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ACCELERATION OF RELAXATION METHODS FOR NON-HERMITIAN LINEAR SYSTEMS*

M. EIERMANN†, W. NIETHAMMER†, AND R. S. VARGA‡

Abstract. Let $A = I - B \in \mathbb{C}^{n,n}$, with $\mathrm{diag}(B) = \mathbf{0}$, denote a nonsingular non-Hermitian matrix. To iteratively solve the linear system $A\mathbf{x} = \mathbf{b}$, two splittings of A, together with induced relaxation methods, have been recently investigated in [W. Niethammer and R. S. Varga, *Results in Math.*, 16 (1989), pp. 308–320]. The *Hermitian splitting* of A is defined by $A = M^h - N^h$, where $M^h := (A + A^*)/2$ is the Hermitian part of A. The *skew-Hermitian splitting* of A is similarly defined by $A = M^s - N^s$ with $M^s := I + (A - A^*)/2$.

This paper considers k-step iterative methods to accelerate the relaxation schemes (involving a relaxation factor ω) that are generated by these two splittings. The primary interest is not to determine the optimal relaxation factor ω that minimizes the spectral radius of the associated iteration operator. Rather, a value of ω is sought such that the resulting relaxation method can be most efficiently accelerated by a k-step method. For the Hermitian splitting, the choice $\omega = 1$ (together with a suitable Chebyshev acceleration) turns out to be optimal in this sense. For the skew-Hermitian splitting, a *hybrid* scheme is proposed that is nearly optimal.

As another application of this latter hybrid procedure, the block Jacobi method arising from a model equation for a convection-diffusion problem is analyzed.

Key words. iterative methods for non-Hermitian matrix equations, relaxation methods, Hermitian splittings, skew-Hermitian splittings, Chebyshev acceleration

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1. Introduction.

To solve a nonsingular linear system of algebraic equations

$$A\mathbf{x} = \mathbf{x} - B\mathbf{x} = \mathbf{b} \qquad (A, B \in \mathbb{C}^{n,n},\ \mathrm{diag}(B) = \mathbf{0},\ \mathbf{b} \in \mathbb{C}^n) \tag{1.1}$$

whose coefficient matrix A is *non-Hermitian*, Niethammer and Varga [11] recently studied relaxation methods based on either the Hermitian or the skew-Hermitian splitting of $A = I - B$. Letting

$$F := (B + B^*)/2 \quad \text{and} \quad G := (B - B^*)/2 \tag{1.2}$$

denote, respectively, the Hermitian and skew-Hermitian parts of B, then the *Hermitian splitting* of A is defined by

$$A = M^h - N^h \quad \text{with } M^h := I - F \quad \text{and} \quad N^h := G \tag{1.3}$$

(here, we assume that M^h is invertible, which is, for instance, guaranteed if the Hermitian part M^h of A is positive definite). The naturally associated *skew-Hermitian splitting* of A is given by

$$A = M^s - N^s \quad \text{with } M^s := I - G \quad \text{and} \quad N^s := F\,. \tag{1.4}$$

It should be mentioned that Concus and Golub [2] earlier introduced the Hermitian splitting (1.3) of A. Under the assumption that M^h is positive definite, they chose M^h as a preconditioner for an associated conjugate gradient method.

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† Institut für Praktische Mathematik, Universität Karlsruhe, D-7500 Karlsruhe 1, Germany.
‡ Institute for Computational Mathematics, Kent State University, Kent, Ohio 44242. The research of this author was supported by the Alexander von Humboldt Foundation.

Each splitting $A = M - N$ of A gives rise to a class of one-parameter relaxation schemes for the solution of (1.1), namely,

$$\{(1-\omega)I + \omega M\}\,\mathbf{x}_m := \{(1-\omega)I + \omega N\}\,\mathbf{x}_{m-1} + \omega\mathbf{b}\,, \tag{1.5}$$

or equivalently,

$$\mathbf{x}_m = \mathcal{T}_\omega \mathbf{x}_{m-1} + \mathbf{c}_\omega \qquad (m = 1, 2, \cdots), \tag{1.6}$$

where $\omega \neq 0$ is an arbitrary complex number for which $(1-\omega)I + \omega M$ is nonsingular and

$$\mathcal{T}_\omega := \{(1-\omega)I + \omega M\}^{-1}\{(1-\omega)I + \omega N\}, \qquad \mathbf{c}_\omega := \omega\{(1-\omega)I + \omega M\}^{-1}\mathbf{b}\,.$$

In this way, the specific splittings defined in (1.3) and (1.4) generate the following two relaxation methods:

$$\mathbf{x}_m := \mathcal{T}_\omega^h \mathbf{x}_{m-1} + \mathbf{c}_\omega^h \qquad (m = 1, 2, \cdots), \tag{1.7}$$

where

$$\mathcal{T}_\omega^h := (I - \omega F)^{-1}\{(1-\omega)I + \omega G\}, \qquad \mathbf{c}_\omega^h := \omega(I - \omega F)^{-1}\mathbf{b},$$

and

$$\mathbf{x}_m := \mathcal{T}_\omega^s \mathbf{x}_{m-1} + \mathbf{c}_\omega^s \qquad (m = 1, 2, \cdots), \tag{1.8}$$

where

$$\mathcal{T}_\omega^s := (I - \omega G)^{-1}\{(1-\omega)I + \omega F\}, \qquad \mathbf{c}_\omega^s := \omega(I - \omega G)^{-1}\mathbf{b},$$

these methods each depending on a single relaxation parameter ω.

Under the assumption (cf. (1.2)) that $I - F$ is Hermitian and positive definite, Niethammer and Varga [11] determined inclusion sets for the eigenvalues of the corresponding relaxation matrices $\mathcal{T}_\omega^h$ and $\mathcal{T}_\omega^s$. To be more precise, they showed that the spectrum $\sigma(\mathcal{T}_\omega^h)$ of $\mathcal{T}_\omega^h$ is contained in a certain rectangle (which degenerates to an interval on the imaginary axis when $\omega = 1$) (cf. [11, Fig. 1]), whereas $\sigma(\mathcal{T}_\omega^s)$ is contained in a *bow-tie region* (cf. [11, Fig. 2]). In this paper, we examine the question of whether these facts can be used to effectively *accelerate* the procedures (1.7) and (1.8) by the application of *k-step iterative methods*, such as the *Chebyshev semi-iterative method* (when $k = 2$).

To go beyond the special schemes (1.7) and (1.8), we need some additional terminology. Let

$$\mathbf{x}_m = T\mathbf{x}_{m-1} + \mathbf{c} \qquad (m = 1, 2, \ldots), \tag{1.9}$$

with $1 \notin \sigma(T)$, be called a *basic iteration* for the solution of (1.1) which results from a splitting of the matrix A. We assume that we have a priori information about the eigenvalues of T of the form

$$\sigma(T) \subseteq \Omega\,, \tag{1.10}$$

where Ω is a compact subset of the complex plane with $1 \notin \Omega$. In addition, with the notation $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, we always require that Ω has no isolated points and that

$\overline{\mathbb{C}} \setminus \Omega$ is of finite connectivity. Note that for $\sigma(\mathcal{T}_\omega^h)$ of (1.7), as well as for $\sigma(\mathcal{T}_\omega^s)$ of (1.8), inclusions of this type are available, as previously mentioned. To (1.9), we now apply the *k-step method* (cf. [10])

$$\text{(1.11)}\quad \mathbf{y}_m := \mu_{m,0}(T\mathbf{y}_{m-1} + \mathbf{c}) + \mu_{m,1}\mathbf{y}_{m-1} + \mu_{m,2}\mathbf{y}_{m-2} + \cdots + \mu_{m,k}\mathbf{y}_{m-k}\,,$$

where $\mu_{m,j} \in \mathbb{C}$ ($\mu_{m,j} := 0$ for $j > m$) and $\sum_{j=0}^{k} \mu_{m,j} = 1$ ($m = 1, 2, \cdots$), in order to accelerate the convergence of the basic iterations (1.9). It is well known that these k-step methods belong to the class of *semi-iterative methods* or *polynomial acceleration methods* applied to (1.9) (cf. Varga [12]). The error vectors $\mathbf{e}_m := (I - T)^{-1}\mathbf{c} - \mathbf{y}_m$, associated with the mth iterate $\mathbf{y}_m$ of (1.11), can be written as $\mathbf{e}_m = p_m(T)\mathbf{e}_0$, where the polynomials $p_m(z) = \sum_{j=0}^{m} \pi_{m,j} z^j$ are recursively defined by $p_m(z) := 0$ ($m < 0$), $p_0(z) := 1$ and

$$p_m(z) := (\mu_{m,0}z + \mu_{m,1})p_{m-1}(z) + \mu_{m,2}p_{m-2}(z) + \cdots + \mu_{m,k}p_{m-k}(z) \quad (m = 1, 2, \cdots).$$

For notational convenience, we collect the Taylor coefficients of each of the polynomials p_m into an infinite lower triangular matrix $P = (\pi_{m,j})_{m \geq j \geq 0}$ which we call the *generating matrix* for the k-step method (1.11). Then it is known (cf. [4]) that, for a given P, the value of

$$\kappa(T, P) := \limsup_{m \to \infty} \sup_{\mathbf{e}_0 \neq \mathbf{0}} \left[\frac{\|\mathbf{e}_m\|}{\|\mathbf{e}_0\|} \right]^{1/m}$$

depends only on the structure of the Jordan canonical form of the matrix T. With regard to (1.10), we therefore define the *asymptotic convergence factor of the k-step method* (1.11), *with respect to the information* $\sigma(T) \subseteq \Omega$, by

$$\text{(1.12)}\qquad \kappa(\Omega, P) := \max\{\kappa(T, P) : T \in \mathbb{C}^{n,n}, n \geq 1, \text{ with } \sigma(T) \subseteq \Omega\}\,.$$

The best, i.e., *smallest*, convergence factor we can hope to achieve by any k-step method ($k = 1, 2, \cdots$) in this worst-case philosophy is the *asymptotic convergence factor* of Ω, defined by

$$\text{(1.13)}\quad \kappa(\Omega) := \inf\{\kappa(\Omega, P) :\ P \text{ generates a } k\text{-step method }, k = 1, 2, \cdots\}\,.$$

The infimum in (1.13) is actually a minimum (cf. [4]), and each k-step method for which this minimum is attained will be called *asymptotically optimal with respect to* Ω. The best-known examples of asymptotically optimal methods are the Chebyshev semi-iterative methods studied by Manteuffel [9]. In [9], Ω is an ellipse with either real foci or complex conjugate foci, with $1 \notin \Omega$, and in these cases there exists a Chebyshev semi-iterative method, i.e., a two-step method, which is asymptotically optimal with respect to Ω. (We note, more generally, that *any* ellipse Ω in $\mathbb{C}$ with $1 \notin \Omega$ admits an asymptotically optimal two-step method (cf. [10]).)

The quantity $\kappa(\Omega)$ of (1.13) has some interesting *capacity-like* properties. For example, it is known that

$$\text{(1.14)}\qquad \kappa(\Omega_1) < \kappa(\Omega_2)$$

if Ω_1 is a proper subset of Ω_2 (cf. [3, Prop. 3]). To compare the convergence factors of two compact sets Ω_1 and Ω_2 with $\Omega_1 \not\subseteq \Omega_2$ and $\Omega_2 \not\subseteq \Omega_1$, another observation is

helpful. Let Ω be a compact subset of $\mathbb{C}$ with $1 \notin \Omega$, and let $t_m \in \Pi_m$ be a polynomial of the exact degree m satisfying $t_m(1) = 1$ and the condition

$$t_m(z) = 1 \quad \text{implies} \quad z \notin \Omega\,.$$

Then it is known (cf. [3, Lemma 4]) that

$$\kappa(\Omega) \leq [\kappa(t_m(\Omega))]^{1/m}\,, \tag{1.15}$$

where $t_m(\Omega)$ denotes the image of Ω under the polynomial transformation $z \mapsto t_m(z)$. Moreover, equality holds in (1.15) if and only if the implication

$$z \notin \Omega \quad \text{implies} \quad t_m(z) \notin t_m(\Omega) \tag{1.16}$$

is valid for every $z \in \mathbb{C}$ (cf. [3, Thm. 6]).

If $\overline{\mathbb{C}} \setminus \Omega$ is simply connected, and if Φ denotes the Riemann mapping function that maps the complement of Ω conformally onto the exterior of the unit circle such that the points at infinity correspond to each other, then $\kappa(\Omega)$ can be expressed explicitly (cf. [4, Thm. 11]) as

$$\kappa(\Omega) = \frac{1}{|\Phi(1)|} < 1\,, \tag{1.17}$$

the last inequality in (1.17) following from the assumption that $1 \notin \Omega$. In what follows, (1.17) turns out to be useful when one must decide whether it is worthwhile to apply a k-step method of the form (1.11) to a given basic iteration (1.9) whose operator T satisfies $\sigma(T) \subseteq \Omega$.

We briefly describe the contents of this paper. For the relaxation method (1.7) associated with the Hermitian splitting of A, we examine in §2 which choice of ω in (1.7) is best. Here, we are not interested in merely minimizing $\rho(\mathcal{T}_\omega^h)$ as a function of ω; rather, we seek a value of ω such that (1.7) can be most efficiently accelerated by a k-step method. It turns out that the choice $\omega = 1$ is best in this sense. For the relaxation method (1.8) associated with the skew-Hermitian splitting of A, we accelerate (1.8), in §3, by means of a new hybrid scheme, consisting of a polynomial transformation of the given linear system, together with the application of a stationary one-step method to the transformed equations. An optimized Chebyshev acceleration of (1.8), as given in Chin and Manteuffel [1], will be shown here to converge more *slowly* than this hybrid method. In §4, we apply our hybrid procedure to the block Jacobi method for a model equation of a convection-diffusion problem. Again, we make use of results due to Chin and Manteuffel [1], who determined sets in the complex plane containing all eigenvalues of the block Jacobi matrix in this example. Finally, a comparison with the associated block successive overrelaxation (SOR) method is given.

2. The Hermitian splitting. We first consider the *Hermitian* splitting (1.3) of A and its associated relaxation methods (1.7). Let $\{\gamma_j\}_{j=1}^n$ denote the eigenvalues of F (cf. (1.2)) with $\alpha := \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_n =: \beta$. We always assume that the Hermitian part of A, namely $I - F$, is positive definite. This assumption, coupled with the hypothesis of (1.1) that the trace of F is zero, implies that

$$\alpha \leq 0 \leq \beta < 1\,. \tag{2.1}$$

For all $\omega \in (0, 1/\beta)$, the matrix $I - \omega F$ is evidently nonsingular and the relaxation matrix $\mathcal{T}_\omega^h$ of (1.7) is thus defined for these values of ω. From [11, (3.13) and Fig. 1],

we further have the inclusion

$$\sigma(\mathcal{T}_\omega^h) \subseteq \Omega_\omega := \begin{cases} \{z \in \mathbb{C} : c \le \operatorname{Re} z \le d,\ |\operatorname{Im} z| \le f\}, & \text{if } 0 < \omega < 1, \\ [-if, if], & \text{if } \omega = 1, \\ \{z \in \mathbb{C} : d \le \operatorname{Re} z \le c,\ |\operatorname{Im} z| \le f\}, & \text{if } 1 < \omega < 1/\beta, \end{cases} \tag{2.2}$$

where

$$c = c(\omega) := \frac{1-\omega}{1-\omega\alpha}, \quad d = d(\omega) := \frac{1-\omega}{1-\omega\beta}, \quad \text{and} \quad f = f(\omega) := \frac{\omega\rho(G)}{1-\omega\beta}.$$

In every case, the rectangle Ω_ω is contained in the open unit disk if and only if $d^2(\omega) + f^2(\omega) < 1$, which is equivalent (cf. [11, Thm. 3.2]) to

$$0 < \omega < \omega_g := \frac{2(1-\beta)}{1+\rho^2(G)-\beta^2}. \tag{2.3}$$

The relaxation method (1.7) is therefore guaranteed to converge for any ω in the interval $(0, \omega_g)$.

A natural question now is which choice of ω is optimal. The classical concept of optimality seeks a minimum of the spectral radius $\rho(\mathcal{T}_\omega^h)$ as a function of ω. With our limited information, $\sigma(\mathcal{T}_\omega^h) \subseteq \Omega_\omega$, i.e.,

$$\rho(\mathcal{T}_\omega^h) \le \hat{\rho}(\omega) := \sqrt{d^2(\omega) + f^2(\omega)},$$

we seek to minimize $\hat{\rho}(\omega)$ for $\omega \in (0, \omega_g)$. An easy calculation shows that $\hat{\rho}$ has exactly one minimum in $(0, \omega_g)$, which is attained at

$$\omega^* = \frac{1-\beta}{1-\beta+\rho^2(G)} \qquad \text{with } \hat{\rho}(\omega^*) = \frac{\rho(G)}{\sqrt{(1-\beta)^2+\rho^2(G)}}. \tag{2.4}$$

The question is whether this definition of optimality really makes sense in our context. Let us consider the following example. For $\rho(G) = 1$ and $\beta = 0.5$, we obtain from (2.4) that $\omega^* = \frac{1}{3}$ and $\rho(\mathcal{T}_{1/3}^h) \le \hat{\rho}(\omega^*) = 2/\sqrt{5} = 0.8944\cdots$. On the other hand, if we choose $\omega = 1$, then Ω_1 reduces to a line segment, i.e., $\Omega_1 = [-2i, 2i]$, and $\mathcal{T}_1^h$ may be divergent. But the Chebyshev semi-iterative method for this interval, or equivalently, the stationary two-step method

$$\mathbf{y}_m := \mu_0(\mathcal{T}_1^h \mathbf{y}_{m-1} + \mathbf{c}_1) + \mu_1 \mathbf{y}_{m-1} + \mu_2 \mathbf{y}_{m-2} \qquad (m = 2, 3, \cdots) \tag{2.5}$$

with

$$\mu_0 := (\sqrt{5}-1)/2, \quad \mu_1 = 0, \quad \text{and} \quad \mu_2 = 1 - \mu_0$$

has an asymptotic convergence factor of $(\sqrt{5}-1)/2 = 0.6180\cdots$ (cf. Niethammer and Varga [10, Ex. 2]). Is $\omega = 1$ therefore a better choice than $\omega = \frac{1}{3}$? At this stage, this could be a hasty conclusion since we can also accelerate $\mathbf{x}_m = \mathcal{T}_{1/3}^h \mathbf{x}_{m-1} + \mathbf{c}_{1/3}^h$ by a Chebyshev procedure or another k-step method, with the goal of constructing an even faster scheme.

After these considerations, we believe that it is more appropriate to determine an ω that minimizes $\kappa(\Omega_\omega)$ (cf. (1.13)), rather than $\hat{\rho}(\omega)$. The assertion of the following theorem is that $\omega = 1$ is optimal in this sense, i.e., the introduction of a relaxation

parameter $\omega \neq 1$ does *not* improve the iterative method based on the Hermitian splitting (1.3) of A.

THEOREM 1. *For each rectangle* Ω_ω (*cf.* (2.2)) *with* $0 < \omega < \omega_g$ (*cf.* (2.3)) *and* $\omega \neq 1$, *there holds*

$$\kappa(\Omega_\omega) > \kappa(\Omega_1) = \frac{\rho(G)}{1-\beta+\sqrt{(1-\beta)^2+\rho^2(G)}}.$$

Proof. Let ω be arbitrary (but fixed) in $(0,\omega_g)$ with $\omega \neq 1$. The interval

$$I_\omega := [d(\omega) - if(\omega),\ d(\omega) + if(\omega)]$$

(cf. (2.2)) is a proper subset of the rectangle Ω_ω. The polynomial (in Π_1) defined by

$$t_1(z) := \frac{1-\omega\beta}{\omega(1-\beta)}(z-1) + 1 \qquad (t_1(1) = 1),$$

induces a bijection of I_ω onto Ω_1. From (1.14) and (1.15), we therefore obtain

$$\kappa(\Omega_\omega) > \kappa(I_\omega) = \kappa(t_1(I_\omega)) = \kappa(\Omega_1)\,.$$

The explicit expression for $\kappa(\Omega_1)$ has been derived in [10, Ex. 2]. □

We conclude this section with a simple example which shows that the bounds of (2.2) may be a considerable *overestimation* of the spectrum of $\mathcal{T}_\omega^h$.

Example 2.1. If we discretize the boundary value problem

$$-u''(t) + \tau u'(t) = f(t) \quad \text{on } (0,1) \quad \text{with given } u(0), u(1) \in \mathbf{R} \tag{2.6}$$

($\tau > 0$), by using central differences with mesh size $h = 1/(n+1)$, a linear system results whose coefficient matrix A is an $n \times n$ Toeplitz tridiagonal matrix. Normalizing its diagonal entries to be equal to 1, we have

$$A = \text{tridiag}[-(1+R)/2,\ 1,\ -(1-R)/2] \in \mathbf{R}^{n,n} \tag{2.7}$$

with the mesh Reynolds number $R := \tau h/2$ (cf. Elman and Golub [5, §2]). Thus, its Hermitian splitting is given (cf. (1.3)) by

$$A = (I - \text{tridiag}[1/2, 0, 1/2]) - \text{tridiag}[R/2, 0, -R/2]. \tag{2.8}$$

Since the eigenvalues of the matrix $\text{tridiag}[a,0,b] \in \mathbf{C}^{n,n}$ are known (cf. [5, Lemma 2]) to be

$$\lambda_k = 2\sqrt{ab}\cos\left(\frac{\pi k}{n+1}\right) \qquad (k = 1, 2, \cdots, n),$$

we deduce from (2.2) the estimate

$$\rho(\mathcal{T}_1^h) \leq \frac{R\cos(\pi h)}{1-\cos(\pi h)} = \frac{\tau}{\pi^2 h} + \mathcal{O}(h) \qquad (h \to 0) \tag{2.9}$$

(where, because of Theorem 1, we only consider $\omega = 1$). On the other hand, in this simple example, the eigenvalues λ of $\mathcal{T}_1^h = (I-F)^{-1}G$ can be computed explicitly. Let λ be such an eigenvalue. If $\lambda \neq 0$, then

$$(I-F) - \frac{1}{\lambda}G = \text{tridiag}\left[-\frac{1-R/\lambda}{2},\ 1,\ -\frac{1+R/\lambda}{2}\right]$$

must be singular, or equivalently, there must be a $k \in \{1, 2, \cdots, n\}$ such that

$$2\sqrt{\frac{1}{4} - \frac{R^2}{4\lambda^2}}\cos(\pi k h) = 1\,.$$

From this, it is easy to see that the eigenvalues of $\mathcal{T}_1^h = (I - F)^{-1}G$ are given by

$$\lambda = \pm R\cot(\pi k h)i, \qquad \begin{cases} k = 1, 2, \cdots, n/2, & \text{if } n \text{ is even,} \\ k = 1, 2, \cdots, (n-1)/2, & \text{if } n \text{ is odd.} \end{cases}$$

(If n is odd, then $\mathcal{T}_1^h$ has in addition the eigenvalue $\lambda = 0$.) This implies that

$$\rho(\mathcal{T}_1^h) = R\cot(\pi h) = \frac{\tau}{2\pi} + \mathcal{O}(h^2) \qquad (h \to 0),$$

which is an order of magnitude smaller than the estimate (2.9).

3. The skew-Hermitian splitting. We turn now to the investigation of the relaxation methods (1.8) induced by the skew-Hermitian splitting $A = (I - G) - F$ (cf. (1.4)) of A. As in the previous section, we again assume that the Hermitian part $M^h = I - F$ of A is positive definite, i.e., for the eigenvalues $\alpha = \gamma_1 \le \cdots \le \gamma_n = \beta$ of F, there holds, as in (2.1),

$$\alpha \le 0 \le \beta < 1\,.$$

For the relaxation matrix $\mathcal{T}_\omega^s$ of (1.8) with $\omega > 0$, Niethammer and Varga [11] derived the eigenvalue inclusion

$$\sigma(\mathcal{T}_\omega^s) \subseteq \tilde{\Omega}_\omega := \{z \in \mathbb{C} : |z - c_1(\omega)| \le |c_1(\omega)| \text{ or } |z - c_2(\omega)| \le |c_2(\omega)|\}, \tag{3.1}$$

where

$$c_1(\omega) := \frac{1 - \omega + \omega\beta}{2}, \qquad c_2(\omega) := \frac{1 - \omega + \omega\alpha}{2}.$$

They actually proved more, namely, that $\sigma(\mathcal{T}_\omega^s)$ is contained in a bow-tie region (cf. [11, Fig. 2]), which is itself contained in $\tilde{\Omega}_\omega$. These bow-tie regions depend on the spectral radius $\rho(G)$ of the skew-Hermitian part G of A, and fill out $\tilde{\Omega}_\omega$ as $\rho(G)$ tends to infinity. The estimate (3.1) has the advantage of being independent of $\rho(G)$.

Note that under the given assumption on α and β, we have

$$c_2(\omega) \le c_1(\omega) < \tfrac{1}{2},$$

i.e., $\rho(\mathcal{T}_\omega^s) < 1$ if $c_2(\omega) > -\frac{1}{2}$, which is equivalent to $\rho(\mathcal{T}_\omega^s) < 1$ for $0 < \omega < \omega_g := 2/(1 - \alpha)$ (cf. [11, Thm. 4.1]). The optimal relaxation factor ω_0, with respect to the information (3.1), occurs when the condition $c_2(\omega_0) = -c_1(\omega_0)$ holds, i.e.,

$$\omega_0 = \frac{2}{2 - (\alpha + \beta)} \quad \text{with } \rho(\mathcal{T}_{\omega_o}^s) \le \frac{\beta - \alpha}{2 - (\alpha + \beta)} \tag{3.2}$$

(cf. [11, Thm. 4.1]). In this latter case, we give the associated bow-tie region in Fig. 1.

We now apply these results to the example already discussed in the last section.

Example 3.1. The skew-Hermitian splitting (1.4) of the matrix A of (2.7) has the form $A = (I - G) - F$, where $F = \text{tridiag}[1/2, 0, 1/2]$ and $G = \text{tridiag}[R/2, 0, -R/2]$ (cf. (2.8)). With $\alpha = -\cos(\pi h)$ and $\beta = \cos(\pi h)$, we have $\omega_0 = 1$ and

$$\rho(\mathcal{T}_1^s) \le \cos(\pi h) < 1\,.$$

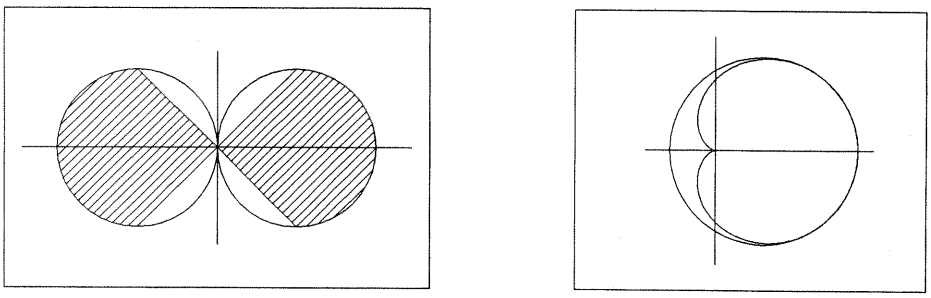

FIG. 1. *Shape of the bow-tie region containing* $\sigma(\mathcal{T}_\omega^s)$ (*cf.* (3.1)) *together with* Λ_c (*cf.* (3.4)), *where* $c_1(\omega) = -c_2(\omega)$ (*left-hand side*). Λ_c^2 (*cf.* (3.9)) *and the enclosing ("embracing") disk* $D(\zeta(\mu_0^*), r(t_0))$ (*cf. Proposition* 3.2) (*right-hand side*).

Thus, the matrix $\mathcal{T}_1^s$ is convergent for *any* value of the parameter τ (cf. (2.6)) and for *any* mesh size h. As in the case of $\mathcal{T}_1^h$, the eigenvalues of $\mathcal{T}_1^s$ can be calculated explicitly in this simple example. With the technique used in Example 1, we see that the spectrum of $\mathcal{T}_1^s$ consists of the points

$$\lambda = \pm \frac{\cos(\pi k h)}{\sqrt{R^2 \cos^2(\pi k h) + 1}}, \qquad \begin{cases} k = 1, 2, \cdots, \frac{n}{2}, & \text{if } n \text{ is even,} \\ k = 1, 2, \cdots, \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

(If n is odd, we again note that $\mathcal{T}_1^s$ has the additional eigenvalue $\lambda = 0$.) Note further that $\sigma(\mathcal{T}_1^s)$ possesses only *real* elements. We thus conclude that

$$\rho(\mathcal{T}_1^s) = \frac{\cos(\pi h)}{\sqrt{R^2 \cos^2(\pi h) + 1}}.$$

To accelerate the convergence of the basic iteration (1.8), the Chebyshev semi-iterative method can be applied to the interval $[-\rho(\mathcal{T}_1^s), \rho(\mathcal{T}_1^s)]$.

The problem we wish to consider now is how we can use the information of (3.1) to accelerate the convergence of the basic iterative method $\mathbf{x}_m := \mathcal{T}_\omega^s \mathbf{x}_{m-1} + \mathbf{c}^s$.

To simplify our notation, we consider the basic iteration method

$$\mathbf{x}_m = T\mathbf{x}_{m-1} + \mathbf{c} \qquad (m = 1, 2, \cdots), \tag{3.3}$$

where we assume that $\sigma(T) \subseteq \Lambda_c$, where

$$\Lambda_c := \{z \in \mathbb{C} : |z - c| \le c \quad \text{or} \quad |z + c| \le c\}, \tag{3.4}$$

with $0 < c < \frac{1}{2}$. (Note that $\tilde{\Omega}_\omega$ of (3.1) has this form for $\omega = \omega_0$.)

We first wish to design a stationary two-step method

$$\mathbf{y}_m = \mu_0(T\mathbf{y}_{m-1} + \mathbf{c}) + \mu_1 \mathbf{y}_{m-1} + \mu_2 \mathbf{y}_{m-2} \qquad (m = 2, 3, \cdots) \tag{3.5}$$

(with $\mu_0 + \mu_1 + \mu_2 = 1$), which is compatible with this information for $\sigma(T)$. Recently, Chin and Manteuffel [1] solved an analogous problem. They determined the optimal relaxation parameter ω of the SOR method under the assumption that the corresponding Jacobi matrix T is weakly cyclic of index 2 and satisfies the condition (3.4). Their result is the following proposition.

PROPOSITION 3.1. [1, §3.4]. *For* $0 < c < \frac{1}{2}$, *define the positive numbers* $t \in (\sqrt{2}, 2)$ *and* $\kappa_2 \in (0,1)$ *by*

$$t^2 := \frac{3+\sqrt{5-4c^2}}{2(1+c^2)} \quad \text{and} \quad \kappa_2 := \sqrt{\frac{t+1}{t-1}} \left(\frac{1-\sqrt{1-c^2t^2}}{ct} \right). \tag{3.6}$$

Then, the asymptotic convergence factor $\kappa(\Lambda_c, P)$ (*cf.* (1.12)) *of any stationary two-step method* (3.5) *satisfies the inequality*

$$\kappa(\Lambda_c, P) \geq \kappa_2 ,$$

with equality holding if and only if the parameters $\{\mu_j\}_{j=0}^2$ *are chosen to be*

$$\mu_0 = 1 + \kappa_2^2, \quad \mu_1 = 0, \quad \text{and} \quad \mu_2 = -\kappa_2^2 .$$

The quantity κ_2 of Proposition 3.1 also represents the *best* asymptotic convergence factor that can be obtained by applying a Chebyshev acceleration to (3.3), since these Chebyshev procedures are asymptotically *stationary* two-step methods (cf. Golub and Varga [7]).

But neither a Chebyshev method nor a stationary two-step method is asymptotically optimal with respect to the information $\sigma(T) \subseteq \Lambda_c$, where Λ_c is defined in (3.4). The asymptotic convergence factor of such a method, i.e., the quantity $\kappa(\Lambda_c)$ of (1.13), is given by

$$\kappa(\Lambda_c) = \frac{1-\cos(\pi c)}{\sin(\pi c)} . \tag{3.7}$$

This follows from (1.17), together with the fact that the exterior mapping function Φ of Λ_c is known in closed form (cf. [8, §5.7]). Knowing Φ, we can construct an asymptotically optimal nonstationary one-step method based, for example, on the Fejér nodes of Λ_c (cf. [6]).

Finally, we present a hybrid scheme that is nearly optimal with respect to the information $\sigma(T) \subseteq \Lambda_c$ $(0 < c < \frac{1}{2})$. Instead of $\mathbf{x} = T\mathbf{x} + \mathbf{c}$, we consider the equivalent linear system

$$\mathbf{x} = T^2\mathbf{x} + T\mathbf{c} + \mathbf{c} . \tag{3.8}$$

The eigenvalues of T^2 are contained in Λ_c^2, whose boundary is the cardioid

$$\partial\Lambda_c^2 = \{ z \in \mathbb{C} : |z| = 2c^2(1+\cos(\arg z)) \} . \tag{3.9}$$

Since Λ_c^2 can be easily enclosed ("embraced") by a circle, we solve (3.8) by the following stationary one-step method, often also called a *stationary first-order Richardson method*:

$$\mathbf{x}_m = \mu_0(T^2\mathbf{x}_{m-1} + T\mathbf{c} + \mathbf{c}) + (1-\mu_0)\mathbf{x}_{m-1} \qquad (m = 1, 2, \cdots) \tag{3.10}$$

$(\mu_0 \in \mathbb{C},\ \mu_0 \neq 0)$. In general, as it certainly is not efficient to compute T^2 explicitly (especially if T is sparse), we divide (3.10) into two half-steps:

$$\begin{aligned} \mathbf{x}_{m-1/2} &= T\mathbf{x}_{m-1} + \mathbf{c}, \\ \mathbf{x}_m &= \mu_0(T\mathbf{x}_{m-1/2} + \mathbf{c}) + (1-\mu_0)\mathbf{x}_{m-1} \qquad (m = 1, 2, \cdots) . \end{aligned} \tag{3.11}$$

Before we answer the question as to which choice of μ_0 is optimal (with respect to the given information $\sigma(T^2) \subseteq \Lambda_c^2$), a remark concerning the asymptotic convergence factor of a hybrid method such as (3.10) or (3.11) should be made. If we apply a k-step method, as in (1.11) for instance, we must perform *one* matrix-vector multiplication by T in each iteration step. The hybrid method (3.11), however, requires *two* such matrix-vector multiplications per step. To compare both schemes fairly, we must compare κ_2 of Proposition 3.1 with $[\kappa(\Lambda_c^2, P(\mu_0))]^{1/2}$, where $P(\mu_0)$ denotes the generating matrix (cf. (1.13)) of the stationary first-order Richardson extrapolation with parameter μ_0. Our new result is found in Proposition 3.2.

PROPOSITION 3.2. *For $0 < c < \frac{1}{2}$, the effective convergence factor*

$$\kappa_h(\mu_0) := [\kappa(\Lambda_c^2, P(\mu_0))]^{1/2}$$

of any hybrid scheme of the form (3.11) *satisfies the inequality*

$$\kappa_h(\mu_0) \geq \frac{c}{1-c^2}\left[\frac{27}{4}(1-c^2)\right]^{1/4} =: \kappa_h(\mu_0^*), \tag{3.12}$$

with equality holding if and only if

$$\mu_0^* = \frac{2+c^2}{2-2c^2}.$$

Proof. Since the residual polynomials associated with a stationary first-order Richardson method are $p_m(z) = [\mu_0 z + 1 - \mu_0]^m$ $(m = 0, 1, \cdots)$, the convergence factor $\kappa(\Lambda_c^2, P(\mu_0))$ is given by

$$\kappa(\Lambda_c^2, P(\mu_0)) = \max_{z \in \Lambda_c^2} |\mu_0 z + 1 - \mu_0| = \max_{z \in \Lambda_c^2} \left|\frac{z-\zeta}{1-\zeta}\right|, \tag{3.13}$$

where $\zeta = \zeta(\mu_0) := 1 - 1/\mu_0$. Writing $\zeta = 2c^2 t$, observe that to minimize (3.13) as a function of ζ, we can confine our attention to those t that are contained in $[0, 2]$. For the function $r(t) := \max_{z \in \Lambda_c^2} |z - 2c^2 t|$, there holds

$$r^2(t) = \begin{cases} 4c^4(t-2)^2 & \text{for } 0 \leq t \leq \frac{2}{3}, \\ 8c^4 \frac{t^3}{2t-1} & \text{for } \frac{2}{3} \leq t \leq 2. \end{cases}$$

Furthermore, $\kappa^2(\Lambda_c^2, P(\mu_0)) = r^2(t)/(1-2c^2t)^2$ (with $\mu_0 = 1/(1-2c^2t)$) is monotonically decreasing for $t \in [0, \frac{2}{3}]$ and attains its minimum in $[\frac{2}{3}, 2]$ at $t_0 = 3/(4+2c^2)$. Therefore,

$$\mu_0^* = \frac{1}{1-2c^2 t_0} = \frac{2+c^2}{2-2c^2}$$

is the optimal extrapolation parameter, and substituting this into (3.13) gives the desired result of (3.12). □

The asymptotic convergence factors of the relaxation method (1.8) with ω of (3.2), of the stationary two-step (or Chebyshev) acceleration (3.5) described in Proposition 3.1, of the hybrid procedure (3.11) (cf. Proposition 3.2), and finally, of an asymptotically optimal method with respect to Λ_c (cf. (3.7)), are compared in Table 1.

TABLE 1

c	Convergence factor of equation (1.8)	(3.6)	(3.12)	(3.7)
0.2	0.4000	0.3420	0.3324	0.3249
0.4	0.8000	0.7451	0.7348	0.7265
0.45	0.9000	0.8661	0.8595	0.8541
0.495	0.9900	0.9859	0.9851	0.9844

The entries of the last two columns of the table below are seen to be more nearly equal as c increases to $\frac{1}{2}$. More precisely, on setting

$$c =: \tfrac{1}{2} - \varepsilon \qquad (0 < \varepsilon < \tfrac{1}{2} \text{ with } \varepsilon \text{ small}),$$

it can be verified from (3.12) that, as a function of ε,

$$\kappa_h(\mu_0^*) = 1 - 3\varepsilon + \frac{25}{6}\varepsilon^2 - \frac{437}{54}\varepsilon^3 + \mathcal{O}(\varepsilon^4) \qquad (\varepsilon \to 0),$$

whereas (3.7), as a function of ε, is

$$\kappa(\Lambda_{1/2-\varepsilon}) = 1 - \pi\varepsilon + \frac{\pi^2}{2}\varepsilon^2 - \frac{\pi^3}{3}\varepsilon^3 + \mathcal{O}(\varepsilon^4) \qquad (\varepsilon \to 0).$$

In terms of *rates of convergence*, we have that

$$\lim_{\varepsilon \to 0} \left\{ \frac{-\log \kappa(\Lambda_{1/2-\varepsilon})}{-\log \kappa_h(\mu_0^*)} \right\} = \frac{\pi}{3} = 1.0471\cdots.$$

Thus for ε small, the *loss* in the rate of convergence of the hybrid method (3.11) over that of the best rate of convergence, i.e., $-\log \kappa(\Lambda_{1/2-\varepsilon})$, is *less than* 5 percent. (In fact, from the numerical evaluation of the quantity in braces above, it appears that this loss in the rate of convergence never exceeds 5 percent for *any* c with $0 < c < \frac{1}{2}$.)

4. An example. The constant coefficient convection-diffusion equation

$$-\Delta u + \tau u_x = f \tag{4.1}$$

($\tau \geq 0$) on the unit square $(0,1) \times (0,1)$, with Dirichlet boundary conditions, is often used to construct test problems for iterative methods (cf., e.g., Chin and Manteuffel [1] or Elman and Golub [5]). The standard central difference discretization with mesh size $h = 1/(n+1)$ in both coordinate directions leads to a linear system whose coefficient matrix A has the block tridiagonal form

$$A = \text{tridiag}[-I, K, -I] \in \mathbf{R}^{n^2,n^2} \quad \text{with} \quad K = \text{tridiag}[-(1+R_x), 4, -(1-R_x)] \in \mathbf{R}^{n,n},$$

where $R_x := \tau h/2$ for the rowwise natural ordering of the mesh points. The corresponding block Jacobi matrix T has the eigenvalues

$$\lambda_{k,l} = \frac{\cos(\pi k h)}{2 - \sqrt{1 - R_x^2}\cos(\pi l h)} \qquad (k,l = 1,2,\cdots,n) \tag{4.2}$$

(cf. [5, Thm. 1]). For $R_x \leq 1$, the eigenvalues of T are therefore all real with

$$\lambda_{k,l} \in \left[-\frac{1}{2 - \sqrt{1 - R_x^2}}\,,\ \frac{1}{2 - \sqrt{1 - R_x^2}} \right] \qquad (k,l = 1,2,\cdots,n).$$

If, however, $R_x \geq 1$, then from (3.4),

$$\sigma(T) \subseteq \Lambda_{1/4} .$$

This inclusion is not sharp: Chin and Manteuffel showed that $\sigma(T)$ is contained in a certain bow-tie region (cf. [1, Fig. 2.2]) whose size depends on R_x. As R_x becomes larger, however, these bow-tie regions fill out $\Lambda_{1/4}$. (Besides that, we have ignored the factor $\cos(\pi h)$, which is not essential for our analysis.)

For $R_x > 1$, we now apply the hybrid scheme (3.11) to the block Jacobi method, which has the asymptotic convergence factor

$$\kappa_h(\mu_0) = 0.4229\cdots \quad \text{for the optimal } \mu_0 = 1.1 \quad (\text{when } c = \tfrac{1}{4})$$

(cf. Proposition 3.2). For the same problem, Chin and Manteuffel [1, (4.15)] found the spectral radius of the associated block SOR matrix $\mathcal{L}_\omega$ (with optimal relaxation parameter) which is $\rho(\mathcal{L}_\omega) = 0.1885\cdots$ if $R_x \geq 1.7177$. Clearly, the straightforward application of the hybrid procedure (3.11) is not competitive with the block SOR method for this model problem.

However, the block Jacobi matrix T in our example is weakly cyclic of index 2 (cf. [13, p. 39]), i.e., there exists an $n \times n$ permutation matrix Q such that

$$\tilde{T} = QTQ^* = \left[\begin{array}{c|c} 0 & T_1 \\ \hline T_2 & 0 \end{array}\right], \tag{4.3}$$

where the null submatrices on the diagonal are square. The transformed system $\mathbf{x} = \tilde{T}^2\mathbf{x} + \tilde{T}\mathbf{c} + \mathbf{c}$, which was the starting point of the hybrid scheme (3.11), then has the form

$$\left[\begin{array}{c} \mathbf{x}_1 \\ \mathbf{x}_2 \end{array}\right] = \left[\begin{array}{c|c} T_1T_2 & 0 \\ \hline 0 & T_2T_1 \end{array}\right] \left[\begin{array}{c} \mathbf{x}_1 \\ \mathbf{x}_2 \end{array}\right] + \left[\begin{array}{c} \tilde{\mathbf{c}}_1 \\ \tilde{\mathbf{c}}_2 \end{array}\right], \tag{4.4}$$

where the vectors $\mathbf{x}$ and $\tilde{\mathbf{c}} := \tilde{T}\mathbf{c} + \mathbf{c}$ are partitioned conformally with respect to the partitioning of (4.3). In view of (4.4), it is sufficient to solve the reduced problem

$$\mathbf{x}_2 = T_2T_1\mathbf{x}_2 + T_2\mathbf{c}_1 + \mathbf{c}_2$$

for the vector $\mathbf{x}_2$. Since $\sigma(T_2T_1) \setminus \{0\} = \sigma(T^2) \setminus \{0\}$, we have $\sigma(T_2T_1) \subseteq \Lambda_{1/4}^2$, and thus the optimal extrapolation parameter μ_0 of the *cyclically reduced hybrid scheme*

$$\begin{aligned} \mathbf{x}_{m-1/2} &= T_1\mathbf{x}_{m-1} + \mathbf{c}_1, \\ \mathbf{x}_m &= \mu_0(T_2\mathbf{x}_{m-1/2} + \mathbf{c}_2) + (1-\mu_0)\mathbf{x}_{m-1} \qquad (m = 1, 2, \cdots) \end{aligned} \tag{4.5}$$

is again given by Proposition 3.2. One step of the iterative method (4.5) requires *one* matrix-vector multiplication by each of the blocks T_1 and T_2. The effective asymptotic convergence factor of (4.5) (with optimal $\mu_0 = 1.1$) is therefore $[\kappa_h(\mu_0)]^2 = 0.1789\cdots$, indicating that (4.5) is marginally *faster* than the block SOR method, since $\rho(\mathcal{L}_\omega) = 0.1885\cdots$ for this latter method when $R_x \geq 1.7177$.

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