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Jensen Polynomials with Applications to the Riemann ξ-Function

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In this paper, generalizations of some known results for Jensen polynomials, pertaining to (i) convexity, (ii) the Turán inequalities, and (iii) the Laguerre inequalities, are established. These results are then applied in general to real entire functions, which are representable by Fourier transforms, and in particular to the Riemann ξ-function. © 1990 Academic Press, Inc.

1. INTRODUCTION

A brief outline of this paper is as follows. In Section 2 we generalize several of those results in [CC; CV3; V1; V2] which pertain to (i) convexity, (ii) the Turán inequalities for Jensen polynomials, and (iii) the Laguerre inequalities. In Section 3 these results are first applied to real entire functions which are representable by Fourier transforms, and then specifically to the Riemann ξ-function.

In Section 2 we consider real entire functions of the form

$$f(x) := \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \qquad (\gamma_k > 0;\, k = 0, 1, 2, \ldots), \tag{1.1}$$

where it is assumed that the γ_k's satisfy the *Turán inequalities*:

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geqslant 0 \qquad (k = 1, 2, 3, \ldots). \tag{1.2}$$

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Under these (weak) hypotheses, we show in Theorem 2.5 not only that the *Laguerre inequalities*,

$$L_p(x) := L_p(x; f) := (f^{(p+1)}(x))^2 - f^{(p)}(x) f^{(p+2)}(x) \geq 0 \qquad (x \geq 0;\ p = 0, 1, 2, ...), \tag{1.3}$$

are valid, but also that

$$L_p^{(v)}(x) \geq 0 \qquad (x \geq 0;\ v, p = 0, 1, 2, ...). \tag{1.4}$$

The *Jensen polynomials* $g_n(t) := g_n(t; f)$, associated with $f(x)$ of (1.1), are defined by

$$g_n(t) := \sum_{k=0}^{n} \binom{n}{k} \gamma_k t^k \qquad (n = 0, 1, 2, ...). \tag{1.5}$$

Proposition 2.6 provides a representation for the *Turán differences* of these Jensen polynomials:

$$\Delta_n(t) := g_n^2(t) - g_{n-1}(t)\, g_{n+1}(t) \qquad (n = 1, 2, 3, ...). \tag{1.6}$$

The main result, Theorem 2.7, of Section 2 is that if (1.2) holds (with $\gamma_k > 0$), then

$$\Delta_n^{(v)}(0) \geq 0 \qquad (v = 0, 1, 2, ...;\ n = 1, 2, 3, ...). \tag{1.7}$$

These results are then related to functions in the Laguerre–Pólya class, which is defined as follows.

DEFINITION 1.1. A real entire function $f(x)$ is said to be in the Laguerre–Pólya class, written $f(x) \in \mathscr{L} - \mathscr{P}$, if $f(x)$ can be expressed in the form

$$f(x) = Ce^{-\alpha x^2 + \beta x} x^n \prod_{j=1}^{\omega} (1 - x/x_j)\, e^{x/x_j} \qquad (\omega \leq \infty), \tag{1.8}$$

where $\alpha \geq 0$, β, and C are real numbers, n is a nonnegative integer and the x_j's are real and nonzero with $\sum_{j=1}^{\omega} x_j^{-2} < \infty$.

In Section 3, we first state in Proposition 3.1 some basic results concerning the Fourier transforms of admissible kernels (cf. Definition 3.1). Our main result, Theorem 3.3, establishes a Laguerre inequality for the Fourier transforms of a large class of admissible kernels. The paper concludes with an application of the foregoing results to the Riemann ξ-function.

2. Main Results

We begin with the following variation of a known result of Vincze [V1].

Lemma 2.1. *For positive integers n and M with* $0<M<n$, *let* $\{A_k\}_{k=0}^n$ *and* $\{B_k\}_{k=0}^n$ *be two sets of real numbers which satisfy*

(i) $\sum_{k=0}^n A_k=0$,

(ii) $A_k<0 \quad (k=0, 1, ..., M)$,
$A_k\geqslant 0 \quad (k=M+1, M+2, ..., n)$, *and*

(iii) $0<B_0\leqslant B_1\leqslant \cdots \leqslant B_n$. (2.1)

Then,

$$\sum_{k=0}^n A_kB_k\geqslant 0. \tag{2.2}$$

Moreover, if $A_n>0$, *then equality holds in* (2.2) *if and only if* $B_0=B_1= \cdots =B_n$.

Proof. From (2.1),

$$\begin{aligned}\sum_{k=0}^n A_kB_k &= \sum_{k=0}^M A_kB_k+\sum_{k=M+1}^n A_kB_k\\ &\geqslant B_M\sum_{k=0}^M A_k+B_{M+1}\sum_{k=M+1}^n A_k\\ &=(B_{M+1}-B_M+B_M)\sum_{k=M+1}^n A_k+B_M\sum_{k=0}^M A_k\\ &=(B_{M+1}-B_M)\sum_{k=M+1}^n A_k+B_M\sum_{k=0}^n A_k \qquad (2.3)\\ &=(B_{M+1}-B_M)\sum_{k=M+1}^n A_k\geqslant 0, \qquad (2.3')\end{aligned}$$

which gives the desired result of (2.2). Next, if $A_n>0$, then equality can hold in (2.3) only if $B_0=B_1= \cdots =B_M$ and if $B_{M+1}=B_{M+2}= \cdots =B_n$. Then, since $\sum_{k=M+1}^n A_k > 0$, equality can hold in (2.3′) only if $B_M=B_{M+1}$, which gives the final result of Lemma 2.1. ∎

To highlight the dependence on n of the numbers $\{A_k\}_{k=0}^n$ and $\{B_k\}_{k=0}^n$, we remark that Lemma 2.1 is employed in Lemma 2.2 and 2.4 in the form

$$A_k:=A_k(n) \quad \text{and} \quad B_k:=B_k(n) \quad (k=0, 1, ..., n). \tag{2.4}$$

Continuing, we need

LEMMA 2.2. (a) *If*

$$z(n) := -1 + \sum_{k=0}^{n} \binom{n}{k} \left[\frac{n+1-2k}{n+1-k} \right] \qquad (n = 0, 1, 2, ...), \tag{2.5}$$

then

$$z(n) = 0 \qquad (n = 0, 1, 2, ...). \tag{2.6}$$

(b) *If m is a nonnegative integer, and if*

$$\begin{aligned} A_0(2m) &:= -1, \\ A_{k+1}(2m) &:= \binom{2m}{k} \left[\frac{2m+2-4(m-k)^2}{(2m+1-k)(k+1)} \right] \quad (k = 0, 1, ..., m-1), \\ A_{m+1}(2m) &:= \binom{2m}{m} \left[\frac{1}{m+1} \right], \end{aligned} \tag{2.7}$$

then

$$\sum_{k=0}^{m+1} A_k(2m) = 0. \tag{2.8}$$

(c) *If m is a positive integer, and if*

$$\begin{aligned} A_0(2m-1) &:= -1, \\ A_{k+1}(2m-1) &:= \binom{2m-1}{k} \left[\frac{6m-4k-4(m-k)^2}{(2m-k)(k+1)} \right] \\ &\qquad (k = 0, 1, ..., m-1), \end{aligned} \tag{2.9}$$

then

$$\sum_{k=0}^{m} A_k(2m-1) = 0. \tag{2.10}$$

Proof. (a) For each fixed, but arbitrary, nonnegative integer n, we have

$$\begin{aligned} z(n) &= -1 + \sum_{k=0}^{n} \binom{n}{k} - \sum_{k=0}^{n} \binom{n}{k} \left[\frac{k}{n+1-k} \right] \\ &= -1 + 2^n - \sum_{k=1}^{n} \binom{n}{k-1} \\ &= -1 + 2^n - (2^n - 1) = 0, \end{aligned}$$

the desired result of (2.6).

(b) Since (2.8) is immediate if $m=0$, consider any positive integer m. Then,

$$\sum_{k=0}^{m+1} A_k(2m) = -1 + \sum_{k=0}^{m} A_{k+1}(2m)$$
$$= -1 + \binom{2m}{m}\left[\frac{1}{m+1}\right]$$
$$+ \sum_{k=0}^{m-1} \binom{2m}{k}\left[\frac{2m+2-4(m-k)^2}{(2m+1-k)(k+1)}\right]. \tag{2.11}$$

Now, a change of indices ($j := 2m-k$) shows that

$$\sum_{k=m+1}^{2m} \binom{2m}{k}\left[\frac{2m+1-2k}{2m+1-k}\right] = -\sum_{j=0}^{m-1} \binom{2m}{j}\left[\frac{2m-2j-1}{j+1}\right]. \tag{2.11'}$$

Hence, using (2.11) and (2.11′), a calculation yields

$$\sum_{k=0}^{m+1} A_k(2m) = -1 + \binom{2m}{m}\left[\frac{1}{m+1}\right] + \sum_{k=0}^{m-1} A_{k+1}(2m)$$
$$= -1 + \binom{2m}{m}\left[\frac{1}{m+1}\right] + \sum_{k=0}^{m-1} \binom{2m}{k}$$
$$\times\left[\frac{2m+1-2k}{2m+1-k} - \frac{2m-2k-1}{k+1}\right]$$
$$= -1 + \binom{2m}{m}\left[\frac{1}{m+1}\right] + \sum_{k=0}^{m-1} \binom{2m}{k}\left[\frac{2m+1-2k}{2m+1-k}\right]$$
$$+ \sum_{k=m+1}^{2m} \binom{2m}{k}\left[\frac{2m+1-2k}{2m+1-k}\right]$$
$$= z(2m),$$

where $z(n)$ is defined by (2.5). Since $z(2m)=0$ from part (a), the desired result of (2.8) follows.

(c) In order to prove (2.10), fix an arbitrary positive integer m, and consider

$$z(2m-1) = -1 + \sum_{k=0}^{2m-1} \binom{2m-1}{k}\left[\frac{2m-2k}{2m-k}\right]. \tag{2.12}$$

Using the change of indices ($k := 2m-1-j$), we have

$$\sum_{k=m}^{2m-1} \binom{2m-1}{k}\left[\frac{2m-2k}{2m-k}\right] = -\sum_{j=0}^{m-1} \binom{2m-1}{j}\left[\frac{2m-2-2j}{j+1}\right], \tag{2.12'}$$

and, consequently, with (2.9) and (2.12′), (2.12) can be expressed as

$$z(2m-1) = -1 + \sum_{k=0}^{m-1} \binom{2m-1}{k} \left[\frac{6m-4k-4(m-k)^2}{(2m-k)(k+1)} \right]$$

$$= -1 + \sum_{k=0}^{m-1} A_{k+1}(2m-1) = \sum_{k=0}^{m} A_k(2m-1).$$

Therefore, the desired result of (2.10) follows from part (a). ▮

The next lemma provides a simple, but useful, characterization of nonnegative sequences that satisfy the Turán inequalities. As usual, $[[x]]$ denotes the greatest integer less than or equal to the real number x (cf. (2.16) and Remark 1).

Lemma 2.3. *Let* $\{\gamma_k\}_{k=0}^{N}$, $N \geqslant 2$, *be a sequence of nonnegative real numbers and let* μ *be a nonnegative integer with* $0 \leqslant \mu < N-1$ *such that*

$$\gamma_k > 0 \quad (k=0, 1, ..., \mu) \quad \text{and} \quad \gamma_j = 0 \quad (j=\mu+1, \mu+2, ..., N). \tag{2.13}$$

Then, the extended Turán inequalities hold, i.e.,

$$\gamma_k \gamma_{j-1} - \gamma_{k-1} \gamma_j \geqslant 0 \qquad (1 \leqslant k \leqslant j \leqslant N), \tag{2.14}$$

if and only if the Turán inequalities hold, i.e.,

$$\gamma_k^2 - \gamma_{k-1} \gamma_{k+1} \geqslant 0 \qquad (k=1, 2, ..., N-1). \tag{2.15}$$

In particular, if (2.14) *holds, then*

$$\gamma_k \gamma_{\mu-k+2} \leqslant \gamma_{k+1} \gamma_{\mu-k+1} \qquad (k=0, 1, ..., [[(\mu+1)/2]]). \tag{2.16}$$

Proof. Since the proof of Lemma 3.2 is easily verified when $\mu=0$ or $\mu=1$, we may assume that $\mu \geqslant 2$. Then, assume that (2.14) holds, which clearly implies the weaker statement

$$\gamma_k \gamma_{j-1} - \gamma_{k-1} \gamma_j \geqslant 0 \qquad (1 \leqslant k \leqslant j \leqslant \mu). \tag{2.14′}$$

Then, letting $j=k+1$ in (2.14′) gives

$$\gamma_k^2 - \gamma_{k-1} \gamma_{k+1} \geqslant 0 \qquad (1 \leqslant k \leqslant \mu-1). \tag{2.15′}$$

But, because $\gamma_j = 0$ for $\mu+1 \leqslant j \leqslant N$ from (2.13), we further have

$$\gamma_k^2 - \gamma_{k-1} \gamma_{k+1} = 0 \qquad (\mu \leqslant k \leqslant N-1). \tag{2.15″}$$

Combining (2.15′) and (2.15″) then gives (2.15). Conversely, assume that

(2.15) holds. It is clear that (2.15′) follows from (2.15). But (2.15′) implies that

$$\frac{\gamma_{\mu-1}}{\gamma_\mu} \geqslant \frac{\gamma_{\mu-2}}{\gamma_{\mu-1}} \geqslant \cdots \geqslant \frac{\gamma_1}{\gamma_2} \geqslant \frac{\gamma_0}{\gamma_1} > 0. \tag{2.16′}$$

Consequently, the above inequalities give

$$\frac{\gamma_{j-1}}{\gamma_j} \geqslant \frac{\gamma_{k-1}}{\gamma_k} \qquad (1 \leqslant k \leqslant j \leqslant \mu),$$

which in turn gives (2.14′). Again, since $\gamma_j = 0$ for $\mu + 1 \leqslant j \leqslant N$ from (2.13), then the inequalities (2.14) follow from (2.14′). Finally, since (2.16) is a special case of (2.14), the proof is complete. ∎

The proof of the next lemma is patterned after the work of Vincze [V1].

LEMMA 2.4. *Let* $\{\gamma_k\}_{k=0}^{\infty}$, *be a sequence of positive real numbers which satisfy the Turán inequalities*

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geqslant 0 \qquad (k = 1, 2, 3, ...). \tag{2.17}$$

If

$$\sigma_{n,p} := \sum_{k=0}^{n} \binom{n}{k} (\gamma_{p+k+1}\gamma_{p+n-k+1} - \gamma_{p+k+2}\gamma_{p+n-k}) \qquad (n, p = 0, 1, 2, ...), \tag{2.18}$$

then

$$\sigma_{n,p} \geqslant 0 \qquad (n, p = 0, 1, 2, ...). \tag{2.19}$$

Moreover equality holds in (2.19) *for some nonnegative integers n and p if and only if all summands in* (2.18) *vanish, i.e.,*

$$\gamma_p\gamma_{p+n+2} = \gamma_{p+1}\gamma_{p+n+1} = \cdots = \gamma_{p+1+[[(n/2)]]}\gamma_{p+n+1-[[(n/2)]]}. \tag{2.19′}$$

Proof. We first prove (2.19) when $p = 0$. On setting

$$\sigma_n := \sigma_{n,0}, \tag{2.20}$$

we can also express σ_n from (2.18) as

$$\sigma_n = -\gamma_0\gamma_{n+2} + \sum_{k=0}^{n} \binom{n}{k} \left[\frac{n+1-2k}{n+1-k}\right] \gamma_{k+1}\gamma_{n-k+1}. \tag{2.21}$$

As in [V], we consider two cases; namely, when n is even and when n is odd.

Case 1. Suppose $n := 2m$, where m is a fixed nonnegative integer. From (2.21)

$$\begin{aligned}\sigma_{2m} &= -\gamma_0\gamma_{2m+2} + \sum_{k=0}^{2m} \binom{2m}{k}\left[\frac{2m+1-2k}{2m+1-k}\right]\gamma_{k+1}\gamma_{2m-k+1} \\ &= -\gamma_0\gamma_{2m+2} + \sum_{k=0}^{m} \binom{2m}{k}\left[\frac{2m+1-2k}{2m+1-k}\right]\gamma_{k+1}\gamma_{2m-k+1} \\ &\quad + \sum_{k=m+1}^{2m} \binom{2m}{k}\left[\frac{2m+1-2k}{2m+1-k}\right]\gamma_{k+1}\gamma_{2m-k+1}.\end{aligned}$$

Now, applying a change of indices to the last sum above (cf. (2.11′)), a straightforward calculation yields

$$\begin{aligned}\sigma_{2m} &= -\gamma_0\gamma_{2m+2} + \binom{2m}{m}\left[\frac{1}{m+1}\right]\gamma_{m+1}^2 \\ &\quad + \sum_{k=0}^{m-1} \binom{2m}{k}\left[\frac{2m+2-4(m-k)^2}{(2m+1-k)(k+1)}\right]\gamma_{k+1}\gamma_{2m-k+1},\end{aligned}$$

which we can write as

$$\sigma_{2m} = \sum_{k=0}^{m+1} A_k(2m)\, B_k(2m), \tag{2.22}$$

where $A_k(2m)$ is defined by (2.7) and where we set

$$B_k(2m) := \gamma_k\gamma_{2m-k+2} \qquad (m=0, 1, 2, ...; k=0, 1, ..., m+1). \tag{2.23}$$

Thus, by Lemma 2.3 (cf. (2.16) with $\mu := 2m$), we have

$$0 < B_0(2m) \leqslant B_1(2m) \leqslant \cdots \leqslant B_{m+1}(2m). \tag{2.24}$$

Next, by (2.8) of Lemma 2.2,

$$\sum_{k=0}^{m+1} A_k(2m) = 0 \qquad (m=0, 1, 2, ...). \tag{2.25}$$

Now, from (2.7), $A_0(2m) = -1$ for any nonnegative integer m, and for a positive integer m, set

$$M := m - \sqrt{\frac{m+1}{2}}, \tag{2.26}$$

so that $M \leqslant m-1$. Then, it is easily seen from (2.7) that $A_{k+1}(2m)<0$ for all nonnegative integers k with $k<M$, while $A_{k+1}(2m) \geqslant 0$ for $M \leqslant k \leqslant m-1$. Thus, the hypotheses of Lemma 2.1 are satisfied. Therefore, by (2.2) of Lemma 2.1, it follows from (2.22) that

$$\sigma_{2m} = \sum_{k=0}^{m+1} A_k(2m)\, B_k(2m) \geqslant 0 \qquad (m=0, 1, 2, \ldots), \tag{2.27}$$

the desired result of (2.19) in the case when $p=0$ and $n=2m$.

Case 2. Suppose $n := 2m-1$, where m is a fixed positive integer. Then a calculation, which is *mutatis mutandis* the same as that of Case 1, shows that

$$\begin{aligned}\sigma_{2m-1} &= -\gamma_0 \gamma_{2m+1} + \sum_{k=0}^{m-1} \binom{2m-1}{k} \left[\frac{2m-2k}{2m-k} - \frac{2m-2-2k}{k+1}\right] \gamma_{k+1} \gamma_{2m-k} \\ &= \sum_{k=0}^{m} A_k(2m-1)\, B_k(2m-1),\end{aligned}$$

where $A_k(2m-1)$ is defined in (2.9) and where we set

$$B_k(2m-1) := \gamma_k \gamma_{2m+1-k} \qquad (k=0, 1, \ldots, m). \tag{2.28}$$

Once again, by Lemmas 2.1–2.3, we deduce that

$$\sigma_{2m-1} \geqslant 0 \qquad (m=1, 2, 3, \ldots). \tag{2.29}$$

Combining (2.27) and (2.29) then gives

$$\sigma_n = \sigma_{n,0} \geqslant 0 \qquad (n=0, 1, 2, \ldots), \tag{2.30}$$

the special case $p=0$ of the desired result (2.19).

Next, to complete the proof of (2.19), we observe that if the sequence $\{\gamma_k\}_{k=0}^{\infty}$ satisfies (2.17), then so does the shifted sequence $\{\gamma_{p+k}\}_{k=0}^{\infty}$ $(p=0, 1, 2, \ldots)$. Consequently, if we apply the foregoing argument to the sequence $\{\gamma_{p+k}\}_{k=0}^{\infty}$ $(p=0, 1, 2, \ldots)$, we obtain the general result of (2.19). Finally, using the sharpened form of (2.2) of Lemma 2.1, it can be shown that equality holds in (2.19) of Lemma 2.4 if and only if all summands in (2.18) vanish, i.e., (2.19′) is valid. ∎

Remark 1. As in the final sharpening of (2.19′) of Lemma 2.4 (derived from the sharpened form of (2.2) of Lemma 2.1), similar sharpenings of subsequent results are also possible. These, however, are left to the reader.

In order to motivate our next result, we remark that the assumption, that the sequence $\{\gamma_k\}_{k=0}^{\infty}$ (with $\gamma_k > 0$) satisfies the Turán inequalities

(2.17), is *equivalent* to the assertion that this sequence is logarithmically concave. More precisely, let $G(x)$ be a C^∞ function defined on $\mathbb{R}$ such that

$$\begin{aligned} &G^{(k)}: \mathbb{R} \to (0, \infty) \qquad (k=0, 1, 2, \ldots), \\ &G^{(k)}(0) := \gamma_k \qquad (k=0, 1, 2, \ldots). \end{aligned} \tag{2.31}$$

Then,

$$\frac{d^2}{dx^2} \log G^{(k)}(x)\bigg|_{x=0} \leqslant 0 \qquad (k=0, 1, 2, \ldots) \tag{2.32}$$

if and only if the Turán inequalities hold:

$$\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geqslant 0 \qquad (k=1, 2, 3, \ldots). \tag{2.33}$$

Next, it easily follows from (2.33) and (2.16′) that

$$0 < \gamma_p \leqslant \left(\frac{\gamma_1}{\gamma_0}\right)^p \gamma_0 \qquad (p=0, 1, 2, \ldots), \tag{2.34}$$

which implies (by the Cauchy–Hadamard formula) that

$$f_0(x) := \sum_{k=0}^{\infty} \gamma_k x^k \tag{2.35}$$

has a radius of convergence R with $0 < R \leqslant \infty$, and that

$$f(x) := \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \tag{2.36}$$

represents a (real) *entire* function. But then, the results of Vincze [V1; V2] (cf. [CC]) show that (2.33) implies that the function $f_0(x)$ of (2.35) is such that $1/f_0^{(p)}(x)$ is *convex* for all $0 \leqslant x < R$ and $p=0, 1, 2, \ldots$, i.e.,

$$\frac{d^2}{dx^2}\left(\frac{1}{f_0^{(p)}(x)}\right) = \frac{2[f_0^{(p+1)}(x)]^2 - f_0^{(p)}(x)\, f_0^{(p+2)}(x)}{[f_0^{(p)}(x)]^3} \geqslant 0$$
$$(0 \leqslant x < R;\ p=0, 1, 2, \ldots). \tag{2.37}$$

Since the γ_k's are positive in (2.35), then $f_0^{(p)}(x) > 0$ on $[0, R)$ and (2.37) imply

$$2[f_0^{(p+1)}(x)]^2 - f_0^{(p)}(x)\, f_0^{(p+2)}(x) \geqslant 0 \qquad (0 \leqslant x < R;\ p=0, 1, 2, \ldots). \tag{2.37′}$$

For the related function $f(x)$ of (2.36), the special case $\nu=0$ of (2.39) below gives

$$[f^{(p+1)}(x)]^2 - f^{(p)}(x)\, f^{(p+2)}(x) \geqslant 0 \qquad (x \geqslant 0;\ p=0, 1, 2, ...), \qquad (2.37'')$$

and, as $f^{(p)}(x) > 0$ on $[0, +\infty)$, we see from (2.37″) that $f^{(p)}(x)$ also satisfies the *weaker* inequalities of (2.37′). For more precise relationships between the Turán inequalities (2.33) and those of (2.37), see [CVi].

THEOREM 2.5. *Let* $\{\gamma_k\}_{k=0}^{\infty}$ *be a sequence of positive real numbers which satisfy the Turán inequalities* (2.33). *If*

$$L_p(x) := L_p(x; f) := [f^{(p+1)}(x)]^2 - f^{(p)}(x) f^{(p+2)}(x) \quad (x \in \mathbb{R};\ p=0, 1, 2, ...), \qquad (2.38)$$

where $f(x)$ *is defined in* (2.36), *then*

$$L_p^{(\nu)}(x) \geqslant 0 \qquad (x \geqslant 0;\ \nu,\ p=0, 1, 2, ...). \qquad (2.39)$$

Proof. In order to prove (2.39), it suffices to show that, for each fixed nonnegative integer p, the Maclaurin series coefficients of $L_p(x)$ of (2.38) are *nonnegative.* Since $f(x)$ of (2.36) is an entire function, the following calculation is readily justified:

$$\begin{aligned} L_p(x) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \gamma_{p+k+1}\gamma_{p+n-k+1} \frac{x^n}{n!} \\ &\quad - \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \gamma_{p+k+2}\gamma_{p+n-k} \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \sigma_{n,p} \frac{x^n}{n!}, \end{aligned}$$

where $\sigma_{n,p}$ is defined in (2.18). Therefore, by (2.19) of Lemma 2.4,

$$\sigma_{n,p} \geqslant 0 \qquad (n,\ p=0, 1, 2, ...),$$

which implies (2.39). ∎

Remark 2. The inequalities

$$L_p(x) \geqslant 0 \qquad (x \in \mathbb{R};\ p=0, 1, 2, ...) \qquad (2.40)$$

are called the *Laguerre inequalities.* It is well known (cf. [Pa] or [S]) that if $f(x)$ is a function in the *Laguerre–Pólya class* (cf. Definition 1.1), then the Laguerre inequalities (2.40) hold *for all real x.* In particular, if $f(x) = \sum_{k=0}^{\infty} \gamma_k (x^k/k!)$ is in $\mathscr{L}-\mathscr{P}$, then the special case $x=0$ of (2.40) gives, from (2.38), the Turán inequalities,

$$\gamma_{p+1}^2 \geqslant \gamma_p \gamma_{p+2} \qquad (p=0, 1, 2, ...). \qquad (2.40')$$

Consequently, if $f(x)=\sum_{k=0}^{\infty}\gamma_k(x^k/k!)$ has $\gamma_k>0$ $(k=0, 1, 2, ...)$, then Theorem 2.5 shows, under the *weaker* assumption (2.40′), that the Laguerre inequalities hold for all $x \geqslant 0$ *and also*, that all the derivatives of $L_p(x)$ are nonnegative for $x \geqslant 0$. Thus, Theorem 2.5 provides a *new* collection of necessary conditions for a real entire function $f(x)$ (cf. (2.36)), with positive Maclaurin coefficients, to belong to the Laguerre–Pólya class.

We next turn to the main result of this section. To this end, we first establish a representation theorem for the Turán differences of the Jensen polynomials (cf. (2.41) and (2.42)) associated with a sequence of positive real numbers.

PROPOSITION 2.6. *Let* $\{\gamma_k\}_{k=0}^{\infty}$ *be a sequence of positive real numbers which satisfy the Turán inequalities* (2.33). *If*

$$\begin{aligned} \Delta_{n,p}(t) &:= g_{n,p}^2(t) - g_{n-1,p}(t)\, g_{n+1,p}(t); \\ \Delta_n(t) &:= \Delta_{n,0}(t) \qquad (t \in \mathbb{R};\, n=1, 2, 3, ...;\, p=0, 1, 2, ...), \end{aligned} \tag{2.41}$$

where

$$\begin{aligned} g_{n,p}(t) &:= \sum_{k=0}^{n} \binom{n}{k} \gamma_{k+p} t^k, \\ g_n(t) &:= g_{n,0}(t) \qquad (n=0, 1, 2, ...), \end{aligned} \tag{2.42}$$

then

$$\Delta_{n,p}(t) := \frac{t^{2n-1}}{2} \sum_{k=0}^{n-1} \binom{n-1}{k} \sum_{j=0}^{k} \binom{k}{j} t^{2k-j} \sum_{i=0}^{j} \binom{j}{i} \psi(i, j, k; p)$$

$$(n=1, 2, 3, ...;\, p=0, 1, 2, ...), \tag{2.43}$$

where

$$\begin{aligned} \psi(i, j, k; p) := {} & 2\gamma_{p+k-j+i+1}\gamma_{p+k-i+1} \\ & - \gamma_{p+k-i+2}\gamma_{p+k-j+i} - \gamma_{p+k-i}\gamma_{p+k-j+i+2}. \end{aligned} \tag{2.44}$$

Proof. We have already observed (cf. (2.36)) that since (2.33) holds, the function

$$f(x) := \sum_{k=0}^{\infty} f^{(k)}(0) \frac{x^k}{k!} = \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!}$$

is an entire function. Therefore, by Cauchy's integral formula, we have

$$\gamma_k = f^{(k)}(0) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w^{k+1}}\, dw \qquad (k=0, 1, 2, ...), \tag{2.45}$$

where C is a positively oriented circle, centered at the origin. Hence, using (2.45), we obtain from (2.42) that

$$g_{n,p}(t)=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{2\pi i}\int_C\frac{t^k f(w)}{w^{k+p+1}}\,dw,$$

which can be expressed as

$$g_{n,p}(t)=\frac{1}{2\pi i}\int_C\left(1+\frac{t}{w}\right)^n f(w)\,\frac{dw}{w^{p+1}}\qquad (n,\ p=0,1,2,\ldots). \tag{2.46}$$

Thus, a calculation shows from (2.46) and (2.41) that

$$\Delta_{n,p}(t)=\frac{t^2}{2}\left(\frac{1}{2\pi i}\right)^2\int_C\int_C b_n(w,z,t)\,\frac{f(w)}{w^{p+1}}\,\frac{f(z)}{z^{p+1}}\,dw\,dz, \tag{2.47}$$

where

$$b_n(w,z,t):=\left(1+\frac{t}{w}\right)^{n-1}\left(1+\frac{t}{z}\right)^{n-1}\left[\frac{2}{wz}-\frac{1}{w^2}-\frac{1}{z^2}\right]\qquad (|z|,\ |w|>0), \tag{2.48}$$

and where the interchanging of the order of integration is readily justified. Now, if we expand $b_n(w, z, t)$ in powers of t, $1/z$, and $1/w$, we find that

$$b_n(w,z,t)=\left[\frac{2}{wz}-\frac{1}{w^2}-\frac{1}{z^2}\right]\sum_{k=0}^{n-1}\binom{n-1}{k}\left[\left(\frac{1}{w}+\frac{1}{z}\right)+\frac{t}{wz}\right]^k t^k,$$

which can also be expressed as

$$b_n(w,z,t)=\left[\frac{2}{wz}-\frac{1}{w^2}-\frac{1}{z^2}\right]\sum_{k=0}^{n-1}\binom{n-1}{k}\times\sum_{j=0}^{k}\binom{k}{j}\frac{t^{2k-j}}{(wz)^{k-j}}\sum_{i=0}^{j}\binom{j}{i}\left(\frac{1}{w}\right)^i\left(\frac{1}{z}\right)^{j-i}. \tag{2.49}$$

Hence, using (2.49), $\Delta_{n,p}(t)$ can be expressed as (cf. (2.47)),

$$\Delta_{n,p}(t)=\frac{t^2}{2}\sum_{k=0}^{n-1}\binom{n-1}{k}\sum_{j=0}^{k}\binom{k}{j}t^{2k-j}\sum_{i=0}^{j}\binom{j}{i}\times\frac{1}{(2\pi i)^2}\int_C\int_C\left\{2\left(\frac{1}{w}\right)^{p+k-j+i+1}\cdot\left(\frac{1}{z}\right)^{p+k-i+1}-\left(\frac{1}{w}\right)^{p+k-j+i}\cdot\left(\frac{1}{z}\right)^{p+k-i+2}-\left(\frac{1}{w}\right)^{p+k-j+i+2}\cdot\left(\frac{1}{z}\right)^{p+k-i}\right\}\frac{f(w)}{w}\frac{f(z)}{z}\,dw\,dz.$$

On applying the formula of (2.45) to the above expression, the desired result of (2.43) is then obtained. ∎

We next combine the foregoing results to prove the following theorem

THEOREM 2.7. *Let* $\{\gamma_k\}_{k=0}^{\infty}$ *be a sequence of positive real numbers which satisfy the Turán inequalities* (2.33). *Then,*

$$\Delta_{n,p}^{(v)}(0) \geqslant 0 \qquad (n=1, 2, 3, ...; v, p=0, 1, 2, ...), \tag{2.50}$$

where $\Delta_{n,p}(t)$ *is defined by* (2.41). *Moreover,*

$$\Delta_{n,p}^{(v)}(t) \geqslant 0 \qquad (t \geqslant 0; n=1, 2, 3, ...; v, p=0, 1, 2, ...). \tag{2.51}$$

Proof. In order to prove (2.50), we use formula (2.43) of Proposition 2.6 for the representation of $\Delta_{n,p}(t)$, and note that it suffices to show that

$$\sum_{i=0}^{j} \binom{j}{i} \psi(i, j, k; p) \geqslant 0. \tag{2.52}$$

With $m := p+k-j$, we find from (2.44) that

$$\begin{aligned}
&\sum_{i=0}^{j} \binom{j}{i} \psi(i, j, k; p) \\
&\quad = \sum_{i=0}^{j} \binom{j}{i} \{[\gamma_{m+i+1}\gamma_{m+j-i+1} - \gamma_{m+j-i+2}\gamma_{m+i}] \\
&\qquad + [\gamma_{m+i+1}\gamma_{m+j-i+1} - \gamma_{m+j-i}\gamma_{m+i+2}]\} \\
&\quad = 2\sum_{i=0}^{j} \binom{j}{i} [\gamma_{m+i+1}\gamma_{m+j-i+1} - \gamma_{m+j-i}\gamma_{m+i+2}] = 2\sigma_{j,m},
\end{aligned}$$

where $\sigma_{j,m}$ is defined in (2.18). But then by Lemma 2.4,

$$\sigma_{j,m} \geqslant 0 \qquad (j, m=0, 1, 2, ...),$$

and, *a fortiori*, inequalities (2.50) and (2.51) hold. ∎

COROLLARY 2.8. *Let* $\{\gamma_k\}_{k=0}^{\infty}$ *be a sequence of positive real numbers which satisfy the Turán inequalities* (2.33). *With the notation of* (2.41) *and* (2.42), *set*

$$g_n^*(t) := t^n g_n\left(\frac{1}{t}\right) \qquad \text{and} \qquad \Delta_n^*(t) := t^{2n}\Delta_n\left(\frac{1}{t}\right) \qquad (n=1, 2, 3, ...). \tag{2.53}$$

Then, the following inequalities hold:

$$\frac{n}{n+1}(g_{n+1}^{*\prime}(t))^2 - g_{n+1}^{*}(t)\, g_{n+1}^{*\prime\prime}(t) \geqslant 0 \qquad (t \geqslant 0;\, n = 1, 2, 3, ...); \tag{2.54}$$

$$\frac{n+1}{n}\, g_{n+1}(t)\, g_n'(t) - g_{n+1}'(t)\, g_n(t) \geqslant 0 \qquad (t \geqslant 0;\, n = 1, 2, 3, ...); \tag{2.55}$$

and

$$\frac{n+1}{n}(g_{n,p}'(t))^2 - g_{n-1,p}(t)\, g_{n+1,p}''(t) \geqslant 0 \qquad (t \geqslant 0;\, n = 1, 2, 3, ...). \tag{2.56}$$

Proof. It can be verified from the definitions of (2.42) and (2.53) that

$$g_n^{*\prime}(t) = n g_{n-1}^{*}(t) \qquad (t \in \mathbb{R};\, n = 1, 2, 3, ...).$$

Consequently, a short calculation shows that inequality (2.54) is an immediate consequence of the special case $v = p = 0$ of (2.51) of Theorem 2.7. In order to prove (2.55), we recall the known fact (cf. [CV3, Proposition 2.1(iii)] or [R, p. 133]) that the Jensen polynomials $\{g_n(t)\}_{n=0}^{\infty}$ satisfy

$$n g_n(t) = n g_{n-1}(t) + t g_n'(t) \qquad (t \in \mathbb{R};\, n = 1, 2, 3, ...). \tag{2.57}$$

Using (2.57), we find that

$$\Delta_n(t) = t\left[\frac{1}{n}\, g_{n+1}(t)\, g_n'(t) - \frac{1}{n+1}\, g_{n+1}'(t)\, g_n(t)\right] \qquad (t \in \mathbb{R};\, n = 1, 2, 3, ...), \tag{2.58}$$

where $\Delta_n(t)$ is defined by (2.41). Now by (2.51) of Theorem 2.7, $\Delta_n(t) \geqslant 0$ ($t \geqslant 0$; $n = 1, 2, 3, ...$), and we thus conclude that (2.55) holds. Finally, in order to prove (2.56), we use the following known results [CV3, Proposition 2.1(iv) and (v)] that the polynomials, $g_{n,p}(t)$ ($n, p = 0, 1, 2, ...$) satisfy

$$g_{n+1,p}(t) = g_{n,p}(t) + t g_{n,p+1}(t) \qquad (t \in \mathbb{R};\, n, p = 0, 1, 2, ...),$$

and thus

$$\Delta_{n,p}(t) = t^2[g_{n-1,p+1}^2(t) - g_{n-1,p}(t)\, g_{n-1,p+2}(t)] \qquad (t \in \mathbb{R};\, n = 1, 2, 3, ...;\, p = 0, 1, 2, ...), \tag{2.59}$$

where $\Delta_{n,p}(t)$ is defined by (2.41). Once again by Theorem 2.7, $\Delta_{n,p}(t) \geqslant 0$ ($t \geqslant 0$; $n = 1, 2, 3, ...$; $p = 0, 1, 2, ...$), and a calculation, using (2.59) and the formula $g_{n,p}'(t) = n g_{n-1,p+1}(t)$ ($n = 1, 2, 3, ...$; $p = 0, 1, 2, ...$), yields the desired inequality (2.56). ∎

Remark 3. The foregoing results have applications in the theory of special functions. While Theorem 2.7 was proved under weak assumptions, we can nonetheless deduce from Theorem 2.7 several known inequalities (cf. [S]) concerning classical polynomials and functions. For example, set

$$\mathscr{L}_n(t) := \sum_{k=0}^{n} \binom{n}{k} \frac{(-t)^k}{k!}, \tag{2.60}$$

so that $\mathscr{L}_n(t)$ is the nth classical Laguerre polynomial (cf. [R, p. 213]). On defining $\{\hat{\gamma}_k := 1/k!\}_{k=0}^{\infty}$, we see that $(\hat{\gamma}_k)^2 - \hat{\gamma}_{k-1}\hat{\gamma}_{k+1} = [(k!)^2(k+1)]^{-1} > 0$ for all $k = 1, 2, \ldots$, so that Turán inequalities (2.33) are satisfied for $\{\hat{\gamma}_k\}_{k=0}^{\infty}$. Thus, from the definition in (2.60), it follows from (2.51) of Theorem 2.7 *not only* that

$$\Delta(t; \mathscr{L}_n) := (\mathscr{L}_n(-t))^2 - \mathscr{L}_{n-1}(-t)\, \mathscr{L}_{n+1}(-t) \geqslant 0$$
$$(t \geqslant 0;\, n = 1, 2, \ldots), \tag{2.61}$$

but *also* that the Maclaurin coefficients of $\Delta_n(t, \mathscr{L}_n)$ are (cf. (2.50)) all nonnegative. More precisely, if $\Delta(t, \mathscr{L}_n) := \sum_{k=2}^{2n} \alpha_j(n)\, t^j$, then with the sharpened form of Lemma 2.4, it follows, for any $n \geqslant 1$, that

$$\alpha_j(n) > 0 \qquad (j = 2, 3, \ldots, 2n).$$

As an application of the foregoing results to *transcendental* (i.e., nonpolynomial) entire functions, consider the particular Mittag–Leffler function

$$E_q(x) := \sum_{n=0}^{\infty} \gamma_n(q) \frac{x^n}{n!}, \quad \text{with} \quad \gamma_n(q) := \frac{n!}{\Gamma(1+qn)}$$
$$(n = 0, 1, \ldots), \tag{2.62}$$

where $q > 1$ is a positive integer. It is known (cf. [ESV, p. 3]) that $E_q(x)$ is an entire function of order $1/q$. (Since $q > 1$, then the order of $E_q(x)$ is positive and less than unity, so that $E_q(x)$ has infinitely many zeros.) From the definition in (2.62), it easily follows that the coefficients $\{\gamma_n(q)\}_{n=0}^{\infty}$ satisfy the strict Turán inequalities,

$$\gamma_n^2(q) - \gamma_{n-1}(q)\, \gamma_{n+1}(q) > 0 \qquad (n = 1, 2, 3, \ldots), \tag{2.63}$$

for any positive integer $q > 1$. Thus, if $g_n(t; q) := \sum_{k=0}^{n} \binom{n}{k} \gamma_k(q)\, t^k$ denotes the Jensen polynomials associated with $E_q(x)$, and if (cf. 2.41) with $p = 0$)

$$\Delta_n(t; q) := g_n^2(t; q) - g_{n-1}(t; q)\, g_{n+1}(t; q) =: \sum_{j=2}^{2n} \alpha_j(n; q)\, t^j$$
$$(n = 1, 2, 3, \ldots),$$

then the sharpened form of Lemma 2.4 coupled with (2.63) gives

$$\alpha_j(n; q) > 0 \qquad (j = 2, 3, ..., 2n; q > 1). \tag{2.64}$$

Remark 4. We claim that Corollary 2.8 is also of theoretical interest, since it provides necessary conditions for a real entire function to belong to the Laguerre–Pólya class. Indeed, it is known (cf. [CV3, Proposition 2.3] or [P1]) that if

$$f(x) := \sum_{k=0}^{\infty} \gamma_k \frac{x^k}{k!} \qquad (\gamma_0 \neq 0),$$

is a real entire function with infinitely many zeros, then a *necessary and sufficient condition* for $f(x)$ to be in the Laguerre–Pólya class is that

$$\Delta_n(t) := g_n^2(t) - g_{n-1}(t)\, g_{n+1}(t) > 0 \qquad (t \in \mathbb{R} - \{0\}; n = 1, 2, 3, ...), \tag{2.65}$$

where

$$g_n(t) := \sum_{k=0}^{n} \binom{n}{k} \gamma_k t^k \qquad (n = 0, 1, 2, ...) \tag{2.66}$$

is the nth Jensen polynomial associated with $f(x)$.

3. Applications

In this section, in conjunction with the authors' earlier work (cf. [CNV; CV1–CV4]), we provide two applications of the results of Section 2. The first application involves a general class of entire functions whose members can be represented by Fourier transform, while the second application pertains to properties of the Riemann ξ-function.

To facilitate the description of a general class of entire functions which we shall consider in the sequel, it is convenient to use the following definition (cf. [CV4]).

Definition 3.1. A function $K: \mathbb{R} \to \mathbb{R}$ is called an *admissible kernel* if it satisfies the following properties:

$$\begin{aligned} &\text{(i)} && K \text{ is integrable over } \mathbb{R}, \\ &\text{(ii)} && K(t) > 0 \qquad (t \in \mathbb{R}), \\ &\text{(iii)} && K(t) = K(-t) \qquad (t \in \mathbb{R}), \\ &\text{(iv)} && K(t) = O(\exp(-|t|^{2+\varepsilon})), \quad \text{for some } \varepsilon > 0, \text{ as } t \to \infty. \end{aligned} \tag{3.1}$$

It is known [P2] that the Fourier transform

$$F(x) := F(x; K) := \int_{-\infty}^{\infty} K(t)\, e^{ixt}\, dt \tag{3.2}$$

of an admissible kernel, $K(t)$, is a real entire function of finite order $\rho := \rho(F(x; K))$, where ρ satisfies

$$\rho \leqslant \frac{\varepsilon+2}{\varepsilon+1} < 2, \qquad \text{for some} \quad \varepsilon := \varepsilon(K) > 0. \tag{3.3}$$

Remark 5. Since an admissible kernel is an even function, it follows that the Maclaurin series expansion of $F(x)$ (cf. (3.2)) is of the form

$$F(x) = \sum_{k=0}^{\infty} b_k \frac{(-x^2)^k}{(2k)!}, \tag{3.4}$$

where (cf. (3.1ii))

$$b_k := b_k(F) := 2 \int_0^{\infty} t^{2k} K(t)\, dt > 0 \qquad (k = 0, 1, 2, ...). \tag{3.5}$$

On setting $z = -x^2$ in (3.4), we obtain the function

$$F_c(z) := F_c(-x^2) := \int_{-\infty}^{\infty} K(t) \cosh(t\sqrt{z})\, dt := \sum_{k=0}^{\infty} \gamma_k \frac{z^k}{k!}, \tag{3.6}$$

where

$$\gamma_k := \gamma_k(F_c) := \frac{k!}{(2k)!} b_k(F) \qquad (k = 0, 1, 2, ...). \tag{3.7}$$

To bring into sharper focus the results proved below, we state the next proposition a property of functions in the Laguerre–Pólya class, $\mathscr{L} - \mathscr{P}$ (cf. Definition 1.1).

PROPOSITION 3.1. (a) *If $f(x) \in \mathscr{L} - \mathscr{P}$, then $f(x)$ satisfies the Laguerre inequalities, i.e.,*

$$L_p(x) := L_p(x; f) := (f^{(p+1)}(x))^2 - f^{(p)}(x)\, f^{(p+2)}(x) \geqslant 0$$
$$(x \in \mathbb{R};\ p = 0, 1, 2, ...). \tag{3.8}$$

(b) *If $K(t)$ is an admissible kernel, then*

$$F(x) := F(x; K) \in \mathscr{L} - \mathscr{P} \quad \text{if and only if} \quad F_c(x) := F_c(x; K) \in \mathscr{L} - \mathscr{P}. \tag{3.9}$$

(c) *If $K(t)$ is an admissible kernel, then*

$$L_0(x; F) \geqslant 0 \text{ for all } x \in \mathbb{R}$$
$$\text{if and only if } L_0(x; F_c) \geqslant 0 \text{ for all } x \leqslant 0. \quad (3.10)$$

Proof. The proof of (3.8) is well known (cf. [S]). Next, to establish (3.9), assume that $F(x)$ is in $\mathscr{L}-\mathscr{P}$. Then, $F(x)$ is even, its order is less than 2, and $F(0) = b_0 > 0$ which follow from (3.4), (3.3), and (3.5), respectively. Thus, from (1.8) of Definition 1.1,

$$F(x) = b_0 \prod_{j=1}^{\omega} \left(1 - \frac{x^2}{x_j^2}\right), \qquad \text{where} \quad x_j > 0 \text{ and } \sum_{j=1}^{\omega} x_j^{-2} =: \beta < \infty.$$

With $z = -x^2$, then, by definition (cf. (3.6)),

$$F_c(z) = b_0 \prod_{j=1}^{\omega} \left(1 + \frac{z}{x_j^2}\right) = b_0 e^{\beta z} \prod_{j=1}^{\omega} \left(1 + \frac{z}{x_j^2}\right) e^{-z/x_j^2}, \qquad (3.9')$$

which, by Definition 1.1, is an element of $\mathscr{L}-\mathscr{P}$. The converse (i.e., that $F_c(x)$ in $\mathscr{L}-\mathscr{P}$ implies $F(x)$ is in $\mathscr{L}-\mathscr{P}$) is similar. Finally, to establish (3.10), note that $F(x)$ is an even function of x from (3.4), which implies that $L_0(x; F)$ is an even function of x. Thus,

$$L_0(x; F) \geqslant 0 \text{ for all } x \in \mathbb{R}$$
$$\text{if and only if } L_0(-x^2; F) \geqslant 0 \text{ for all } x \leqslant 0. \quad (3.9'')$$

Since from (3.6) $F_c(z) = F(x)$, where $z = -x^2$, a short calculation shows that

$$L_0(-x^2; F) = L_0(z; F_c),$$

and the desired result (3.10) then follows from (3.9''). ∎

Remark 6. The equivalence in (3.10) is *best possible* in the sense that there are polynomials $p_c(x)$ and $p(x) := p_c(-x^2)$ such that

$$L_0(x; p) \geqslant 0 \quad \text{for all} \quad x \in \mathbb{R}, \qquad (3.11)$$

$$L_0(x; p_c) \geqslant 0 \quad \text{for all} \quad x \leqslant 0, \qquad (3.12)$$

but

$$L_0(x_0; p_c) < 0 \quad \text{for some} \quad x_0 > 0. \qquad (3.13)$$

As a concrete example, consider the polynomial $p_c(x) := x(x^2 - 1)$ $[(x - 1/2)^2 + 0.2]$ and set $p(x) := p_c(-x^2)$. Then, a calculation shows that

(3.11) and (3.12) are valid, and (3.13) holds with $x_0 := 0.5$. In this case, the polynomial $L_0(x; p) := (p'(x))^2 - p(x)\, p''(x)$ is given explicitly by

$$L_0(x; p) = 10x^{18} + 14x^{16} + 8x^{14} + \tfrac{33}{2}x^{12} + \tfrac{5843}{200}x^{10} + 15x^8 + \tfrac{101}{50}x^6 + \tfrac{9}{10}x^4 + \tfrac{81}{200}x^2,$$

which clearly satisfies (3.11).

The proof of our main result (cf. Theorem 3.3 below) of this section hinges on the following known result.

THEOREM 3.2 [CV1, Theorem 2.4]. *Suppose that*

$$K(t) \text{ is an admissible kernel,} \tag{3.14}$$

and that

$$\log(K(\sqrt{t})) \text{ is strictly concave for } 0 < t < \infty. \tag{3.15}$$

Let $f(z)$ be an even entire function in the Laguerre-Pólya class, normalized by $f(0) := 1$, and set

$$b_m := b_m(K, f) := 2 \int_0^\infty t^{2m} f(it)\, K(t)\, dt \qquad (m = 0, 1, 2, \ldots). \tag{3.16}$$

Then,

$$b_m^2 > \frac{2m-1}{2m+1}\, b_{m-1} b_{m+1} \qquad (m = 1, 2, 3, \ldots). \tag{3.17}$$

Preliminaries aside, we next prove

THEOREM 3.3. *Set*

$$F_c(z) := F_c(z; K, f) := \int_{-\infty}^{\infty} f(it)\, K(t) \cosh(t\sqrt{z})\, dt, \tag{3.18}$$

where the kernel $K(t)$ satisfies (3.14)–(3.15), *and assume that $f(z) \in \mathcal{L} - \mathcal{P}$, with $f(z)$ even and with $f(0) := 1$. If*

$$L_p(x; F_c) := (F_c^{(p+1)}(x))^2 - F_c^{(p)}(x)\, F_c^{(p+2)}(x) \qquad (p = 0, 1, 2, \ldots), \tag{3.19}$$

then

$$L_p^{(\nu)}(x; F_c) \geqslant 0 \qquad (x \geqslant 0;\ \nu, p = 0, 1, 2, \ldots). \tag{3.20}$$

Proof. Set

$$F(x) := F(x, K, f) := \int_{-\infty}^{\infty} e^{ixt} f(it)\, K(t)\, dt. \tag{3.21}$$

Since, by hypothesis, f is an element of $\mathscr{L}-\mathscr{P}$ and f is even, then f is an entire function whose order is less than or equal to 2, and it is easy to verify that $f(it)\,K(t)$ is also an admissible kernel (cf. (3.1)). Therefore, $F(x)$ is a real entire function and hence so is $F_c(z)$ (cf. (3.18)). Now, a calculation shows that the Maclaurin series expansion of $F_c(z)$ is given by

$$F_c(z) = \sum_{k=0}^{\infty} \gamma_k \frac{z^k}{k!}, \tag{3.22}$$

where

$$\gamma_k := \gamma_k(F) := \frac{k!\, b_k}{(2k)!} \qquad (k = 0, 1, 2, ...), \tag{3.23}$$

and where b_k is defined in (3.16). But then, it follows from (3.17) of Theorem 3.2 that all the Turán inequalities

$$\gamma_k^2 > \gamma_{k-1}\gamma_{k+1} \qquad (k = 1, 2, 3, ...) \tag{3.24}$$

hold. In addition, since $f(it)\,K(t)$ is an admissible kernel, (3.5) and (3.23) give that $\gamma_k > 0$ $(k = 0, 1, 2, ...)$ and therefore, by Theorem 2.5, the Laguerre inequalities (3.20) are all valid for $x \geqslant 0$. ∎

Remark 7 *and an Open Problem.* By Proposition 3.1, we know that $F(x; K) \in \mathscr{L}-\mathscr{P}$ if and only if $F_c(x; K) \in \mathscr{L}-\mathscr{P}$ and hence (cf. (3.8)), in *this case*

$$L_p^{(\nu)}(x; F_c) \geqslant 0 \qquad \textit{for all} \quad x \in \mathbb{R} \quad (\nu, p = 0, 1, 2, ...). \tag{3.25}$$

However, if we do not assume that $F(x; K)$ belongs to the Laguerre–Pólya class, then it is an open problem to characterize those admissible kernels, $K(t)$, for which the Laguerre inequalities $L_p(x; F) \geqslant 0$ $(x \in \mathbb{R};\ p = 0, 1, 2, ...)$ are valid. The significance of a solution of this problem is that it could lead to the characterization of admissible kernels whose Fourier transforms belong to the Laguerre–Pólya class.

We conclude this proper with a specific application to the Riemann ξ-function, $\xi(x)$, which can be expressed (cf. [P1]) as

$$\xi\left(\frac{x}{2}\right) := 8 \int_0^{\infty} \Phi(t) \cos(xt)\, dt, \tag{3.26}$$

and where the kernel $\Phi(t)$ is the *Jacobi theta* function

$$\Phi(t) := \sum_{n=1}^{\infty} (2n^4\pi^2 e^{9t} - 3n^2\pi e^{5t}) \exp(-n^2\pi e^{4t}). \tag{3.27}$$

With the notations of (3.2) and (3.6), we thus set

$$F(x; \Phi) := \int_{-\infty}^{\infty} \Phi(t)\, e^{ixt}\, dt, \tag{3.28}$$

and

$$F_c(z; \Phi) := \int_{-\infty}^{\infty} \Phi(t) \cosh(t\sqrt{z})\, dt. \tag{3.29}$$

Now, it is well known (see, for example, [CNV, Theorem A]) that $\Phi(t)$ satisfies the properties (3.1), so that $\Phi(t)$ is an admissible kernel. In addition, the authors (cf. [CV1]) have recently proved that $\log(\Phi(\sqrt{t}))$ is strictly concave for $0 < t < \infty$. Therefore, as a direct application of Theorem 3.3 we obtain (with $f(t) \equiv 1$) the following corollary.

COROLLARY 3.4. *The function $F_c(x; \Phi)$ and all its derivatives satisfy the Laguerre inequalities for all nonnegative values of x, i.e.,*

$$L_p^{(\nu)}(x; F_c) \geqslant 0 \qquad (x \geqslant 0;\ \nu,\ p = 0, 1, 2, ...). \tag{3.30}$$

Remarks 8. (1) As an immediate consequence of Theorem 3.3, we also obtain the following result (cf. (3.32)) which is more general than Corollary 3.4. Indeed, set

$$\tilde{F}_c(z) := \tilde{F}_c(z; \Phi, f) := \int_{-\infty}^{\infty} f(it)\, \Phi(t) \cosh(t\sqrt{z})\, dt, \tag{3.31}$$

where $f(t) \in \mathscr{L} - \mathscr{P}$, with $f(t)$ even and $f(0) := 1$. Then

$$L_p^{(\nu)}(x; \tilde{F}_c)) \geqslant 0 \qquad (x \geqslant 0;\ p = 0, 1, 2, ...). \tag{3.32}$$

(2) In light of the authors' earlier work, we next point out a relationship between the Laguerre inequalities and the Riemann Hypothesis. For this relationship, set $f_\mu(t) := \cosh(\mu t)$, $h_\mu(t) := \sinh(\mu t)$ $(\mu \in \mathbb{R})$, and set

$$P_\mu(x) := 2\int_0^{\infty} f_\mu(t)\, \Phi(t) \cos(xt)\, dt$$

and

$$Q_\mu(x) := 2\int_0^{\infty} h_\mu(t)\, \Phi(t) \sin(xt)\, dt.$$

Then, (cf. [CV4, Lemma 3.3 and Corollary 3.5]), the Riemann Hypothesis is true *if and only if*

$$L_0(x; P_\mu) + L_0(x; Q_\mu) \geqslant 0 \qquad (x \geqslant 0, 0 \leqslant \mu < 1), \tag{3.33}$$

where $L_0(x; f) := (f'(x))^2 - f(x) f''(x)$. We also note that, in the proof of (3.33), use was made of the known fact (cf. [CNV, Theorem A]) that all the zeros of $F(x; \Phi)$ (cf. (3.2)) lie in the strip $S(1) := \{z \in \mathbb{C} : |\text{Im}\, z| < 1\}$. ∎

COROLLARY 3.5. *With* (3.29), *set*

$$g_{n,p}(t; F_c) := \sum_{k=0}^{n} \binom{n}{k} \gamma_{k+p} t^k, \tag{3.34}$$

where

$$\gamma_k := \frac{k! b_k}{(2k)!} \text{ and } b_k := \int_0^\infty t^{2k} \Phi(t)\, dt \qquad (k = 0, 1, 2, ...). \tag{3.35}$$

If

$$\Delta_{n,p}(t; F_c) := (g_{n,p}(t; F_c))^2 - g_{n-1,p}(t; F_c)\, g_{n+1,p}(t; F_c)$$
$$(n = 1, 2, 3, ..., p = 0, 1, 2, ...), \tag{3.36}$$

then

$$\Delta_{n,p}^{(\nu)}(t; F_c) \geqslant 0 \qquad \begin{pmatrix} t \geqslant 0;\, \nu, p = 0, 1, 2, ..., \\ n = 1, 2, 3, ... \end{pmatrix}. \tag{3.37}$$

Proof. Since it is known (cf. [CNV] or [CV1]) that the γ_k's, defined by (3.35), satisfy the Turán inequalities $\gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geqslant 0$ $(k = 1, 2, 3, ...)$, then (3.37) follows from Theorem 2.7. ∎

Remark 9. Since $F_c(x; \Phi)$ of (3.29) is an entire function of order $\frac{1}{2}$, it follows from the Hadamard factorization theorem that $F_c(x; \Phi)$ has infinitely many zeros. Therefore, by Remark 4 of Section 2, we see that the Riemann Hypothesis is valid *if and only if*

$$\Delta_{n,0}^{(0)}(t; F_c) > 0 \qquad (t \in \mathbb{R} - \{0\}; n = 1, 2, 3, ...). \tag{3.38}$$

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