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Analysis

SOME 2-PERIODIC TRIGONOMETRIC INTERPOLATION PROBLEMS ON EQUIDISTANT NODES

A. SHARMA, J. SZABADOS[1] and R.S. VARGA[2]

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ABSTRACT: The problem to be considered here is the determination of necessary and sufficient conditions for the uniqueness of 2-periodic lacunary trigonometric interpolation on equidistant nodes. Our main results are new necessary and sufficient conditions in particular cases which depend only on the total number of even and odd integers in the derivatives which define the trigonometric interpolation process. As such, these new conditions can be readily checked, as they avoid the evaluation of determinants.

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1. Introduction.

Given any positive integer n, define

$$x_k = x_k(n) := \frac{k\pi}{n} \qquad (k = 0, 1, \cdots, 2n-1), \tag{1.1}$$

so that $\{x_k\}_{k=0}^{2n-1}$ is a set of $2n$ equidistant *nodes* in $[0, 2\pi)$. Next, with p and q arbitrary

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positive integers, assume that the associated vectors

$$\mathbf{m} := (m_1, m_2, \cdots, m_p), \text{ and } \mathbf{m}' := (m_1', m_2', \cdots, m_q'), \tag{1.2}$$

have nonnegative integer components which satisfy

$$0 =: m_1 < m_2 < \cdots < m_p, \text{ and } 0 \le m_1' < m_2' < \cdots < m_q'. \tag{1.3}$$

We consider here the following *2-periodic trigonometric interpolation problem* for the vectors $(\mathbf{m}; \mathbf{m}')$. For arbitrary data consisting of complex numbers $\{\alpha_{j,\nu}\}_{j=0,\nu=1}^{n-1,p}$ and $\{\alpha_{j,\nu}'\}_{j=0,\nu=1}^{n-1,q}$, we ask if there is a *unique* complex trigonometric polynomial of the form

$$t_M(x) = a_0 + \sum_{k=1}^{M} (a_k \cos kx + b_k \sin kx), \tag{1.4}$$

or of the form

$$t_M(x) = a_0 + \sum_{k=1}^{M-1} (a_k \cos kx + b_k \sin kx) + a_M \cos(Mx + \frac{\varepsilon\pi}{2}) \tag{1.4'}$$

(where $\varepsilon = 0$ or $\varepsilon = 1$), such that

$$\begin{cases} t_M^{(m_\nu)}(x_{2j}) &= \alpha_{j,\nu} \quad (j = 0, 1, \cdots, n-1;\ \nu = 1, 2, \cdots, p), \text{ and} \\ t_M^{(m_\nu')}(x_{2j+1}) &= \alpha_{j,\nu}' \quad (j = 0, 1, \cdots, n-1;\ \nu = 1, 2, \cdots, q). \end{cases} \tag{1.5}$$

Note that as the number of nodes in (1.1) is *even* (namely, $2n$), we see that the interpolation conditions of (1.5) are broken into interpolation conditions on *two* disjoint sets of n nodes, from which the term, *2-periodic* trigonometric interpolation, is derived.

The total number of interpolation conditions in (1.5) is evidently

$$N := n(p+q). \tag{1.6}$$

If N is odd (which implies that n is odd and that $p+q$ is odd), then the sought trigonometric interpolant $t_M(x)$ is necessarily of the form (1.4) (which has an odd number of parameters), and in this case, $M = (N-1)/2$. If N is even (which implies that at least one of the numbers, n and $p+q$, is even), the sought trigonometric interpolant $t_M(x)$ is necessarily of the form

(1.4′) with $M = N/2$, where $\varepsilon (= 0$ or $1)$ is to be appropriately determined. To summarize,

$$(1.7) \qquad \begin{cases} M = (N-1)/2 & \text{if } N \text{ is odd, and} \\ M = N/2 & \text{if } N \text{ is even.} \end{cases}$$

We say that this $(\mathbf{m}; \mathbf{m}')$ 2-periodic trigonometric interpolation problem of (1.5) is *regular*, if, for arbitrary data, the interpolation problem (1.5) admits a *unique* solution $t_M(x)$, where $t_M(x)$ is of the appropriate form (1.4) or (1.4′).

This $(\mathbf{m}; \mathbf{m}')$ 2-periodic trigonometric interpolation problem on $2n$ equidistant nodes is a *special case* of the more general s-periodic trigonometric interpolation problem on sn equidistant nodes in $[0, 2\pi)$, considered in Sharma, Smith, and Tzimbalario [5]. In [5], necessary and sufficient conditions for the regularity of this s-periodic trigonometric interpolation problem were derived, in terms of the nonvanishing of several determinants (of possibly large order). In this form, these necessary and sufficient conditions are not in general easy to apply.

Recently, however, two papers (cf. Sharma, Szabados, and Varga [6] and Sharma and Varga [7]) have treated special cases of 2-periodic trigonometric interpolation on equidistant nodes, and, in each of these latter two papers, new necessary and sufficient conditions for regularity were derived which depend *only* on the total number of even and odd integers in the components of the vectors $\mathbf{m}$ and $\mathbf{m}'$ of (1.2). As such, these necessary and sufficient conditions can be *easily checked*, in contrast with the necessary and sufficient determinantal conditions of [5]. Thus, our goal here is to extend the results of [6] and [7] by finding necessary and sufficient conditions, for the regularity of the 2-periodic lacunary trigonometric interpolation problem, which similarly depend only on the total number of even and odd integers in the components of each of the vectors $\mathbf{m}$ and $\mathbf{m}'$ of (1.2). To this end, we use throughout the notation of (cf. (1.3))

$$(1.8) \quad \begin{cases} E & := \text{number of even integers in the components of } \mathbf{m} = (0, m_2, \cdots, m_p), \\ O & := \text{number of odd integers in the components of } \mathbf{m} = (0, m_2, \cdots, m_p), \\ E' & := \text{number of even integers in the components of } \mathbf{m}' = (m_1', m_2', \cdots, m_q'), \\ O' & := \text{number of odd integers in the components of } \mathbf{m}' = (m_1', m_2', \cdots, m_q'), \end{cases}$$

so that

$$E + O = p \text{ and } E' + O' = q. \tag{1.9}$$

2. Statements of Theorems 1 and 2.

With the notation of section 1, we now state our basic results, Theorem 1 (when N is odd) and Theorem 2 (when N is even). We begin with the case when N is odd.

THEOREM 1. *Let N be odd, so that* (cf. (1.6)) $n =: 2r+1$ *and* $p+q =: 2s+1$ *are both odd. Then, a necessary condition for the* $(\mathrm{m};\mathrm{m}')$ *2-periodic trigonometric interpolation problem* (1.5) *to be regular is that* (cf. (1.8))

$$E + E' - 1 = s = O + O'. \tag{2.1}$$

Moreover, if the components of m and m' *are, respectively, alternately even and odd integers, i.e.* (cf. (1.2) and (1.3)),

$$m_i + m_{i+1} \text{ is odd } (i = 0, 1, \cdots, p-1), \text{ and } m'_i + m'_{i+1} \text{ is odd } (i = 0, 1, \cdots, q-1), \tag{2.2}$$

then (2.1) *is both necessary and sufficient for this* $(\mathrm{m};\mathrm{m}')$ *2-periodic trigonometric interpolation problem to be regular.*

It is not yet known if condition (2.1) is, by itself, *both* necessary and sufficient for the regularity of this trigonometric interpolation problem when N is odd, though we tend to believe that it *is*. Indeed, with N odd and with (2.1) holding, there are examples showing that the trigonometric interpolation problem is regular, *without* (2.2) holding. Explicit examples of this will be given in section 4.

We continue with the case when N of (1.6) is even. Here, there are three cases to be dealt with:

$$\begin{cases} i) & n \text{ is even } (n =: 2r), \text{ and } p+q \text{ is odd } (p+q =: 2s+1), \\ ii) & n \text{ is even } (n =: 2r), \text{ and } p+q \text{ is even } (p+q =: 2s+2), \\ iii) & n \text{ is odd } (n =: 2r+1), \text{ and } p+q \text{ is even } (p+q =: 2s+2). \end{cases} \tag{2.3}$$

Because N is now even, the sought interpolant $t_M(x)$ is necessarily of the form (1.4′), and $\varepsilon(= 0$ or $1)$ must also be appropriately chosen in each case of (2.3). We also use the notation $[[x]]$ to denote the integer part of the real number x.

THEOREM 2. *i) Let N be even, with* (cf. (2.3*i*)) $n =: 2r$ *even and* $p + q =: 2s + 1$ *odd. Then, a necessary condition for the* $(\mathrm{m}; \mathrm{m}')$ *2-periodic trigonometric interpolation problem* (1.5) *to be regular is that*

$$E + E' - 1 = s = O + O'. \tag{2.4i}$$

Moreover, if the alternation condition (2.2) *holds, then* (2.4*i*) *is both necessary and sufficient for this* $(\mathrm{m}; \mathrm{m}')$ *2-periodic trigonometric interpolation problem to be regular, with* ε *of* (1.4′) *being given by*

$$\varepsilon = \begin{cases} 0, & if\ p\ is\ odd; \\ 1, & if\ p\ is\ even. \end{cases} \tag{2.5i}$$

ii) Let N be even, with (cf. (2.3*ii*)) $n =: 2r$ *even and* $p + q =: 2s + 2$ *even. Then, a necessary condition for the* $(\mathrm{m}; \mathrm{m}')$ *2-periodic trigonometric interpolation problem* (1.5) *to be regular is that*

$$E = [[(p+1)/2]], \text{ and } E' = [[(q+1)/2]]. \tag{2.4ii}$$

Moreover, if the alternation condition (2.2) *holds, then* (2.4*ii*) *is both necessary and sufficient for this* $(\mathrm{m}; \mathrm{m}')$ *2-periodic trigonometric interpolation problem to be regular, with* ε *of* (1.4′) *being given by*

$$\varepsilon = \begin{cases} 0, & if\ p\ and\ q\ are\ both\ odd; \\ 1, & if\ p\ and\ q\ are\ both\ even. \end{cases} \tag{2.5ii}$$

iii) Let N be even, with (cf. (2.3*iii*)) $n =: 2r + 1$ *odd and* $p + q =: 2s + 2$ *even. Then, a necessary condition for the* $(\mathrm{m}; \mathrm{m}')$ *2-periodic trigonometric interpolation problem* (1.5) *to be regular is that*

$$E + E' - (O + O') = 0 \ or\ 2. \tag{2.4iii}$$

Moreover, if the alternation condition (2.2) *holds, then* (2.4*iii*) *is both necessary and sufficient for this* ($\mathbf{m};\mathbf{m}'$) *interpolation problem to be regular, with* ε *of* (1.4′) *being given by*

$$\varepsilon = \begin{cases} 0, & if \quad E+E'=s+2 \quad and \quad O+O'=s; \\ 1, & if \quad E+E'=s+1 \quad = \quad O+O'. \end{cases} \tag{2.5iii}$$

Having stated above our main results, we next outline the remaining portions of this paper. In section 3, we give some useful results on determinants having special forms. Then in section 4, we derive in detail necessary and sufficient *determinantal* conditions for the regularity of this ($\mathbf{m};\mathbf{m}'$) 2-periodic trigonometric interpolation problem when N is odd, from which the necessary condition of (2.1) of Theorem 1 is obtained. The remainder of section 4 gives a detailed treatment of the sufficiency part of Theorem 1 when N is odd. Because the methods used in establishing the results of Theorem 2 (when N is even), are similar to those used in establishing Theorem 1 (when N is odd), only the first part i) of Theorem 2 is given in detail in section 5.

To conclude this section, we now comment on the relationship of Theorem 1 and 2 here to known results in the literature. In [1], Cavaretta, Sharma, and Varga considered what can be analogously called the *1-periodic trigonometric interpolation problem* on the equidistant nodes $\{x_k(n)\}_{k=0}^{n-1}$, with $x_k(n) := 2k\pi/n$, where the single vector

$$\mathbf{m} := (m_1, m_2, \cdots, m_q) \tag{2.6}$$

has nonnegative integer components with $0 =: m_1 < m_2 < \cdots < m_q$. Here, n is either an odd or even positive integer, and for arbitrary complex numbers $\{\alpha_{k\nu}\}_{k=0,\nu=0}^{n-1,q}$, the associated 1-periodic lacunary interpolation problem was defined by

$$t_M^{m_\nu}(x_k) = \alpha_{k,\nu} \qquad (k=0,1,\cdots,n-1;\ \nu=0,1,\cdots,q), \tag{2.7}$$

where $t_M(x)$ is a trigonometric polynomial of the appropriate form (1.4) or (1.4′). In [1], necessary and sufficient conditions, for the interpolation problem (2.7) to be regular, were derived, and remarkably, these conditions depend only on the number of odd and even integers in $\mathbf{m}$. (For a recent survey of lacunary trigonometric interpolation, see Chapter 11 of Lorentz, Jetter, and Riemenschneider [4].)

Curiously, if $\mathbf{m} = \mathbf{m}'$ in (1.2), one would expect that this would cover completely the case treated in [1]. This, however, is not the case because the number of nodes in (1.1) is always even, namely $2n$, whereas the case treated in [1] allows the total number of nodes to be either odd or even. In this sense, the results in this paper can be viewed as being *complementary* to those of [1]. In addition, the necessary and sufficient conditions of Theorems 1 and 2 of this paper require the alternation condition of (2.2), a condition which doesn't appear in [1].

3. Some Lemmas on Determinants.

In subsequent sections, we shall need results for deciding whether certain determinants are nonzero. For this purpose, we collect in this section some elementary but useful results on determinants. We begin with the following well known result for generalized Vandermonde determinants.

LEMMA 1. *Let $\{t_j\}_{j=1}^{s}$ be positive real numbers with $0 < t_1 < t_2 < \cdots < t_s$, and let $\{\mu_j\}_{j=1}^{s}$ be nonnegative integers with $0 \le \mu_1 < \mu_2 < \cdots < \mu_s$. Then, the determinant D of order s, defined by*

$$D := \begin{vmatrix} t_1^{\mu_1} & t_2^{\mu_1} & \cdots & t_s^{\mu_1} \\ t_1^{\mu_2} & t_2^{\mu_2} & & t_s^{\mu_2} \\ \vdots & & & \vdots \\ t_1^{\mu_s} & t_2^{\mu_s} & \cdots & t_s^{\mu_s} \end{vmatrix}, \tag{3.1}$$

satisfies $D > 0$. Similarly, if the positive real numbers $\{t_j\}_{j=1}^{s}$ in (3.1) are strictly decreasing, then $D \neq 0$ with

$$sgn D = (-1)^{s(s-1)/2}. \tag{3.2}$$

Proof. The first part of Lemma 1 is of course the well known result for generalized Vandermonde determinants (cf. Gantmacher [2, p. 99]). For the second part of Lemma 1, assume $\{t_j\}_{j=1}^{s}$ is a strictly decreasing sequence of positive real numbers. On interchanging the columns of D of (3.1) so as to achieve a rearrangement of $\{t_j\}_{j=1}^s$ which is a strictly increasing sequence of positive real numbers, then part 1 of Lemma gives that this new determinant is nonzero, and, on counting the number of column interchanges necessary to achieve this rearrangement, it follows that (3.2) is valid. □

The next result is similar to the result of Lemma 2 of Cavaretta, Sharma, and Varga [1].

LEMMA 2. *Let* $\{t_j\}_{j=1}^s$ *be positive real numbers with* $0 < t_1 < t_2 < \cdots < t_s$, *and let* $\{m_j\}_{j=1}^p$ *and* $\{m'_j\}_{j=1}^q$ *be any two sets of nonnegative integers with* $0 \le m_1 < m_2 < \cdots < m_p$ *and* $0 \le m'_1 < m'_2 < \cdots < m'_q$, *where* $p + q = s > 0$. *Then, the determinant* D *of order* s, *defined by*

$$D := \begin{vmatrix} t_1^{m_1} & t_2^{m_1} & \cdots & t_s^{m_1} \\ \vdots & & & \vdots \\ t_1^{m_p} & t_2^{m_p} & \cdots & t_s^{m_p} \\ t_1^{m'_1} & -t_2^{m'_1} & \cdots & (-1)^{s-1} t_s^{m'_1} \\ \vdots & & & \vdots \\ t_1^{m'_q} & -t_2^{m'_q} & \cdots & (-1)^{s-1} t_s^{m'_q} \end{vmatrix}, \tag{3.3}$$

satisfies $D \neq 0$ *with*

$$sgn\ D = (-1)^{q(s+p-1)/2}. \tag{3.4}$$

Proof. The Laplace expansion of D of (3.3) about its first p rows, is given (cf. Karlin [3, p. 6]) by

$$\sum_{1 \le j_1 < \cdots < j_p \le s} (-1)^{(1+2+\cdots+p)+(j_1+j_2+\cdots+j_p)}\, D_1 \begin{pmatrix} 1, & 2, & \cdots, & p \\ j_1, & j_2, & \cdots, & j_p \end{pmatrix} \cdot D_2 \begin{pmatrix} p+1, & p+2, & \cdots, & p+q \\ j'_1, & j'_2, & \cdots, & j'_q \end{pmatrix}, \tag{3.5}$$

where $D_1 \begin{pmatrix} 1, & 2, & \cdots, & p \\ j_1, & j_2, & \cdots, & j_p \end{pmatrix}$ is the determinant formed from the rows $1, 2, \cdots, p$ and columns $j_1, j_2, \cdots, j_p$ of D (with $1 \le j_1 < j_2 < \cdots < j_p \le s$), and $D_2 \begin{pmatrix} p+1, & p+2, & \cdots, & p+q \\ j'_1, & j'_2, & \cdots, & j'_q \end{pmatrix}$

is the determinant formed from the complementary rows $p+1, p+2, \cdots, p+q$ and complementary columns $j_1', j_2', \cdots, j_q'$ of D (with $1 \le j_1' < j_2' < \cdots < j_q' \le s$). From part 1 of Lemma 1, $sgn\ D_1 \begin{pmatrix} 1, & 2, & \cdots, & p \\ j_1, & j_2, & \cdots, & j_p \end{pmatrix} = 1$, and, on factoring out $(-1)^{j_k'-1}$ from each k-th column of $D_2 \begin{pmatrix} p+1, & p+2, & \cdots & p+q \\ j_1', & j_2', & \cdots & j_q' \end{pmatrix}$, part 1 of Lemma 1 similarly gives that

$$sgn\ D_2 \begin{pmatrix} p+1, & p+2, & \cdots, & p+q \\ j_1', & j_2', & \cdots, & j_q' \end{pmatrix} = (-1)^{j_1'+j_2'+\cdots+j_q'-q}.$$

It then follows that each term of the sum (3.5) is nonzero with the *same* sign, this sign being given in (3.4). $\square$

We remark that Lemma 2 is also valid when the set $\{m_j'\}_{j=1}^q$ is empty (i.e., $q=0$) since Lemma 2 reduces to the first part of Lemma 1 in this case. Similarly, Lemma 2 is also valid when the set $\{m_j\}_{j=1}^p$ is empty (i.e., $p=0$).

LEMMA 3. *Let $\{t_j\}_{j=1}^k$ be positive real numbers with $t_1 > t_2 > \cdots > t_k > 0$, let $\{u_j\}_{j=1}^\ell$ be positive real numbers with $0 < u_1 < u_2 < \cdots < u_\ell$, and let $k + \ell =: s > 0$. If $\{\mu_j\}_{j=0}^s$ are nonnegative integers with $0 \le \mu_1 < \mu_2 < \cdots < \mu_s$, then the determinant D_s of order s, defined by*

$$D_s := \begin{vmatrix} t_1^{\mu_1} & t_2^{\mu_1} & \cdots & t_k^{\mu_1} & u_1^{\mu_1} & u_2^{\mu_1} & \cdots & u_\ell^{\mu_1} \\ t_1^{\mu_2} & t_2^{\mu_2} & \cdots & t_k^{\mu_2} & -u_1^{\mu_2} & -u_2^{\mu_2} & \cdots & -u_\ell^{\mu_2} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ t_1^{\mu_s} & t_2^{\mu_s} & \cdots & t_k^{\mu_s} & \sigma u_1^{\mu_s} & \sigma u_2^{\mu_s} & \cdots & \sigma u_\ell^{\mu_s} \end{vmatrix} \qquad (\sigma := (-1)^{s-1}), \tag{3.6}$$

satisfies $D_s \neq 0$ with

$$sgn\ D_s = (-1)^{s(s-1)/2}. \tag{3.7}$$

Under the same hypotheses, if

$$\tag{3.8} \tilde{D}_s := \begin{vmatrix} t_1^{\mu_1} & t_2^{\mu_1} & \cdots & t_k^{\mu_1} & u_1^{\mu_1} & u_2^{\mu_1} & \cdots & u_\ell^{\mu_1} \\ -t_1^{\mu_2} & -t_2^{\mu_2} & & -t_k^{\mu_2} & u_1^{\mu_2} & u_2^{\mu_2} & & u_\ell^{\mu_2} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ \sigma t_1^{\mu_s} & \sigma t_2^{\mu_s} & \cdots & \sigma t_k^{\mu_s} & u_1^{\mu_s} & u_2^{\mu_s} & \cdots & u_\ell^{\mu_s} \end{vmatrix} \qquad (\sigma := (-1)^{s-1}),$$

then

$$\tag{3.9} \tilde{D}_s > 0.$$

Proof. Here, we use the Laplace expansion of D_s about its first k columns. Then as in (3.5),

$$\tag{3.10} D_s = \sum_{1 \le i_1 < \cdots < i_k \le s} (-1)^{1+2\cdots+k+\sum_{\nu=1}^{k} i_\nu} D_1 \begin{pmatrix} i_1, & i_2, & \cdots, & i_k \\ 1, & 2, & \cdots, & k \end{pmatrix} \cdot D_2 \begin{pmatrix} i'_1, & i'_2, & \cdots, & i'_\ell \\ k+1, & k+2, & \cdots, & k+\ell \end{pmatrix}.$$

From the second part of Lemma 1, it follows from (3.2) and the hypotheses on $\{t_j\}_{j=1}^k$ that

$$sgn\ D_1 \begin{pmatrix} i_1, & i_2, & \cdots, & i_k \\ 1, & 2, & \cdots, & k \end{pmatrix} = (-1)^{\frac{k(k-1)}{2}},$$

while, on factoring out $(-1)^{i'_\nu - 1}$ from each ν-th row of D_2, it similarly follows from the first part of Lemma 1 that

$$sgn\ D_2 \begin{pmatrix} i'_1, & i'_2, & \cdots, & i'_\ell \\ k+1, & k+2, & \cdots, & k+\ell \end{pmatrix} = (-1)^{\sum_{\nu=1}^{\ell}(i'_\nu - 1)}.$$

Thus, each term of the sum (3.10) is nonzero with the same sign, this sign being given in (3.7). Finally, for the determinant $\tilde{D}_s$ of (3.8), simply factoring out $(-1)^{j-1}$ from each jth row $(j = 1, 2, \cdots, s)$, exactly gives the determinant D_s of (3.6). Thus, we have $\tilde{D}_s = (-1)^{s(s-1)/2} D_s$, and with (3.7), it follows that $\tilde{D}_s > 0$. □

As in the remark following Lemma 2, we note that both parts of Lemma 3 are valid when either the set $\{u_\ell\}_{j=1}^{\ell}$ is empty (i.e., $\ell = 0$) or when the set $\{t_j\}_{j=1}^{k}$ is empty (i.e., $k = 0$).

This will be used in the proof of Theorem 1 in section 4.

4. Necessary and Sufficient Conditions for Regularity: N Odd.

With the definition of (1.6), we assume here that $N = n(p+q)$ is odd, so that n and $p+q$ are also both odd, and we write

$$n =: 2r + 1 \text{ and } p + q =: 2s + 1. \tag{4.1}$$

In this case, we have (cf. (1.7))

$$M = \frac{N-1}{2} = ns + r, \tag{4.2}$$

and the sought trigonometric polynomials must be of the form (1.4).

The derivation of necessary and sufficient conditions for a *unique* trigonometric interpolant in (1.5) is based on the elementary notion that the interpolation of null data is unique if and only if certain associated determinants are *nonzero*. To begin, we use the familiar device of identifying (in a $1-1$ way) trigonometric polynomials (in the real variable x) with algebraic polynomials (in the complex variable z), through the transformation $z = e^{ix}$. Then, any trigonometric polynomial of the form (1.4) can be expressed as

$$t_M(x) = z^{-M} q_{2M}(z) \qquad (q_{2M} \in \pi_{2M}), \tag{4.3}$$

where π_k denotes the set of all complex algebraic polynomials of degree at most k. Thus, with (4.1) and (4.2), we can express (4.3) as

$$t_M(x) = z^{-M} \sum_{\lambda=0}^{2s} z^{\lambda n} Q_\lambda(z), \tag{4.4}$$

where

$$Q_\lambda(z) := \sum_{j=0}^{n-1} a_{\lambda,j} z^j \qquad (\lambda = 0, 1, \cdots, 2s).$$

Then, since $z = e^{ix}$ implies $d/dx = iz d/dz$, the interpolation conditions of (1.5), applied to

(4.4) for null data, reduce, respectively with $z_{2k} := e^{ix_{2k}}$ and $z_{2k+1} := e^{ix_{2k+1}}$, to

$$\tag{4.5}\begin{cases} \sum_{j=0}^{n-1} z_{2k}^j \left[\sum_{\lambda=0}^{2s} a_{\lambda,j} (j+\lambda n - M)^{m_\nu} \right] = 0, & (\nu = 1,2,\cdots,p;\ k = 0,1,\cdots,n-1), \\ \sum_{j=0}^{n-1} z_{2k+1}^j \left[\sum_{\lambda=0}^{2s} (-1)^\lambda a_{\lambda,j} (j+\lambda n - M)^{m'_\nu} \right] = 0, & (\nu = 1,2,\cdots,q;\ k = 0,1,\cdots,n-1), \end{cases}$$

where we have used the facts (cf. (1.1)) that $z_{2k}^n = 1$ and $z_{2k+1}^n = -1$. Next, if we define the quantity in square brackets in the first display of (4.5) to be B_j and if we set $p_{n-1}(z) := \sum_{j=0}^{n-1} B_j z^j$, then (4.5) implies that $p_{n-1}(z_{2k}) = 0$ for $k = 0,1,\cdots,n-1$. But as the n associated nodes $\{z_{2k}\}_{k=0}^{n-1}$ are all distinct, then $p_{n-1}(z) \equiv 0$; whence, $B_j = 0$ for all $j = 0,1,\cdots,n-1$. As the same argument applies to the second display of (4.5), we thus have

$$\tag{4.6}\begin{cases} \sum_{\lambda=0}^{2s} a_{\lambda,j} (j+\lambda n - M)^{m_\nu} = 0, & (\nu = 1,2,\cdots,p;\ j = 0,1,\cdots,n-1), \\ \sum_{\lambda=0}^{2s} (-1)^\lambda a_{\lambda,j} (j+\lambda n - M)^{m'_\nu} = 0, & (\nu = 1,2,\cdots,q;\ j = 0,1,\cdots,n-1). \end{cases}$$

Recalling from (4.2) that $M = ns + r$ and from (4.1) that $n = 2r+1$, then on dividing out a factor of n^{m_ν} and $n^{m'_\nu}$, respectively, from each of the two equations of (4.6), and on setting

$$\tag{4.7} \alpha_j := \frac{j-r}{2r+1} \qquad (j = 0,1,\cdots,2r),$$

it follows from (4.6) that, for *each* $j = 0,1,\cdots,2r$,

$$\tag{4.8}\begin{cases} \sum_{\lambda=0}^{2s} a_{\lambda,j} (\alpha_j + \lambda - s)^{m_\nu} = 0 & (\nu = 1,2,\cdots,p), \\ \sum_{\lambda=0}^{2s} (-1)^\lambda a_{\lambda,j} (\alpha_j + \lambda - s)^{m'_\nu} = 0 & (\nu = 1,2,\cdots,q). \end{cases}$$

But for each choice of $j = 0,1,\cdots,2r$, the above equations represent $2s+1$ homogeneous linear equations in the $2s+1$ variables $\{a_{\lambda,j}\}_{\lambda=0}^{2s}$. If $\Delta(\alpha_j)$ denotes the determinant of order $2s+1$ of the coefficients of $\{a_{\lambda,j}\}_{\lambda=0}^{2s}$ in (4.8), then $\Delta(\alpha_j) \neq 0$ for *each* $j = 0,1,\cdots,n-1$ if and only if $t_M(x) \equiv 0$ in (4.3). Consequently, we immediately have the result of

PROPOSITION 1. *Let $N = n(p+q)$ be odd, so that n is odd (with $n =: 2r+1$) and $p+q$ is odd (with $p+q =: 2s+1$). Then, in order for the* $(\mathrm{m};\mathrm{m}')$ *2-periodic trigonometric interpolation problem (1.5) to be regular on the $2n$ equidistant nodes $\{x_k\}_{k=0}^{2n-1}$ in $[0,2\pi)$, it is necessary and sufficient that*

$$\Delta(\alpha_j) \neq 0 \qquad \left(\alpha_j := \frac{j-r}{2r+1};\ j = 0,1,\cdots,2r\right), \tag{4.9}$$

where $\Delta(\alpha_j)$ denotes the determinant of order $2s+1$ of the coefficients of $\{a_{\lambda,j}\}_{\lambda=0}^{2s}$ in (4.8).

We remark from (4.7) that the α_j's of (4.7) all satisfy

$$|\alpha_j| < \frac{1}{2} \qquad (j = 0,1,\cdots,2r). \tag{4.10}$$

Next, for convenience, let $\Delta(\alpha)$ denote the determinant (of order $2s+1$) of the coefficients of $\{a_{\lambda,j}\}_{\lambda=0}^{2s}$ in (4.8) when α_j is replaced by the variable α. Then, this determinant $\Delta(\alpha)$ can be expressed in usual row-column notation as

$$\Delta(\alpha) = |c_{i,j}(\alpha)| \qquad (1 \le i,j \le 2s+1), \tag{4.11}$$

where, from (4.8),

$$c_{i,j}(\alpha) := \begin{cases} (\alpha+j-1-s)^{m_i} & (1 \le i \le p;\, 1 \le j \le 2s+1), \\ \\ (-1)^{j-1}(\alpha+j-1-s)^{m'_{i-p}} & (p+1 \le i \le p+q = 2s+1;\, 1 \le j \le 2s+1). \end{cases} \tag{4.12}$$

Thus, the determinant $\Delta(\alpha)$ is explicitly given by

$$\Delta(\alpha) = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ (\alpha-s)^{m_2} & (\alpha+1-s)^{m_2} & & \alpha^{m_2} & (\alpha+1)^{m_2} & & (\alpha+s)^{m_2} \\ \vdots & & & & & & \vdots \\ (\alpha-s)^{m_p} & (\alpha+1-s)^{m_p} & \cdots & \alpha^{m_p} & (\alpha+1)^{m_p} & \cdots & (\alpha+s)^{m_p} \\ (\alpha-s)^{m'_1} & -(\alpha+1-s)^{m'_1} & \cdots & (-1)^s\alpha^{m'_1} & (-1)^{s-1}(\alpha+1)^{m'_1} & \cdots & (\alpha+s)^{m'_1} \\ \vdots & & & & & & \vdots \\ (\alpha-s)^{m'_q} & -(\alpha+1-s)^{m'_q} & \cdots & (-1)^s\alpha^{m'_q} & (-1)^{s-1}(\alpha+1)^{m'_q} & \cdots & (\alpha+s)^{m'_q} \end{vmatrix}, \tag{4.13}$$

where the first p rows all have entries with positive signs, while the last q rows have entries which are alternately positive and negative in sign.

Next, we derive a useful general relationship between $\Delta(\alpha)$ and $\Delta(-\alpha)$. Starting with $\Delta(-\alpha)$ from (4.13), interchange its first and last $(2s+1)$st columns, its second and the $2s$th columns, etc., a total of s interchanges. This gives from (4.13) that

$$\Delta(-\alpha) = (-1)^s \begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ (-\alpha+s)^{m_2} & & (-\alpha+1)^{m_2} & (-\alpha)^{m_2} & & (-\alpha-s)^{m_2} \\ \vdots & & & & & \vdots \\ (-\alpha+s)^{m_p} & & (-\alpha+1)^{m_p} & (-\alpha)^{m_p} & & (-\alpha-s)^{m_p} \\ (-\alpha+s)^{m'_1} & & (-1)^{s-1}(-\alpha+1)^{m'_1} & (-1)^s(-\alpha)^{m'_1} & & (-\alpha-s)^{m'_1} \\ \vdots & & & & & \vdots \\ (-\alpha+s)^{m'_q} & \cdots & (-1)^{s-1}(-\alpha+1)^{m'_q} & (-1)^s(-\alpha)^{m'_q} & \cdots & (-\alpha-s)^{m'_q} \end{vmatrix}.$$

If m_j is even, the j-th row of the determinant above is the same as the j-th row of $\Delta(\alpha)$ of (4.13), the same being true for the appropriate rows of these determinants if m'_j is even. But if m_j is odd, the j-th row of the determinant above is the *negative* of the j-th of $\Delta(\alpha)$ of (4.13), the analogous property holding when m'_j is odd. On factoring out (-1) from each row where m_j or m'_j is odd, it follows from the definition of (1.8) that

$$\Delta(-\alpha) = (-1)^{s+O+O'}\Delta(\alpha) \qquad (\alpha \text{ arbitrary}). \tag{4.14}$$

It is evident from (4.7) that $\alpha_r = 0$. From Proposition 1, it thus follows that $\Delta(0) \neq 0$ is a *necessary* condition for this (m; m$'$) 2-periodic interpolation problem (1.5) to be regular in the case when N is odd. Assuming that $m'_1 > 0$ (the treatment of the case $m'_1 = 0$ being similar), take the case $\alpha = 0$ of the determinant of (4.13) (of order $2s+1$), and subtract its first column from its last $(2s+1)$-st column, its second column from its $2s$-th column, etc.

This leaves $\Delta(0)$ unchanged and gives

$$(4.15)\qquad \Delta(0) = \begin{vmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ (-s)^{m_2} & (-s+1)^{m_2} & & 0 & 1-(-1)^{m_2} & & s^{m_2}-(-s)^{m_2} \\ \vdots & & & \vdots & & & \vdots \\ (-s)^{m_p} & (-s+1)^{m_p} & \cdots & 0 & 1-(-1)^{m_p} & \cdots & s^{m_p}-(-s)^{m_p} \\ (-s)^{m'_1} & -(-s+1)^{m'_1} & \cdots & 0 & (-1)^{s-1}[1-(-1)^{m'_1}] & \cdots & s^{m'_1}-(-s)^{m'_1} \\ \vdots & & & \vdots & & & \vdots \\ (-s)^{m'_q} & -(-s+1)^{m'_q} & \cdots & 0 & (-1)^{s-1}[1-(-1)^{m'_q}] & \cdots & s^{m'_q}-(-s)^{m'_q} \end{vmatrix}.$$

Note that the last s columns of $\Delta(0)$ in (4.15) have a zero row whenever m_j or m'_j is even. Thus, the last s columns of $\Delta(0)$ have, with the notation of (1.8), $E+E'$ zero rows, while the last $s+1$ columns of $\Delta(0)$ have $E+E'-1$ zero rows. Next, assume that $E+E' \geq s+2$, so that the last $s+1$ columns of $\Delta(0)$ have at least $s+1$ zero rows. Now, the Laplace expansion of $\Delta(0)$, in terms of its first s columns, is given by the sum of (3.10) with $k := s$ and $\ell := s+1$. But, the fact that the last $s+1$ columns of $\Delta(0)$ have at least $s+1$ zero rows implies that the final determinant $D_2\begin{pmatrix} i'_1, & i'_2, & \cdots, & i'_{s+1} \\ s+1, & s+2, & \cdots, & 2s+1 \end{pmatrix}$ of (3.10) must vanish, since it has at least one zero row, for *any* choice of $1 \leq i'_1 < i'_2 < \cdots < i'_{s+1} \leq 2s+1$. Hence, $E+E' \geq s+2$ implies $\Delta(0) = 0$. Thus, a necessary condition for regularity is that

$$E+E' \leq s+1.$$

In a similar fashion (on adding appropriate columns of $\Delta(0)$ in (4.15)), we deduce that a necessary condition for regularity is that (cf. (1.8))

$$O+O' \leq s.$$

But since $(E+E')+(O+O') = p+q = 2s+1$ from (1.9) and (4.1), it must follow from the above two inequalities that

$$(4.16)\qquad E+E' = s+1 \text{ and } O+O' = s,$$

which establishes the necessary condition (2.1) of Theorem 1.

We remark that (4.16) is a necessary condition for the regularity of this $(\mathbf{m}; \mathbf{m}')$ 2-periodic interpolation problem when N is odd. In fact, we now show that (4.16) *implies* $\Delta(0) \neq 0$ when N is odd. Because the derivatives in the interpolation conditions of (1.5) can be applied in any order, we may assume (without loss of generality) that

$$(4.17) \qquad \begin{cases} 0 = m_1, m_2, \cdots, m_\ell \text{ are even, and } m_{\ell+1}, m_{\ell+2}, \cdots, m_p \text{ are odd, and} \\ \quad m_1', m_2', \cdots, m_k' \text{ are even, and } m_{k+1}', m_{k+2}', \cdots, m_q' \text{ are odd,} \end{cases}$$

where, from (4.16),

$$(4.18) \qquad \ell + k - 1 = s = p - \ell + q - k.$$

From our previous construction, we know that the last s columns of $\Delta(0)$ of (4.15) have s rows of zeros, corresponding to the even integers $m_2, \cdots, m_\ell, m_1', \cdots, m_k'$. Thus, the Laplace expansion of $\Delta(0)$ of (4.15) about its center column, namely $(1, 0, \cdots, 0)^T$, results in a single determinant, of order $2s$, equal to $\pm\Delta(0)$, which can be expressed (up to a nonzero constant factor) as the product

$$(4.19) \qquad \Delta(0) = D_1 \cdot D_2,$$

where, after elementary column operations, D_1 and D_2 are determinants, each of order s, given by

$$(4.20) \qquad D_1 = \begin{vmatrix} 1 & 2^{m_2} & \cdots & (s-1)^{m_2} & s^{m_2} \\ \vdots & & & & \vdots \\ 1 & 2^{m_\ell} & \cdots & (s-1)^{m_\ell} & s^{m_\ell} \\ (-1)^{s-1} & (-1)^s 2^{m_1'} & \cdots & -(s-1)^{m_1'} & s^{m_1'} \\ \vdots & & & & \vdots \\ (-1)^{s-1} & (-1)^s 2^{m_k'} & \cdots & -(s-1)^{m_k'} & s^{m_k'} \end{vmatrix},$$

and

$$(4.21)\qquad D_2 = \begin{vmatrix} 1 & 2^{m_{\ell+1}} & \cdots & (s-1)^{m_{\ell+1}} & s^{m_{\ell+1}} \\ \vdots & & & & \vdots \\ 1 & 2^{m_p} & \cdots & (s-1)^{m_p} & s^{m_p} \\ (-1)^{s-1} & (-1)^s 2^{m'_{k+1}} & \cdots & -(s-1)^{m'_{k+1}} & s^{m'_{k+1}} \\ \vdots & & & & \vdots \\ (-1)^{s-1} & (-1)^s 2^{m'_q} & \cdots & -(s-1)^{m'_q} & s^{m'_q} \end{vmatrix}.$$

Upon multiplying all entries in the last k rows of D_1 and the last $q-k$ rows of D_2 by $(-1)^{s-1}$, then Lemma 2 directly shows that both D_1 and D_2 are nonzero. Thus (cf. (4.19)), $\Delta(0) \neq 0$. This gives us the following result, which establishes the first part of Theorem 1.

PROPOSITION 2. *Let $N = n(p+q)$ be odd, so that n is odd (with $n =: 2r+1$) and $p+q$ is odd (with $p+q =: 2s+1$). Then, a necessary and sufficient condition that $\Delta(0) \neq 0$ is that* (cf. (1.8))

$$(4.22)\qquad E + E' - 1 = s = O + O'.$$

We now complete the proof of Theorem 1. We assume that (4.22) is valid, and that the components of $\mathbf{m}$ and $\mathbf{m}'$ are, respectively, alternately even and odd integers (cf. (2.2)). To establish that this $(\mathbf{m},\mathbf{m}')$ 2-periodic trigonometric interpolation problem is regular, it is sufficient, from Proposition 1 and (4.10), to show that $\Delta(\alpha) \neq 0$ for any real α with $0 < |\alpha| < 1/2$. But from (4.22) , we have $O + O' = s$, which implies from (4.14) that $\Delta(-\alpha) = \Delta(\alpha)$. Thus, to establish Theorem 1, it suffices to show that

$$(4.23)\qquad \Delta(\alpha) \neq 0 \text{ for any } 0 < \alpha < 1/2.$$

Continuing, since $p+q = 2s+1$ is odd from (4.1) when N is odd, assume that p is even, i.e., $p = 2\ell$, and that q is odd, i.e., $q = 2k+1$ (the remaining case of p odd and q even being similar). With the alternation hypothesis of (2.2), then

$$(4.24)\qquad \begin{cases} 0 = m_1 < m_3 < \cdots < m_{2\ell-1} & \text{are even integers, and} \\ \quad m_2 < m_4 < \cdots < m_{2\ell} & \text{are odd integers.} \end{cases}$$

Next, with the alternation hypothesis of (2.2), assume that

$$(4.24') \qquad \begin{cases} m_1' < m_3' < \cdots < m_{2k+1}' & \text{are even integers, and} \\ m_2' < m_4' < \cdots < m_{2k}' & \text{are odd integers;} \end{cases}$$

(the remaining case of m_1' odd again being similar). From (4.24) and (4.24′), we have (cf. (1.8)) that $E = \ell$, $O = \ell$, $E' = k+1$, and $O' = k$, so that (2.1) is satisfied.

With the hypotheses of (4.24) and (4.24′), the determinant $\Delta(\alpha)$ of (4.13), of order $2s+1$, can now be brought into a more suitable form. Indeed, if we factor out $(-1)^j$ from each j-th column ($j = 1, 2, \cdots, 2s+1$) of $\Delta(\alpha)$ of (4.13), and if we factor out $(-1)^i$ from each i-th row of the first p rows ($i = 1, 2, \cdots, p$) and change the signs in each of the remaining rows ($i = p+1, p+2, \ldots, p+q$) of the resulting determinant, we obtain the determinant $\hat{\Delta}(\alpha)$, of order $2s+1$, given by

$$\hat{\Delta}(\alpha) := \begin{vmatrix} 1 & -1 & \cdots & (-1)^s & (-1)^{s+1} & \cdots & 1 \\ (s-\alpha)^{m_2} & -(s-1-\alpha)^{m_2} & & (-1)^{s+1}\alpha^{m_2} & (-1)^{s+2}(1+\alpha)^{m_2} & & -(s+\alpha)^{m_2} \\ (s-\alpha)^{m_3} & -(s-1-\alpha)^{m_3} & & (-1)^{s}\alpha^{m_3} & (-1)^{s+1}(1+\alpha)^{m_3} & & (s+\alpha)^{m_3} \\ \vdots & & & \vdots & & & \vdots \\ (s-\alpha)^{m_p} & -(s-1-\alpha)^{m_p} & \cdots & (-1)^{s+1}\alpha^{m_p} & (-1)^{s+2}(1+\alpha)^{m_p} & \cdots & -(s+\alpha)^{m_p} \\ (s-\alpha)^{m_1'} & (s-1-\alpha)^{m_1'} & \cdots & \alpha^{m_1'} & (1+\alpha)^{m_1'} & \cdots & (s+\alpha)^{m_1'} \\ -(s-\alpha)^{m_2'} & -(s-1-\alpha)^{m_2'} & & \alpha_2^{m_2'} & (1+\alpha)^{m_2'} & & (s+\alpha)^{m_2'} \\ \vdots & & & \vdots & & & \vdots \\ (s-\alpha)^{m_q'} & (s-1-\alpha)^{m_q'} & \cdots & \alpha^{m_q'} & (1+\alpha)^{m_q'} & \cdots & (s+\alpha)^{m_q'} \end{vmatrix},$$

(4.25)

where $\hat{\Delta}(\alpha)$ differs from $\Delta(\alpha)$ by a multiplicative factor of ± 1. Thus, we must show that $\hat{\Delta}(\alpha) \neq 0$ for any α with $0 < \alpha < 1/2$. In fact, we shall show that

$$(4.26) \qquad sgn\hat{\Delta}(\alpha) = 1 \text{ for any } \alpha \text{ with } 0 < \alpha < 1/2.$$

Applying the Laplace expansion of $\hat{\Delta}(\alpha)$ in terms of its first p rows, then

$$\hat{\Delta}(\alpha) = \sum_{1\le j_1<\cdots<j_p\le 2s+1} (-1)^{\frac{p(p+1)}{2}+\sum_{\nu=1}^{p} j_\nu} D\begin{pmatrix} 1, & 2, & \cdots, & p \\ j_1, & j_2, & \cdots, & j_p \end{pmatrix} D\begin{pmatrix} p+1, & \cdots, & p+q \\ j_1', & \cdots, & j_q' \end{pmatrix}, \tag{4.27}$$

where $(j_1', j_2', \cdots, j_q')$ is complementary to $(j_1, j_2, \cdots, j_p)$ with $1 \le j_1' < j_2' < \cdots < j_q' \le 2s+1$. Now, the determinant $D\begin{pmatrix} p+1, & \cdots, & p+q \\ j_1', & \cdots, & j_q' \end{pmatrix}$ can be seen, from (4.25), to be *exactly* of the form of the determinant in (3.8) of Lemma 3, where k (possibly zero) in (3.8) is the number of indices of $(j_1', \cdots, j_q')$ less than $s+1$. (It is here where the remark after Lemma 3 applies.) Note also that the strict monotonicity requirements, for the positive real numbers t_j's and u_j's in (3.8), follow from (4.25) and the fact that $0 < \alpha < 1/2$. Thus, from (3.9) of Lemma 3,

$$D\begin{pmatrix} p+1, & \cdots, & p+q \\ j_1', & \cdots, & j_q' \end{pmatrix} > 0. \tag{4.28}$$

Next, consider $D\begin{pmatrix} 1, & 2, & \cdots, & p \\ j_1, & j_2, & \cdots & j_p \end{pmatrix}$ of (4.27). On factoring out $(-1)^{j_k-1}$ from the kth column of this determinant (where $k = 1, 2, \ldots, p$), we have

$$D\begin{pmatrix} 1, & 2, & \cdots, & p \\ j_1, & j_2, & \cdots, & j_p \end{pmatrix} = (-1)^{\sum_{k=1}^{p}(j_k-1)}\, D^*\begin{pmatrix} 1, & 2, & \cdots, & p \\ j_1, & j_2, & \cdots, & j_p \end{pmatrix}, \tag{4.29}$$

where the determinant $D^*\begin{pmatrix} 1, & 2, & \cdots, & p \\ j_1, & j_2, & \cdots, & j_p \end{pmatrix}$ is *exactly* of the form of the determinant of (3.6) of Lemma 3, where k (possibly zero) in (3.6) is the number of indices of $(j_1, \ldots, j_p)$ less than $s+1$. Again, the monotonicity requirements for applying (3.7) of Lemma 3 are fulfilled, and so from (3.7),

$$sgn D^*\begin{pmatrix} 1, & 2, & \cdots, & p \\ j_1, & j_2, & \cdots, & j_p \end{pmatrix} = (-1)^{p(p-1)/2}. \tag{4.30}$$

Hence, with (4.28)-(4.30), it follows that each term of the sum in (4.27) has the same sign, given by

$$sgn\hat{\Delta}(\alpha) = (-1)^{\frac{p(p+1)}{2}+2\sum_{\nu=1}^{p} j_\nu - p + \frac{p(p-1)}{2}} = 1 \qquad (0 < \alpha < 1/2),$$

which establishes (4.26) and Theorem 1. □

We conclude this section with examples showing that regularity, in this case when N is odd, *can* hold with (2.1) but without the alternation condition (2.2) holding. Specifically, assume again that $N = n(p+q)$ is odd, so that n is odd and $p+q := 2s+1$ is odd. Choose $s = 1$, and set

$$\mathbf{m} := (0, 2\ell) \text{ and } \mathbf{m}' := (2k+1) \qquad (\text{any } \ell \geq 1, \text{ any } k \geq 0). \tag{4.31}$$

Now, (2.1) is satisfied (with $E + E' - 1 = 1 = O + O'$), but the alternation condition of (2.2) clearly does not hold. We shall show that this $(\mathbf{m}, \mathbf{m}')$ 2-periodic trigonometric interpolation problem defined by (4.31) is however *regular*. In this case, $\Delta(\alpha)$ is given by (cf. (4.13))

$$-\Delta(\alpha) = \begin{vmatrix} 1 & 1 & 1 \\ (1-\alpha)^{2\ell} & \alpha^{2\ell} & (1+\alpha)^{2\ell} \\ (1-\alpha)^{2k+1} & \alpha^{2k+1} & -(1+\alpha)^{2k+1} \end{vmatrix}, \tag{4.32}$$

where, from (4.14), $\Delta(-\alpha) = \Delta(\alpha)$. Moreover, it is readily seen, on expanding $\Delta(\alpha)$, that

$$-\Delta(\alpha) = \left[(1-\alpha)^{2\ell} - \alpha^{2\ell}\right]\left[(1+\alpha)^{2k+1} + \alpha^{2k+1}\right] + \left[(1-\alpha)^{2k+1} - \alpha^{2k+1}\right]\left[(1+\alpha)^{2\ell} - \alpha^{2\ell}\right] . \tag{4.33}$$

But for any $\ell \geq 1$, $k \geq 0$, and α with $0 \leq \alpha < 1/2$, the quantities in brackets are all *positive*, and this establishes that these $((0, 2\ell); (2k+1))$ 2-periodic interpolation problems are regular.

5. Necessary and Sufficient Conditions for Regularity: N Even.

If (cf. (1.6)) $N = n(p+q)$ is even, then one of the three conditions of (2.3) necessarily holds, and we first consider the case (cf. (2.3*i*)) when

$$n \text{ is even } (n =: 2r), \text{ and } p+q \text{ is odd } (p+q =: 2s+1), \tag{5.1}$$

so that (cf. (1.7))

$$M = N/2 = ns + r. \tag{5.2}$$

Since N is even, we necessarily consider trigonometric interpolants of the form (1.4′), where $\varepsilon(= 0 \text{ or } 1)$ in (1.4′) is to be appropriately chosen. Following the procedure of §4 of using the transformation $z = e^{ix}$, we can similarly write any $t_M(x)$ of the form (1.4′) as

$$t_M(x) = z^{-M}\left\{\sum_{\lambda=0}^{2s} z^{n\lambda}Q_\lambda(z) + cz^{(2s+1)n}\right\}, \tag{5.3}$$

where

$$Q_\lambda(z) := \sum_{j=0}^{n-1} a_{\lambda,j}z^j \qquad (\lambda = 0, 1, \cdots, 2s),$$

and where, because of (1.4′), we have

$$a_{0,0} + c(-1)^{\varepsilon+1} = 0. \tag{5.4}$$

With

$$\alpha_j := \frac{j-r}{2r} \qquad (j = 0, 1, \cdots, 2r), \tag{5.5}$$

we obtain, exactly as in (4.8) in the interpolation of null data, the following homogeneous $2s + 1$ linear equations

$$\begin{cases} \displaystyle\sum_{\lambda=0}^{2s} a_{\lambda,j}(\alpha_j + \lambda - s)^{m_\nu} = 0 & (\nu = 1, 2, \cdots, p), \\ \\ \displaystyle\sum_{\lambda=0}^{2s} (-1)^\lambda a_{\lambda,j}(\alpha_j + \lambda - s)^{m'_\nu} = 0 & (\nu = 1, 2, \cdots, q), \end{cases} \tag{5.6}$$

in the $2s + 1$ variables $\{a_{\lambda,j}\}_{\lambda=0}^{2s}$, for *each* $j = 1, 2 \cdots, n-1$. The departure from the case treated in §4 comes in the case $j = 0$ which involves the unknown constant c of (5.3) For this case $j = 0$, the associated homogeneous $2s + 2$ linear equations for the unknowns c and $\{a_{\lambda,0}\}_{\lambda=0}^{2s}$ are given by (5.4) and

$$\begin{cases} \displaystyle\sum_{\lambda=0}^{2s} a_{\lambda,0}(\alpha_0 + \lambda - s)^{m_\nu} + c(\alpha_0 + s + 1)^{m_\nu} = 0 & (\nu = 1, 2, \cdots, p), \\ \displaystyle\sum_{\lambda=0}^{2s} (-1)^\lambda a_{\lambda,0}(\alpha_0 + \lambda - s)^{m'_\nu} - c(\alpha_0 + s + 1)^{m'_\nu} = 0 & (\nu = 1, 2, \cdots, q). \end{cases} \tag{5.7}$$

Clearly, the system of equations (5.6) is the same as those in (4.8), except that here we have in (5.5) that $\alpha_j = \frac{j-r}{2r}$, whereas in (4.8), $\alpha_j = \frac{j-r}{2r+1}$. If, as in §4, $\Delta(\alpha_j)$ denotes the determinant, of order $2s+1$, of the homogeneous system (5.6) for $j = 1, 2, \cdots, n-1$, and if Δ_0 denotes the determinant, of order $2s+2$, of the homogeneous system (5.7) and (5.4) for $j = 0$, then in complete analogy with Proposition 1 of §4, we have

PROPOSITION 3. *Let $N = n(p+q)$ be even with $n =: 2r$ even and $p+q =: 2s+1$ odd. Then, in order for the $(\mathbf{m};\mathbf{m}')$ 2-periodic trigonometric interpolation problem (1.5) to be regular on the $2n$ equidistant nodes $\{x_k\}_{k=0}^{2n-1}$ in $[0, 2\pi)$, it is necessary and sufficient that*

$$\Delta(\alpha_j) \neq 0 \qquad \left(\alpha_j = \frac{j-r}{2r},\ j = 1, 2, \cdots, 2r\right), \tag{5.8}$$

and that

$$\Delta_0 \neq 0 \qquad \left(\alpha_0 = -\frac{1}{2},\ j = 0\right). \tag{5.9}$$

Next, since $\alpha_r = 0$ from (5.8) and since $\Delta(0)$ in this section is the same as that in §4 (except for the value of n), we immediately obtain the results of (4.14) and (4.16), which directly give the first part of the following analogue of Proposition 2 of section 4.

PROPOSITION 4. *Let $N = n(p+q)$ be even with $n =: 2r$ even and $p+q =: 2s+1$ odd. Then, a necessary and sufficient condition that $\Delta(0) \neq 0$ is that*

$$E + E' - 1 = s = O + O'. \tag{5.10}$$

Moreover, a necessary and sufficient condition that $\Delta_0 \neq 0$ is that either

$$\begin{cases} E + O' = s+1 \quad \text{and} \quad E' + O = s \quad \text{with} \quad \varepsilon = 0, \text{ or} \\ E + O' = s \quad \text{and} \quad E' + O = s+1 \quad \text{with} \quad \varepsilon = 1. \end{cases} \tag{5.11}$$

For the proof of the last part of Proposition 4, recalling from (5.5) that $\alpha_0 = -1/2$, we may assume (without loss of generality) that the components of $\mathbf{m}$ and $\mathbf{m}'$ are ordered as in (4.17). Then, up to a nonzero multiplative factor, the determinant Δ_0 from (5.4) and (5.7), of order $2s+2$, becomes

$$
(5.12)\quad \Delta_0 = \left|\begin{array}{cccccccc}
1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\
(2s+1)^{m_2} & & 3^{m_2} & 1 & 1 & 3^{m_2} & & (2s+1)^{m_2} \\
\vdots & & & \vdots & \vdots & & & \vdots \\
(2s+1)^{m_\ell} & \cdots & 3^{m_\ell} & 1 & 1 & 3^{m_\ell} & \cdots & (2s+1)^{m_\ell} \\
-(2s+1)^{m_{\ell+1}} & \cdots & -3^{m_{\ell+1}} & -1 & 1 & 3^{m_{\ell+1}} & \cdots & (2s+1)^{m_{\ell+1}} \\
\vdots & & & \vdots & \vdots & & & \vdots \\
-(2s+1)^{m_p} & \cdots & -3^{m_p} & -1 & 1 & 3^{m_p} & \cdots & (2s+1)^{m_p} \\
-(2s+1)^{m'_1} & \cdots & (-1)^s 3^{m'_1} & (-1)^{s+1} & (-1)^s & (-1)^{s+1} 3^{m'_1} & \cdots & (2s+1)^{m'_1} \\
\vdots & & & \vdots & \vdots & & & \vdots \\
-(2s+1)^{m'_k} & \cdots & (-1)^s 3^{m'_k} & (-1)^{s+1} & (-1)^s & (-1)^{s+1} 3^{m'_k} & \cdots & (2s+1)^{m'_k} \\
(2s+1)^{m'_{k+1}} & \cdots & (-1)^{s+1} 3^{m'_{k+1}} & (-1)^s & (-1)^s & (-1)^{s+1} 3^{m'_{k+1}} & \cdots & (2s+1)^{m'_{k+1}} \\
\vdots & & & \vdots & \vdots & & & \vdots \\
(2s+1)^{m'_q} & & (-1)^{s+1} 3^{m'_q} & (-1)^s & (-1)^s & (-1)^{s+1} 3^{m'_q} & & (2s+1)^{m'_q} \\
1 & \cdots & 0 & 0 & 0 & 0 & \cdots & (-1)^{1+\epsilon}
\end{array}\right| .
$$

It is evident that the first ℓ rows, involving the exponents $m_1, \cdots, m_\ell$, and the $q-k$ rows, involving the exponents $m'_{k+1}, \cdots, m'_q$, of the determinant Δ_0 of (5.12) are *symmetric* about its vertical center, while its remaining interior rows are *antisymmetric* about its vertical center. Thus, if we subtract the first column of Δ_0 of (5.12) from its last $(2s+2)$-nd column, its second column from its $(2s+1)$-st column, etc., thereby forming $\hat{\Delta}_0$ (where $\hat{\Delta}_0 = \Delta_0$), then the last $s+1$ columns of this determinant have at least $E+O'$ zero rows. If we assume that

$$E + O' \geq s+2,$$

then applying the Laplace expansion to this $\hat{\Delta}_0$ about its last $s+1$ columns shows that $\hat{\Delta}_0 = 0$. Thus, a necessary condition that $\Delta_0 \neq 0$ is that

$$(5.13)\qquad E + O' \leq s+1,$$

and, in a completely analogous manner, it similarly follows (on adding appropriate columns of Δ_0) that a necessary condition that $\Delta_0 \neq 0$ is also that

$$(5.14)\qquad E' + O \leq s+1.$$

However, since $E + E' + O + O' = 2s + 1$ from (5.1), then equality cannot simultaneously hold in (5.13) and (5.14). Hence, either $E + O' = s + 1$ and $E' + O = s$, or $E + O' = s$ and $E' + O = s + 1$.

If $E + O' = s + 1$ and $E' + O = s$, then the determinant $\hat{\Delta}_0$, formed above by subtracting the first column of Δ_0 from its last column, etc., has the last $s + 1$ entries of its final row given by

$$(0, 0, \cdots, 0, -1 + (-1)^{1+\varepsilon}). \tag{5.15}$$

Clearly, the choice $\varepsilon = 1$ would make a zero row in (5.15) above, which, by the argument above, would result in $\hat{\Delta}_0 = 0$. Thus, in this first case of (5.11), a *necessary* condition that $\hat{\Delta}_0 \neq 0$ is that $\varepsilon = 0$.

We now show that the choice $\varepsilon = 0$, when $E + O' = s + 1$ and $E' + O = s$, is a *sufficient* condition that $\hat{\Delta}_0 \neq 0$. For the determinant $\hat{\Delta}_0$, of order $2s + 2$, constructed above, we have that the last $s + 1$ columns of $\hat{\Delta}_0$ has exactly $s + 1$ zero rows. Thus, in the Laplace expansion of $\hat{\Delta}_0$ in terms of its last $s + 1$ columns, there is at most one term which has to be examined, namely the product to two subdeterminants, each of order $s + 1$, where the first subdeterminant is taken from the last $s + 1$ columns of $\hat{\Delta}_0$ and rows corresponding to the integers of O and E' (where $E' + O = s$) and the final row of $\hat{\Delta}_0$ (which is zero from (5.15) except for its final entry which is -1, since $\varepsilon = 0$), and the second subdeterminant is complementary to the first subdeterminant, corresponding to the first $s + 1$ columns and the rows corresponding to the integers of E and O' (where $E' + O = s + 1$). But on applying Lemma 2, it is readily seen that neither of these subdeterminants is zero; whence, $\hat{\Delta}_0 \neq 0$. This establishes the first necessary and sufficient condition of (5.11). We remark that in the same manner (on adding columns in (5.12)), one similarly obtains the second necessary and sufficient condition of (5.11).

To complete the proof of Theorem in the case (2.3*i*), we note that the proof of §4 shows that the assumptions, that the components of $\mathbf{m}$ and $\mathbf{m}'$ satisfy (5.10) and the alternation condition of (2.2), give that $\Delta(\alpha) \neq 0$ for any $0 < |\alpha| \leq 1/2$. Thus, it remains only to show that (2.5*i*) of Theorem 2 is valid, i.e., $\varepsilon = 1$ if p is even, and $\varepsilon = 0$ if p is odd. First, assume that p is even, say $p := 2\ell$. Then, $q := 2t + 1$ is odd since $p + q = 2s + 1$ is odd. In this case, $s := \ell + t$. Because $p = 2\ell$ is even and because of the alternation condition (2.2), it necessarily follows that $E = \ell = O$. With $q = 2t + 1$, assume that $E' = t + 1$ and if $O' = t$. Then, $E + E' - 1 = s = O + O'$ which satisfies (5.10), and $E + O' = s$ and $E' + O = s + 1$. Thus from (5.11), $\varepsilon = 1$. Again with $q = 2\ell + 1$, assume on the other hand that $E' = t$ and $O' = t + 1$. But then, $E + E' - 1 = s - 1 \neq O + O' = s + 1$, which violates the condition (5.10). Hence, when p is even, we can only satisfy the conditions of (5.10) and (5.11), with the latter case of (5.11) so that $\varepsilon = 1$, which gives the first part of (2.5*i*). The case when p is odd is completely similar and is omitted.

Finally, because the remaining cases *ii*) and *iii*) of Theorem 2 follow along the lines shown above, we leave these verifications to the reader.

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A. Sharma
Department of Mathematics
University of Alberta
Edmonton, Alberta
T6G 2H1
Canada

J. Szabados
Mathematical Institute
Hungarian Academy of Sciences
Budapest, P.O. Box 127
H-1364
Hungary

R.S. Varga
Institute for Computational Mathematics
Kent State University
Kent, Ohio
44242
USA