros of functions related to the real entire function

$$F(x;\Phi) := \int_0^\infty \Phi(t)\cos(xt)\,dt. \tag{3.1}$$

We prove results when the transform (3.1) is replaced by the Fourier sine transform of $\Phi(t)$ and when the kernel $\Phi(t)$ is replaced by $t\Phi(t)$ (cf. Proposition 3.1) or by the related function $\Phi_j(t)$, where $\Phi_j(t)$ is defined by (3.11). In particular, we prove, for all positive integers j, that the Fourier cosine transform of $\Phi_j(t)$ is positive on the real axis (cf. Theorem 2.4) and that $\Phi_j(t)$ is convex for all $t \geq 0$ (cf. (3.24)). These considerations lead us to some interesting geometric interpretations concerning the zeros of $F(x;\Phi)$. In addition, by altering the interval of integration, we prove that the Fourier cosine integral of $\Phi(t)$ on $[0, 0.11]$ is a function in the Laguerre–Pólya class.

Second, we state in this section three open problems which are of independent interest and which may shed light on the nature of the distribution of zeros of $F(x;\Phi)$ in (3.1). In particular, Open Problem 3 appears to be tractable in light of the author's recent investigations (cf. [8, 9]).

PROPOSITION 3.1. *Set*

$$f_1(x) := \int_0^\infty \Phi(t)\sin(xt)\,dt \qquad (x \in \mathbb{R}), \tag{3.2}$$

where

$$\Phi(t) := \sum_{n=1}^{\infty} a_n(t)$$

and

$$a_n(t) := \pi n^2(2\pi n^2 e^{4t} - 3)\exp(5t - \pi n^2 e^{4t}) \qquad (t \in \mathbb{R};\ n = 1,2,3,\ldots). \tag{3.3}$$

Then, the following assertions hold:

(a) *For* $x_0 \in \mathbb{R}$, $f_1(x_0) = 0$ *if and only if* $x_0 = 0$.

(b) *The function*

$$f_1'(x) = \int_0^\infty t\Phi(t)\cos(xt)\,dt \tag{3.4}$$

cannot have an infinite number of zeros in any horizontal strip of the form

$$S(\tau) := \{z = x + iy \in \mathbb{C}: |y| \leq \tau\} \qquad (\tau > 0). \tag{3.5}$$

In particular, $f_1'(x)$ has an infinite number of nonreal zeros and at most a finite number of real zeros.

Proof. (a) By virtue of the properties of $\Phi(t)$ (cf. Theorem A), we find from (3.2), using integration by parts, that for $x \neq 0$,

$$xf_1(x) = \Phi(0) + \int_0^{\infty} \Phi'(t)\cos(xt)\,dt. \tag{3.6}$$

Since $\Phi'(t) < 0$ for $t > 0$ (cf. Theorem A), it follows that, for $x \neq 0$,

$$xf_1(x) > \Phi(0) + \int_0^{\infty} \Phi'(t)\,dt = \Phi(0) - \Phi(0) = 0.$$

Thus, $f_1(x) \neq 0$ if $x \neq 0$. Also, as it is clear from (3.2) that $f_1(0) = 0$, then (a) is established.

(b) Fix $\tau > 0$ and suppose that $f_1'(x)$ has an infinite number of zeros, say $\{z_k\}_{k=1}^{\infty}$, in $S(\tau)$, with $\lim_{k\to\infty} z_k = \infty$. Then, three integrations by parts applied to the integral for $f_1'(z)$ in (3.4) yield

$$z^2 f_1'(z) = -\Phi(0) + \frac{1}{z}\int_0^{\infty}\left[\frac{d^3}{dt^3}(t\Phi(t))\right]\sin(zt)\,dt \qquad (z \neq 0). \tag{3.7}$$

Since $|\sin(zt)| \leq e^{\tau t}$ for all $z \in S(\tau)$ and all $t \geq 0$, it follows from Theorem A that the integral on the right-hand side of (3.7) tends to zero as z tends to infinity in the strip $S(\tau)$. But $f_1'(z_k) = 0$ $(k = 1, 2, 3, \ldots)$ and so

$$\lim_{k\to\infty} z_k^2 f_1'(z_k) = 0 = -\Phi(0). \tag{3.8}$$

But as $\Phi(0) > 0.466\,696\ldots$ (cf. [6, Lemma 3.6]), this contradiction to (3.8) shows that $f_1'(x)$ cannot have an infinite number of zeros in $S(\tau)$.

Finally, it is evident from (3.4) that $f_1'(z)$ is an *even* entire function, and it is not difficult to prove that $f_1'(z)$ is an entire function of order 1. (Minor modifications of the proof in Appendix A of [7] will justify this assertion.) Hence, it follows from the Hadamard factorization theorem that $f_1'(z)$ has an infinite number of zeros. Since $f_1'(z)$ can have at most a finite number of zeros in $S(\tau)$, for any $\tau > 0$, we conclude that $f_1'(z)$ must have an infinite number of nonreal zeros. □

A result which is more general than the first assertion of Proposition 3.1(b) is given by the following theorem, due to Pólya [21].

THEOREM 3.2 [21]. *Let $K\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a C^∞ function such that*

$$\lim_{t\to\infty} t^{-1}\log|K^{(n)}(t)| = -\infty \qquad (n = 0, 1, 2, \ldots). \tag{3.9}$$

If the entire function

$$f(z;K) := \int_0^\infty K(t)\cos(zt)\,dt \tag{3.10}$$

has an infinite number of zeros in the strip $S(\tau)$ (*cf.* 3.5)), *then* $K(t)$ *is an even function.*

In the sequel, we will also require the notation

$$\Phi_j(t) := \sum_{n=j+1}^{\infty} a_n(t) \qquad (j = 1,2,3,\dots), \tag{3.11}$$

where $a_n(t)$ is given by (3.3). (Our notations here are consistent with the notations used in [6, 8].) Now, it is easy to see that the functions $a_n(t)$ $(n = 1,2,3,\dots)$ and $\Phi_j(t)$ $(j = 1,2,3,\dots)$ are *not* even functions. Consequently, by the foregoing results, each of the entire functions

$$F(x;a_n) := \int_0^\infty a_n(t)\cos(xt)\,dt \qquad (n = 1,2,3,\dots) \tag{3.12}$$

and

$$F(x;\Phi_j) := \int_0^\infty \Phi_j(t)\cos(xt)\,dt \qquad (j = 1,2,3,\dots) \tag{3.13}$$

is an even entire function (of order 1) with an infinite number of nonreal zeros, and each has at most a finite number of real zeros. But as we shall see below, more precise information can be obtained by a careful analysis of the functions $a_n(t)$ and $\Phi_j(t)$. In fact, in light of the results in Section 2, one is interested in conditions which ensure that the more general functions, of the form

$$H_{\mu,j}(x) := \int_0^\infty \Phi_j(t)\cosh(\mu t)\cos(xt)\,dt \qquad (x,\mu \in \mathbb{R};\ j = 1,2,3,\dots), \tag{3.14}$$

are *nonnegative*. Our analysis will use the following technical lemma.

LEMMA 3.3. *Set*

$$p_\mu(y) := 32y^3 - (16\mu + 224)y^2 + (2\mu^2 + 60\mu + 330)y - (3\mu^2 + 30\mu + 75) \qquad (\mu, y \in \mathbb{R}). \tag{3.15}$$

If

$$\mu_0 = 26.457\,09\ldots \tag{3.16}$$

is the smallest real zero of

$$(8\pi - 3)\mu^2 - (256\pi^2 - 240\pi + 30)\mu$$
$$+(2{,}048\pi^3 - 3{,}584\pi^2 + 1{,}320\pi - 75), \tag{3.17}$$

then

$$p_\mu(y) > 0 \qquad \textit{for all } y \geq 4\pi \textit{ and all } 0 \leq \mu < \mu_0. \tag{3.18}$$

Proof. From (3.15), it can be verified that

$$p_\mu(s + 4\pi) := c_3(\mu)s^3 + c_2(\mu)s^2 + c_1(\mu)s + c_0(\mu), \tag{3.19}$$

where

$$\begin{aligned}
c_3(\mu) &:= 32,\\
c_2(\mu) &:= (384\pi - 224) - 16\mu,\\
c_1(\mu) &:= 2\mu^2 - (128\pi - 60)\mu + (1{,}536\pi^2 - 1{,}792\pi + 330),\\
c_0(\mu) &:= (8\pi - 3)\mu^2 - (256\pi^2 - 240\pi - 30)\mu\\
&\quad +(2{,}048\pi^3 - 3{,}584\pi^2 + 1{,}320\pi - 75).
\end{aligned} \tag{3.20}$$

Because these coefficients $c_i(\mu)$ are at most quadratic polynomials in μ, then an elementary calculation shows that

$$\begin{aligned}
c_3(\mu) &> 0 \qquad \text{for all } 0 \leq \mu < \infty,\\
c_2(\mu) &> 0 \qquad \text{for all } 0 \leq \mu < 61.932\,22\ldots,\\
c_1(\mu) &> 0 \qquad \text{for all } 0 \leq \mu < 36.688\,80\ldots,\\
c_0(\mu) &> 0 \qquad \text{for all } 0 \leq \mu < 26.457\,09\ldots =: \mu_0.
\end{aligned} \tag{3.21}$$

Since the interval above associated with $c_0(\mu)$ is contained in all of the other intervals of (3.21), it follows that $p_\mu(s + 4\pi) > 0$ for all $0 \leq \mu < \mu_0$ and all $s \geq 0$, i.e.,

$$p_\mu(y) > 0 \qquad \text{for all } y \geq 4\pi \text{ and all } 0 \leq \mu < \mu_0, \tag{3.22}$$

the desired result of (3.18). □

THEOREM 3.4. *With $H_{\mu,j}(x)$ defined by* (3.14),

$$H_{\mu,j}(x) > 0 \qquad \textit{for all } x \in \mathbb{R},\ j = 1,2,3,\ldots, \textit{ and } \mu \in [0, \mu_0), \tag{3.23}$$

where μ_0 is given by (3.16).

Proof. We first claim that it suffices to prove that the function $\Phi_j(t)\cosh(\mu t)$ ($\mu \in [0, \mu_0)$; $j = 1, 2, 3, \dots$) is strictly convex for $t \geq 0$; that is,

$$\frac{d^2}{dt^2}\left[\Phi_j(t)\cosh(\mu t)\right] > 0 \qquad (t \geq 0;\ \mu \in [0, \mu_0);\ j = 1, 2, 3, \dots). \tag{3.24}$$

Indeed, if (3.24) holds, then integrating by parts twice in (3.14) yields, for $x \neq 0$,

$$\begin{aligned} x^2 H_{\mu,j}(x) &= -\Phi_j'(0) - \int_0^\infty \left[\frac{d^2}{dt^2}\left(\Phi_j(t)\cosh(\mu t)\right)\right]\cos(xt)\,dt \\ &> -\Phi_j'(0) - \int_0^\infty \left[\frac{d^2}{dt^2}\left(\Phi_j(t)\cosh(\mu t)\right)\right]dt \\ &= -\Phi_j'(0) + \Phi_j'(0) = 0, \end{aligned} \tag{3.25}$$

where we have used the elementary fact (from (iv) of Theorem A) that $\lim_{t\to\infty} \Phi_j'(t) = 0$ ($j = 1, 2, 3, \dots$). Thus, (3.25) shows that for $x \neq 0$, $H_{\mu,j}(x) > 0$. Since $\Phi_j(t) > 0$ for all $t \geq 0$ (just use the definition (3.11) and the fact that each $a_n(t) > 0$ for all $t \geq 0$ from (i) of Theorem A), it follows that $H_{\mu,j}(0) > 0$. Therefore, it remains to prove the convexity condition (3.24).

First, from (iv) of Theorem A and (3.11), we note that $\Phi_j'(t) < 0$ for all $t \geq 0$ and $j = 1, 2, 3, \dots$. Thus, for $\mu \geq 0$ and $t \geq 0$, we have

$$\begin{aligned} \frac{d^2}{dt^2}\left[\Phi_j \cosh(\mu t)\right] &= \Phi_j''(t)\cosh(\mu t) \\ &\quad + 2\mu\Phi_j'(t)\sinh(\mu t) + \mu^2\Phi_j\cosh(\mu t) \\ &> \cosh(\mu t)\left[\Phi_j''(t) + 2\mu\Phi_j'(t) + \mu^2\Phi_j\right], \end{aligned} \tag{3.26}$$

where we have used the fact that $\sinh(\mu t) < \cosh(\mu t)$ for $\mu \geq 0$ and $t \geq 0$. By (3.11) and (3.26), we see that it suffices to prove that

$$E_n(t;\mu) := a_n''(t) + 2\mu a_n'(t) + \mu^2 a_n(t) > 0 \qquad (t \geq 0;\ \mu \in [0, \mu_0);\ n \geq 2). \tag{3.27}$$

Using the definition of $a_n(t)$ (cf. (3.3)), we find that, with $y := \pi n^2 e^{4t}$,

$$E_n(t;\mu) = \left(\pi n^2\right)^{-1/4} y^{5/4} e^{-y} p_\mu(y) \qquad (n = 2, 3, \dots),$$

where $p_\mu(y)$ is defined by (3.15). For $n \geq 2$, $\pi n^2 e^{4t} \geq 4\pi$ for all $t \geq 0$, and hence it follows from Lemma 3.3 that $p_\mu(\pi n^2 e^{4t}) > 0$ for all $t \geq 0$, $n \geq 2$, and $\mu \in [0, \mu_0)$. Hence we have the desired result that $E_n(t; u) > 0$ $(t \geq 0;\ \mu \in [0, \mu_0);\ n \geq 2)$. □

Remarks. If $\mu = 0$, then $H_{0,j}(x)$ (cf. (3.23) and (3.13)) reduces to $F(x; \Phi_j)$ and Theorem 3.4 states, in particular, that

$$F(x;\Phi_1) = \int_0^\infty \Phi_1(t)\cos(xt)\,dt > 0 \qquad (x \in \mathbb{R}). \tag{3.28}$$

Since $F(x) := F(x; \Phi) = F(x; a_1) + F(x; \Phi_1)$, the question whether or not $F(x)$ has only real zeros depends on the nature of the intersections of the curves $y = F(x; a_1)$ and $y = -F(x; \Phi_1)$. These two curves have an infinite number of points of intersections, since we know that $F(x)$ has an infinite number of real zeros (cf. Hardy [11] or Pólya [19]). Now, a numerical computation shows that $F(x; a_1)$ has at least one real zero, since

$$F(28;a_1)\cdot F(29;a_1) < 0. \tag{3.29}$$

(Indeed, $F(28; a_1) = 8.887419\ldots \times 10^{-6}$ and $F(29; a_1) = -6.683033\ldots \times 10^{-5}$.) The foregoing results and numerical experimentations lead us to conjecture that the function $F(x; a_1)$ has precisely *one positive* (real) zero. It is particularly interesting to apply the above analysis to the function (cf. (2.31))

$$\begin{aligned}\frac{1}{2}H_\mu(x) &:= \int_0^\infty \Phi(t)\cosh(\mu t)\cos(xt)\,dt \qquad (x \in \mathbb{R})\\ &= \int_0^\infty a_1(t)\cosh(\mu t)\cos(xt)\,dt + H_{\mu,1}(x),\end{aligned} \tag{3.30}$$

since for $\mu \geq 1$, the function $H_\mu(x)$ has *only* real zeros (cf. the discussion following (2.31)). Moreover, by Theorem 3.4, $H_{\mu,1}(x) > 0$ for all $x \in \mathbb{R}$ and all $\mu \in [0, \mu_0]$, where μ_0 is given by (3.16).

Open Problem 3.1. Let $\{z_n\}_{n=0}^\infty$, $z_n := x_n + iy_n$, denote the sequence of nonreal zeros of $F(x; a_1)$ (cf. (3.12)). For each $\tau > 0$, determine the number of nonreal zeros of $F(x; a_1)$ in the strip $S(\tau)$, defined by (3.5).

We next turn to the problem of the distribution of the zeros of the real entire function

$$F(x;R) := \int_0^R \Phi(t)\cos(xt)\,dt \qquad (0 < R < \infty). \tag{3.31}$$

Again, an analysis of the behavior of $\Phi(t)$ will show (cf. Theorem 3.6) that $F(x; R)$ has only real zeros if $0 < R < 0.11$. In order to establish this, we first prove a lemma which extends our previous work (cf. [8, Lemmas 3.1 and 3.2]).

LEMMA 3.5. *With* (3.3), *we have*

$$\Phi''(t) < 0 \qquad \text{for all } t \in I := [0, 0.11]. \tag{3.32}$$

Proof. By a known result (cf. [8, Lemma 3.1]),

$$|\Phi_1''(t)| < (1.031) \cdot 2^{13} \cdot \pi^4 \cdot \exp(17t - 4\pi e^{4t}) \qquad (t \geq 0), \tag{3.33}$$

and (cf. [8, Eq. (3.24)]),

$$|\Phi_1''(t)| < 2.869{,}080\ldots \qquad (t \geq 0). \tag{3.34}$$

Also since $a_1''(t)$ is strictly increasing for $0 \leq t < 0.203\,249\ldots$ (cf. [8, Eq. (3.26)]), it follows that for all $t \in I$,

$$\begin{aligned} \Phi''(t) &\leq a_1''(t) + |\Phi_1''(t)| < a_1''(0.11) + 2.869\,080\ldots \\ &< -3.359\,151\ldots + 2.869\,080\ldots \\ &< -0.490\,071\ldots, \end{aligned} \tag{3.25}$$

which establishes (3.32). □

THEOREM 3.6. *For each R in $I_1 := (0, 0.11]$, the real entire function*

$$F(x; R) := \int_0^R \Phi(t)\cos(xt)\, dt, \tag{3.36}$$

where $\Phi(t)$ is defined by (3.3), *has only real zeros.*

Proof. It is clear that $F(x; R)$ has only real zeros if and only if the function

$$F_1(x; R) := \int_0^1 \varphi_R(t)\cos(xt)\, dt \qquad (R > 0), \tag{3.37}$$

where $\varphi_R(t) := \Phi(Rt)$ $(t \in \mathbb{R})$, has only real zeros. But it is known (cf. Pólya–Szegö [24, Chap. V, Problem 173]) that $F_1(x; R)$ has only real zeros if $\varphi_R(t)$ is a C^2 function with $\varphi_R(t) > 0$, $\varphi_R'(t) < 0$, $\varphi_R''(t) < 0$ for $0 \leq t \leq 1$. By Theorem A (cf. Section 1), $\varphi_R(t) > 0$ and $\varphi_R'(t) < 0$ for all $t > 0$ and for all $R > 0$. By Lemma 3.5, $\varphi_R''(t) < 0$ for $0 \leq t \leq 1$ and $R \in I_1 := (0, 0.11]$. □

We conclude this paper with two additional open problems.

Open Problem 2. If $f(x)$ is a function in the Laguerre–Pólya class, let $\{h_n(t)\}_{n=0}^{\infty}$ be the sequence of polynomials generated by

$$e^{-x^2}f(xt) := \sum_{n=0}^{\infty} h_n(t)\frac{x^n}{n!}. \tag{3.38}$$

Characterize the distribution of zeros of the polynomials $h_n(t)$ $(n = 0, 1, 2, \dots)$.

Comments. In the theory of special functions, the polynomials $h_n(t) := h_n(t; e^{-x^2}, f(x))$ are called the *Brenke polynomials* associated with e^{-x^2} and $f(x)$ (cf. Boas and Buck [2, p. 51]) and, in general, these polynomials are defined as follows: Let $A(x)$ and $B(x)$ be two holomorphic functions defined in a neighborhood of the origin. Then the polynomials $h_n(t) := h_n(t; A, B)$, generated by

$$A(x)\cdot B(xt) := \sum_{n=0}^{\infty} h_n(t; A, B)\frac{x^n}{n!}, \tag{3.39}$$

are the Brenke polynomials associated with the functions $A(x)$ and $B(x)$. In the special case when *both* $A(x)$ and $B(x)$ are real entire functions in the Laguerre–Pólya class, and whence the product $A(x)B(x)$ is also in this class, the polynomials $h_n(t; A, B)$ have been investigated in a series of papers by Iliev and several European and Russian mathematicians. (Since most of these papers are not available in English, we refer the reader to Iliev's recent book [13] and the references contained therein.) If $A(x) := e^x$ and if $B(x)$ is an arbitrary real entire function, then the associated Brenke polynomials $h_n(t; A, B) = g_n(t)$ are precisely the Jensen polynomials discussed in Section 2.

Open Problem 3. It is known from Skovgaard [27] that a *necessary condition* for a real entire function $f(x)$ to belong to the Laguerre–Pólya class is that $f(x)$ satisfy the *Laguerre inequalities*; that is,

$$L_2\left(f^{(p)}(x)\right) := \left(f^{(p)}(x)\right)^2 - f^{(p-1)}(x)f^{(p+1)}(x) \geq 0 \quad (x \in \mathbb{R};\ p = 1, 2, 3, \dots). \tag{3.40}$$

Prove that $L_2(F^{(p)}(x)) \geq 0$ and $L_2(F_c^{(p)}(x)) \geq 0$ $(p = 1, 2, 3, \dots;\ x \in \mathbb{R})$, where

$$F(x) := \int_0^{\infty} \Phi(t)\cos(xt)\,dt \qquad \text{and} \qquad F_c(x) := \int_0^{\infty} \Phi(t)\cosh(t\sqrt{x})\,dt \tag{3.41}$$

and where $\Phi(t)$ is defined by (3.3).

Comments. The Laguerre inequalities are closely related to the results proved in Section 2 (cf. Theorems 2.9 and 2.10). In fact, by (2.71) we see that (3.40) is a special case of a collection of necessary *and* sufficient conditions for a function to belong to the Laguerre–Pólya class. Similarly, inequality (2.73) reduces to (3.40) (with $p = 1$), when $z := x + iy$ is restricted to the real axis. We also remark that for $x = 0$, the Laguerre inequalities

$$L_2\big(F_c^{(p)}(0)\big) > 0 \qquad (p = 1, 2, 3, \dots) \tag{3.42}$$

are known to be valid (cf. [6, 8]), since the inequalities (3.42) are equivalent to the Turán inequalities $\gamma_p^2 - \gamma_{p-1}\gamma_{p+1} > 0$ $(p = 1, 2, 3, \dots)$, where the γ_k's are defined by (2.56) and (2.57). (For related results, see also (2.60).)

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