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Some numerical results on best uniform rational approximation of x^α on [0,1]

Richard S. Varga [1,*] and Amos J. Carpenter [2**]

[1] *Institute for Computational Mathematics, Kent State University, Kent, OH 44242, U.S.A.*
[2] *Dept. of Math. and Computer Science, Butler University, Indianapolis, IN 46208, U.S.A.*

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Let α be a positive number, and let $E_{n,n}(x^\alpha; [0, 1])$ denote the error of best uniform rational approximation from $\pi_{n,n}$ to x^α on the interval [0, 1]. We rigorously determined the numbers $\{E_{n,n}(x^\alpha; [0, 1])\}_{n=1}^{30}$ for six values of α in the interval (0, 1), where these numbers were calculated with a precision of at least 200 significant digits. For each of these six values of α, Richardson's extrapolation was applied to the products $\{e^{\pi\sqrt{4\alpha n}} E_{n,n}(x^\alpha; [0, 1])\}_{n=1}^{30}$ to obtain estimates of

$$\lambda(\alpha) := \lim_{n\to\infty} e^{\pi\sqrt{4\alpha n}} E_{n,n}(x^\alpha; [0, 1]) \ (\alpha > 0).$$

These estimates give rise to two interesting new conjectures in the theory of rational approximation.

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1. Introduction

Let $\pi_{m,n}$ denote the set of all real rational functions $p_m(x)/q_n(x)$, where $p_m(x)$ and $q_n(x)$ are real polynomials of degrees at most m and n, respectively. We assume that $p_m(x)$ and $q_n(x)$ have no common factors, that $q_n(x)$ does not vanish on $[-1, +1]$, and that $q_n(x)$ is normalized by $q_n(0) = 1$. For any positive number α, let

$$E_{n,n}\left(|x|^{2\alpha}; [-1, +1]\right) := \inf\left\{\left\| |x|^{2\alpha} - r_{n,n}(x)\right\|_{L_\infty[-1,+1]}: r_{n,n}(x) \in \pi_{n,n}\right\} \tag{1.1}$$

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denote the error of best uniform rational approximation from $\pi_{n,n}$ to $|x|^{2\alpha}$ on $[-1, +1]$.

The special case $\alpha = 1/2$ attracted the attention of many mathematicians. The first result was obtained by Newman [9], who showed that

$$\frac{1}{2\,\mathrm{e}^{9\sqrt{n}}} \leqslant E_{n,n}(|x|; [-1, +1]) \leqslant \frac{3}{\mathrm{e}^{\sqrt{n}}} \quad (n = 4, 5, \ldots). \tag{1.2}$$

Newman's result has been studied and/or improved by many people, among them Bulanov [3], Ganelius [4,5], Gonchar [6], Tzimbalario [13], and Vjacheslavov [15,16]. The strongest result improving the inequalities of (1.2) was obtained by Vjacheslavov [15], who showed that there exist positive constants M_1 and M_2 such that

$$M_1 \leqslant \mathrm{e}^{\pi\sqrt{n}} E_{n,n}(|x|; [-1, +1]) \leqslant M_2 \quad (n = 1, 2, \ldots). \tag{1.3}$$

Because $|x|$ is an even function on the interval $[-1, +1]$, it is easy to verify (cf. [14], Proposition 3) that

$$E_{2n,2n}(|x|; [-1, +1]) = E_{2n+1,2n+1}(|x|; [-1, +1]) = E_{n,n}(\sqrt{x}; [0, 1]), \tag{1.4}$$

so that (1.3) takes the form

$$M_1 \leqslant \mathrm{e}^{\pi\sqrt{2n}} E_{n,n}(\sqrt{x}; [0, 1]) \leqslant M_2 \quad (n = 1, 2, \ldots). \tag{1.5}$$

What is of considerable interest is the precise asymptotic behavior of $\mathrm{e}^{\pi\sqrt{2n}} E_{n,n}(\sqrt{x}; [0, 1])$, as $n \to \infty$, i.e., the determination of the positive constants $\underline{M}$ and $\overline{M}$, where

$$\underline{M} := \varliminf_{n\to\infty} \mathrm{e}^{\pi\sqrt{2n}} E_{n,n}(\sqrt{x}; [0, 1]) \leqslant \varlimsup_{n\to\infty} \mathrm{e}^{\pi\sqrt{2n}} E_{n,n}(\sqrt{x}; [0, 1]) =: \overline{M}. \tag{1.6}$$

On the urging of Academician A.A. Gonchar, very high precision calculations of the products $\{\mathrm{e}^{\pi\sqrt{2n}} E_{n,n}(\sqrt{x}; [0, 1])\}_{n=1}^{40}$ were carried out by Varga, Ruttan and Carpenter [14]. On using the Richardson extrapolation method on these products, there was strong numerical evidence for the conjecture in [14] that $\underline{M}$ and $\overline{M}$ in (1.6) are both 8, i.e.,

$$8 \stackrel{?}{=} \lim_{n\to\infty} \mathrm{e}^{\pi\sqrt{2n}} E_{n,n}(\sqrt{x}; [0, 1]). \tag{1.7}$$

Subsequently, Stahl [12] has proved that this conjecture of (1.7) is *true*!

Now there has also been interest in the behavior of $E_{n,n}(x^\alpha; [0, 1])$, as a function of n for a fixed $\alpha > 0$, which generalizes the case $\alpha = 1/2$ of (1.5).

The case $\alpha \neq 1/2$ has also been studied by several people, among them Ganelius [4,5], Tzimbalario [13], and Vjacheslavov [16]. In this case, the strongest result was obtained by Ganelius [4], who showed that if $\alpha = \sigma/\tau$ is a positive proper rational number, then

$$|\sin(\pi\alpha)|\, e^{-2\pi(\alpha+2)} \leqslant e^{\pi\sqrt{4\alpha n}} E_{n,n}(x^\alpha;\ [0,\ 1]) \leqslant B_{\sigma,\tau}, \tag{1.8}$$

where $B_{\sigma,\tau}$ is a positive constant depending on σ and τ.

Our goal here, inspired by the results of (1.7) and (1.8), was to see if numerical results might also predict a *sharper* asymptotic form for Ganelius's result (1.8), namely, the existence of a positive constant $\lambda(\alpha)$ such that

$$\lambda(\alpha) \overset{?}{=} \lim_{n\to\infty} e^{\pi\sqrt{4\alpha n}} E_{n,n}(x^\alpha;\ [0,\ 1]) \quad (\alpha > 0). \tag{1.9}$$

To this end, the values of $\{E_{n,n}(x^{j/8};\ [0,\ 1])\}_{n=1}^{30}$ were each computed here to at least 200 significant digits for each of the six values $j = 1, 2, 3, 5, 6, 7$ (the omitted case $j = 4$ was previously given in [14]). To the associated products $\{e^{\pi\sqrt{jn/2}} E_{n,n}(x^{j/8};\ [0,\ 1])\}_{n=1}^{30}$ from (1.8), we then applied (cf. §3) the Richardson extrapolation method, where, as in [14], this extrapolation method was used with $x_n := 1/\sqrt{n}$. These extrapolations produced numerical estimates (to at least six decimal digits) for $\lambda(\alpha)$ of (1.9) for these six values of α.

As can be readily understood, this was a *nontrivial computation*, since, for each such α and each n $(1 \leqslant n \leqslant 30)$, the number $E_{n,n}(x^\alpha;\ [0,\ 1])$ was determined only after several iterations of the second Remez algorithm (cf. §2). The total amount of these calculations consumed approximately 3300 cpu hours on the Alliant FX/8 and the Encore Multimax at the Argonne National Laboratory.

In Table 1.1, we give our estimates, rounded to six decimal digits, of $\lambda(\alpha)$ for the seven values of α given by $\{j/8\}_{j=1}^{7}$. Moreover, noting for any positive integer s that $E_{n,n}(x^s;\ [0,\ 1]) = 0$ for all $n \geqslant s$, then $\lambda(s) = 0$, which is also consistent with the lower bound of (1.8) when $\alpha = s$ is a positive integer. But as this lower bound of (1.8) also vanishes when $\alpha = 0$, we have also included $\lambda(0) = 0$ and $\lambda(1) = 0$ in Table 1.1.

Table 1.1
Estimates of $\lambda(\alpha)$ to six decimal digits

α	$\lambda(\alpha)$
0.000	0.000000
0.125	1.820359
0.250	4.000000
0.375	6.215096
0.500	8.000000
0.625	8.789473
0.750	8.000000
0.875	5.148754
1.000	0.000000

Table 1.2
Estimates of $\lambda(\alpha)$ and values of $2^{2(\alpha+1)}|\sin(\alpha\pi)|$ for $\alpha=\{j/8\}_{j=0}^{8}$

α	$\lambda(\alpha)$	$2^{2(\alpha+1)}\|\sin(\alpha\pi)\|$
0.000	0.000000	0.000000000000000
0.125	1.820359	1.820359442248909
0.250	4.000000	4.000000000000000
0.375	6.215096	6.215095896120149
0.500	8.000000	8.000000000000000
0.625	8.789473	8.789472907742480
0.750	8.000000	8.000000000000000
0.875	5.148754	5.148754023244661
1.000	0.000000	0.000000000000000

We remark that the value $\lambda(1/2)$ was actually determined in [14] to be 8, when rounded to ten decimal digits, rather than the six decimal digits given in Table 1.1. This is because the calculations in this case for the associated products $e^{\pi\sqrt{2n}}E_{n,n}(x^{1/2}; [0, 1])$, were carried out in [14] in high precision up to $n=40$, rather than up to $n=30$ as was done here. It is our belief that extending our calculations here for $e^{\pi\sqrt{jn/2}}E_{n,n}(x^{j/8}; [0, 1])$ from $n=30$ to $n=40$ for $\{j\}_{j=1}^{7}$ will similarly produce values of $\lambda(j/8)$ to ten decimal digits also.

But, it is also our belief that further extended high-precision calculations are *totally unnecessary*, since the numbers of Table 1.1, on close examination, are *exactly* represented (to six decimal digits) by the function

$$2^{2(\alpha+1)}|\sin(\alpha\pi)|. \tag{1.10}$$

This is indicated in Table 1.2, where the function of (1.10) is evaluated to fifteen decimal digits for comparison with the numerical estimates of $\lambda(\alpha)$ from Table 1.1.

The agreement between the last two columns of Table 1.2 gives strong numerical evidence for our new

$$\text{CONJECTURE: } \lambda(\alpha) \stackrel{?}{=} 2^{2(\alpha+1)}|\sin(\alpha\pi)| \quad (\alpha \geqslant 0). \tag{1.11}$$

The truth of this conjecture would clearly simultaneously generalize the case $\alpha=1/2$ of (1.7), as well as sharpen Ganelius's result (1.8). We fully intend to show in a subsequent paper, using techniques similar to those of Stahl [12], that this conjecture of (1.11) is also valid for all $\alpha \geqslant 0$.

In Figure 1.1, we have graphed the function $2^{2(\alpha+1)}|\sin(\alpha\pi)|$ given by our conjecture in (1.11), for $0 \leqslant \alpha \leqslant 3$, while in Figure 1.2, this same function is graphed for $0 \leqslant \alpha \leqslant 5$. Because of the agreement (to six decimal digits) in Table 1.2 between the numerical estimates for $\lambda(\alpha)$ and the evaluations of $2^{2(\alpha+)1)}|\sin(\alpha\pi)|$, the points of Table 1.1 are of course, to plotting accuracy, *on* the graph of $2^{2(\alpha+1)}|\sin(\alpha\pi)|$.

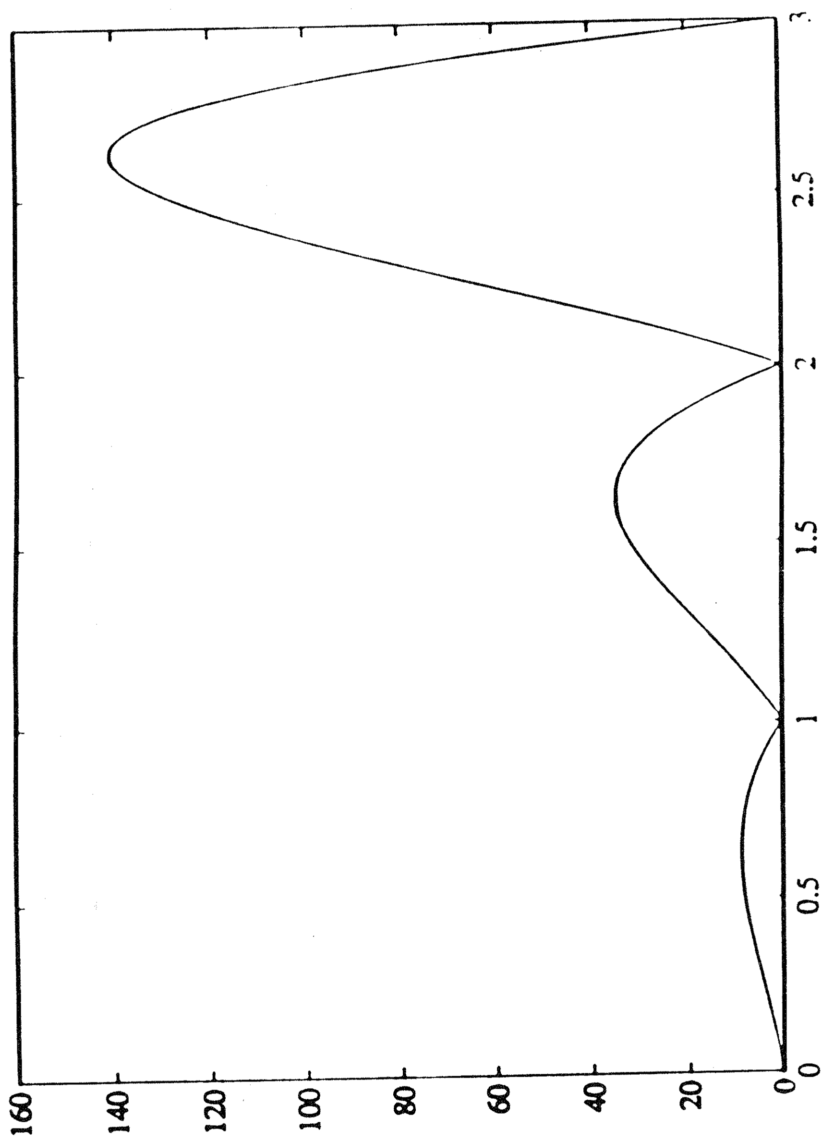

Fig. 1.1. Graph of the function $2^{2(\alpha+1)}|\sin(\alpha\pi)|$ for $0 \leqslant \alpha \leqslant 3$.

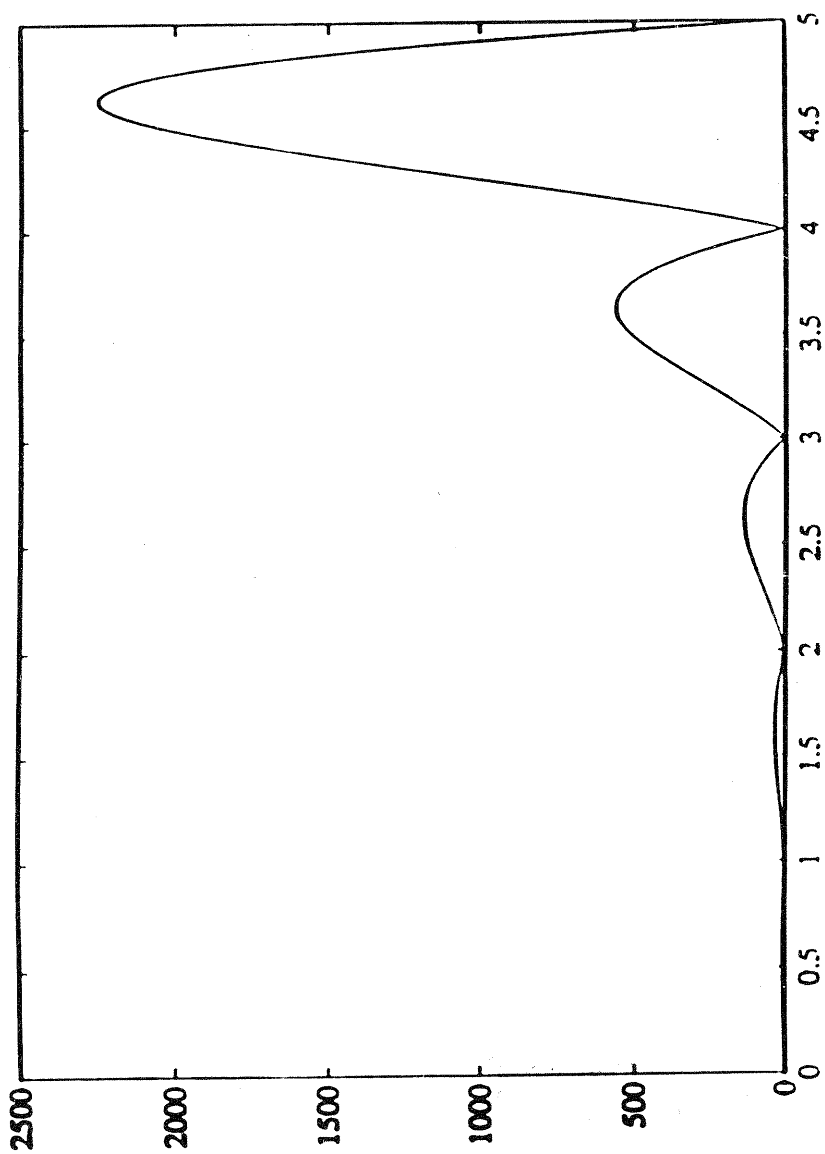

Fig. 1.2. Graph of the function $2^{2(\alpha+1)}|\sin(\alpha\pi)|$ for $0 \leqslant \alpha \leqslant 5$.

In Section 2, we give the theoretical background and the description of how the numbers $\{E_{n,n}(x^{j/8}; [0, 1])\}_{n=1}^{30}$ were computed for $\{j\}_{j=1}^{7}$. Then, in Section 3, the Richardson extrapolation method is applied to the products $\{e^{\pi\sqrt{jn/2}} E_{n,n}(x^{j/8}; [0, 1])\}_{n=1}^{30}$, which gives strong numerical evidence for a second new conjecture concerning the asymptotic behavior of $e^{\pi\sqrt{jn/2}} E_{n,n}(x^{j/8}; [0, 1])$ as $n \to \infty$, which goes beyond the conjecture in (1.11).

2. Computing the products $e^{\pi\sqrt{4\alpha n}} E_{n,n}(x^{g}a; [0, 1])\}_{n=1}^{30}$

As remarked in [14], the function x^α, for any real number α with $0 < \alpha < 1$, is *hypernormal* (cf. Loeb [7]) on the interval [0, 1], i.e., for any pair (m, n) of nonnegative integers,

$$W_{m,n}(\alpha) := \text{span}\{1, x, \ldots, x^m; x^\alpha, x^{1+\alpha}, \ldots, x^{n+\alpha}\} \tag{2.1}$$

is a *Haar space* of dimension $m + n + 2$ (i.e., any function not identically zero in $W_{m,n}(\alpha)$ has at most $m + n + 1$ distinct zeros in [0, 1]). Consequently (cf. [7] or Meinardus [8, p. 165]), for any pair (m, n) of nonnegative integers and for each α in (0, 1), there is a unique best uniform approximation $r_{m,n}^*(x; \alpha) := p^*/q^*$ in $\pi_{m,n}$ for which

$$E_{m,n}(x^\alpha; [0, 1]) := \inf\{ \| x^\alpha - h \|_{L_\infty[0,1]} : h \in \pi_{m,n}\} = \| x^\alpha - p^*/q^* \|_{L_\infty[0,1]} \tag{2.2}$$

is valid, where $\partial p^* = m$ and $\partial q^* = n$ (where ∂g denotes the exact degree of a polynomial g). Moreover, the longest alternation set for $x^\alpha - r_{m,n}^*(x; \alpha)$ on [0, 1] consists of $m + n + 2$ points. For our purposes here, we restrict attention to the cases when $m = n$ $(n = 1, 2, \ldots)$.

It turns out that Propositions 1–3 of [14] all carry over in a straightforward way from the uniform rational approximation of $x^{1/2}$ on [0, 1] to that of x^α on [0, 1], where $0 < \alpha < 1$. In particular, if (cf. (2.2)) $p_n^*(x; \alpha)/q_n^*(x; \alpha)$ is the best uniform approximation to x^α from $\pi_{n,n}$ on [0, 1], and if we write

$$p_n^*(x; \alpha) := \sum_{j=0}^{n} a_j^*(n; \alpha) x^j \text{ and } q_n^*(x; \alpha) := 1 + \sum_{j=1}^{n} b_j^*(n; \alpha) x^j$$
$$(n = 1, 2, \ldots), \tag{2.3}$$

then (cf. [14, Proposition 1])

$$a_j^*(n; \alpha) > 0 \ (j = 0, 1, \ldots, n) \text{ and } b_j^*(n; \alpha) > 0$$
$$(j = 1, 2, \ldots, n)(0 < \alpha < 1), \tag{2.4}$$

which is a useful check when applying the second Remez algorithm to find $E_{n,n}(x^\alpha; [0, 1])$ in high precision. We remark that (2.4) *fails* to hold for $\alpha > 1$, as the computations of best uniform approximations showed that (cf. (2.4)) $a_0^*(n; \frac{5}{4}) < 0$ and $b_n^*(n; \frac{5}{4}) < 0$ for $n = 1, 2, 3, 4$.

Our procedure for finding the numbers $\{E_{n,n}(x^\alpha; [0, 1])\}_{n=1}^{30}$ began with a fairly standard application of the second Remez algorithm (cf. Rivlin [11, p. 136]), to find the first few values of n, namely $\{E_{n,n}(x^\alpha; [0, 1])\}_{n=1}^{3}$, to at least 200 significant digits using Brent's multiple-precision (MP) package [1]. The *ad hoc* procedure, described in some detail in [14] for the case $\alpha = 1/2$, was also used here for values of n satisfying $3 < n \leqslant 30$. The products $\{e^{\pi\sqrt{4\alpha n}} E_{n,n}(x^\alpha; [0, 1])\}_{n=1}^{30}$ were all determined (for the six values of α considered) to an accuracy of at least 200 significant digits. We remark that we stopped for convenience in all cases at $n = 30$ in the determination of the products $e^{\pi\sqrt{4\alpha n}} E_{n,n}(x^\alpha; [0, 1])$, rather than from a breakdown of this procedure. At most

Table 2.1

The numbers $\{E_{n,n}(x^{1/8}; [0, 1])\}_{n=1}^{30}$ and the products $\{e^{\pi\sqrt{n/2}} E_{n,n}(x^{1/8}; [0, 1])\}_{n=1}^{30}$

n	$E_{n,n}(x^{1/8}; [0, 1])$	$e^{\pi\sqrt{n/2}} E_{n,n}(x^{1/8}; [0, 1])$
1	1.49828 08035 89663 87999 54215E-01	1.38150 66734 36674 35964 33656E+00
2	6.52474 26057 62697 04217 68307E-02	1.50987 06314 79538 53867 38048E+00
3	3.34452 02903 28863 40977 67729E-02	1.56800 25948 88750 62245 48366E+00
4	1.88502 76587 26292 65630 87890E-02	1.60264 47703 22252 68890 48372E+00
5	1.13217 75965 43014 31474 17743E-02	1.62623 38204 31939 01915 09553E+00
6	7.12243 24391 00481 20666 25559E-03	1.64360 51896 39952 59936 19837E+00
7	4.64331 06038 89882 31598 35927E-03	1.65707 50072 67713 00280 19286E+00
8	3.11472 44092 16732 27986 74664E-03	1.66790 89303 94862 75612 03873E+00
9	2.13903 86414 99507 06195 68749E-03	1.67686 44042 15708 17512 34043E+00
10	1.49835 13198 42128 14870 77259E-03	1.68442 59707 51716 17174 51447E+00
11	1.06752 20179 95193 76148 93174E-03	1.69091 98306 39287 16418 69800E+00
12	7.71875 20172 73668 09760 19058E-04	1.69657 46458 23960 06164 06250E+00
13	5.65399 51930 25125 74354 11498E-04	1.70155 60557 39932 21618 85061E+00
14	4.18962 62481 01534 67057 57207E-04	1.70598 73586 64283 16549 27247E+00
15	3.13680 88023 74740 18841 83386E-04	1.70996 24658 83472 72866 52091E+00
16	2.37060 27801 16198 31712 01085E-04	1.71355 43229 55196 57069 79303E+00
17	1.80683 48672 14170 47721 76504E-04	1.71682 05599 26271 38070 50825E+00
18	1.38787 62643 59577 07765 91803E-04	1.71980 73868 91096 82300 03980E+00
19	1.07369 78351 54813 05491 86893E-04	1.72255 23441 82199 32579 20557E+00
20	8.36130 29363 39236 81498 88633E-05	1.72508 62848 02405 38691 98399E+00
21	6.55114 16484 69390 44968 11879E-05	1.72743 48300 38855 22697 46818E+00
22	5.16209 65523 07165 82667 41385E-05	1.72961 94560 00053 80128 16262E+00
23	4.08918 61066 97453 55171 76665E-05	1.73165 83167 38461 73403 34024E+00
24	3.25538 43340 88091 28165 37236E-05	1.73356 68762 02743 35794 24681E+00
25	2.60369 62806 87298 99861 44951E-05	1.73535 83993 36293 36639 77902E+00
26	2.09161 62204 78658 54708 60202E-05	1.73704 43379 58531 04290 70109E+00
27	1.68721 14007 51029 39208 76459E-05	1.73863 46370 56227 67929 49284E+00
28	1.36632 75436 57877 85960 70515E-05	1.74013 79801 73066 96995 82873E+00
29	1.11057 67213 70122 84350 02666E-05	1.74156 19877 01112 56427 17205E+00
30	9.05888 49765 86564 44007 45104E-06	1.74293 68084 76185 10896 20917E+00

Table 2.2
The numbers $\{E_{n,n}(x^{1/4}; [0, 1])\}_{n=1}^{30}$ and the products $\{e^{\pi\sqrt{n}} E_{n,n}(x^{1/4}; [0, 1])\}_{n=1}^{30}$

n	$E_{n,n}(x^{1/4}; [0, 1]))$	$e^{\pi\sqrt{n}} E_{n,n}(x^{1/4}; [0, 1])$
1	9.74936 00796 69884 60523 59831E-02	2.25606 94496 99292 05812 45374E+00
2	3.11164 91331 95702 97391 16749E-02	2.64551 46094 58555 87266 20818E+00
3	1.23476 21377 90590 27908 04235E-02	2.84939 37639 93140 44146 33171E+00
4	5.56616 22934 10498 29940 39704E-03	2.98063 34614 17909 01795 72865E+00
5	2.73477 89254 65926 32673 32551E-03	3.07440 09000 53693 38946 68021E+00
6	1.43121 72280 88229 07124 61831E-03	3.14580 23996 71582 20065 65589E+00
7	7.86499 08986 14140 07662 42400E-04	3.20257 08868 72148 44885 39555E+00
8	4.49499 29075 80371 67817 26528E-04	3.24913 75581 95114 38596 11188E+00
9	2.65360 29968 40596 67450 84624E-04	3.28825 13758 88095 88943 66265E+00
10	1.61000 18208 48266 34400 21941E-04	3.32172 16034 37379 18062 84949E+00
11	1.00005 39102 78070 47555 52771E-04	3.35079 49391 08217 39913 27902E+00
12	6.34031 36380 17134 43651 94854E-05	3.37636 25365 24262 50012 14634E+00
13	4.09291 37045 92382 72291 46804E-05	3.39908 08192 75763 24569 92631E+00
14	2.68489 20730 01484 39364 41965E-05	3.41944 55718 77712 04260 00292E+00
15	1.78681 12294 99415 74509 02736E-05	3.43783 92812 60643 95821 79443E+00
16	1.20472 41956 49388 19444 97412E-05	3.45456 24506 99604 89025 33042E+00
17	8.21946 40130 88623 78795 44697E-06	3.46985 49334 84714 73789 99569E+00
18	5.66902 53773 20533 52501 96186E-06	3.48391 08433 51485 68811 85153E+00
19	3.94914 63260 82219 51761 25823E-06	3.49688 92108 45681 95413 32909E+00
20	2.77649 65319 46224 24338 05908E-06	3.50892 17506 66938 27867 15810E+00
21	1.96878 99371 84431 13869 98915E-06	3.52011 86230 35538 68308 29091E+00
22	1.40719 02439 37134 96610 15771E-06	3.53057 27744 54427 35834 22621E+00
23	1.01327 81951 68927 64881 12411E-06	3.54036 32544 64356 37898 96960E+00
24	7.34721 04928 70032 21762 92301E-07	3.54955 77824 05091 04132 82135E+00
25	5.36229 11081 61952 52824 10847E-07	3.55821 47568 77301 80408 28600E+00
26	3.93773 56145 26974 20028 45776E-07	3.56638 48456 06232 21802 32564E+00
27	2.90844 39062 44345 80853 49000E-07	3.57411 22555 59957 34489 90922E+00
28	2.16001 17385 58900 65694 85936E-07	3.58143 57567 06472 73784 46278E+00
29	1.61253 09333 90873 17970 45586E-07	3.58838 95140 09878 46189 15527E+00
30	1.20976 85518 22123 49779 13450E-07	3.59500 37687 54245 96754 63686E+00

fifteen iterations of the Remez algorithm were needed in all the cases considered. For completeness, we list in Tables 2.1–2.7 the numbers $\{E_{n,n}(x^\alpha; [0, 1])\}_{n=1}^{30}$ as well as the products $\{e^{\pi\sqrt{4\alpha n}} E_{n,n}(x^\alpha; [0, 1])\}_{n=1}^{30}$, each number having been rounded to 25 decimal digits, for the seven values of α given by $\{j/8\}_{j=1}^{7}$. (The numbers for the case $j = 4$ were taken from Varga, Ruttan and Carpenter [14].)

3. Extrapolation of the products $\{e^{\pi\sqrt{4\alpha n}} E_{n,n}(x^\alpha; [0, 1])\}_{n=1}^{30}$

As can be seen, the convergence of the products $\{e^{\pi\sqrt{4\alpha n}} E_{n,n}(x^\alpha; [0, 1])\}_{n=1}^{30}$ in Tables 2.1–2.7 is *extremely* slow, so that Richardson extrapolation (cf. Brezinski

Table 2.3
The numbers $\{E_{n,n}(x^{3/8}; [0, 1])\}_{n=1}^{30}$ and the products $\{e^{\pi\sqrt{3n/2}}E_{n,n}(x^{3/8}; [0, 1])\}_{n=1}^{30}$

n	$E_{n,n}(x^{3/8}; [0, 1])$	$e^{\pi\sqrt{3n/2}}E_{n,n}(x^{3/8}; [0, 1])$
1	6.53179 31608 21055 16019 07789E-02	3.06228 32981 63796 15125 18065E+00
2	1.60657 09254 91065 26236 06192E-02	3.70739 67822 64548 63527 14750E+00
3	5.17708 27067 50954 71414 74107E-03	4.05848 94261 40358 06334 53164E+00
4	1.95150 39600 07019 41106 83728E-03	4.28938 78859 73818 31881 90767E+00
5	8.17554 10262 00637 91819 03633E-04	4.45671 67378 86604 88074 28440E+00
6	3.70044 17847 32473 83866 97451E-04	4.58545 71330 10351 11891 62647E+00
7	1.77812 20715 22841 98219 25258E-04	4.68863 31623 23767 47176 99707E+00
8	8.96451 69757 98577 19791 11346E-05	4.77381 10451 87966 49676 63360E+00
9	4.70241 44007 69987 01038 63826E-05	4.84573 57218 73035 46059 52137E+00
10	2.55069 50501 68409 59874 71230E-05	4.90755 79407 69542 66353 00210E+00
11	1.42392 45642 93467 06605 47495E-05	4.96146 49842 08419 00507 68152E+00
12	8.15070 22836 88636 32735 29672E-06	5.00903 03318 58222 54758 61454E+00
13	4.76969 47558 96025 33057 21348E-06	5.05141 94367 27943 90206 24869E+00
14	2.84654 88254 80289 78173 49182E-06	5.08951 68360 53618 14240 35914E+00
15	1.72903 78237 29854 56924 18799E-06	5.12400 78870 20876 78503 94152E+00
16	1.06712 03216 76606 74772 47117E-06	5.15543 31351 27448 52870 75685E+00
17	6.68222 39351 66674 82266 72297E-07	5.18422 55409 71505 42949 84124E+00
18	4.24025 15859 32478 63117 51394E-07	5.21073 66167 18368 24701 58485E+00
19	2.72371 27750 39532 62282 73685E-07	5.23525 51825 76462 43906 17685E+00
20	1.76939 69097 88316 27329 04018E-07	5.25802 10896 78971 89066 87133E+00
21	1.16152 27923 40180 15507 75862E-07	5.27923 54340 73403 20136 95290E+00
22	7.69935 02621 42369 33721 29724E-08	5.29906 82767 12012 43618 46952E+00
23	5.15020 35658 60133 38801 30453E-08	5.31766 45596 57341 43706 97653E+00
24	3.47446 87122 47582 20349 24543E-08	5.33514 86970 52538 59395 94457E+00
25	2.36276 87689 73263 11568 73686E-08	5.35162 81784 85639 06823 82575E+00
26	1.61889 95074 43110 71667 51669E-08	5.36719 64267 53903 51220 46759E+00
27	1.11712 17518 96471 58488 29165E-08	5.38193 50860 15964 95938 87856E+00
28	7.76060 90431 38447 07197 52203E-09	5.39591 58700 14356 32938 28203E+00
29	5.42566 67785 17255 36744 27491E-09	5.40920 20670 98460 38020 88446E+00
30	3.81621 34515 15303 66382 70120E-09	5.42184 97750 12679 75442 62031E+00

[2, p. 6]), as successfully used in [14] for the case $\alpha = 1/2$, was again employed to speed up the convergence for all six values of α in (0, 1) that we considered.

To briefly describe the Richardson extrapolation method, let $\{S_i\}_{i=1}^{n}$, for $n \geqslant 2$, be a given finite sequence of real numbers. Then, the 0^{th} and $(k+1)^{\text{th}}$ columns of the Richardson extrapolation table are defined by

$$T_0^{(i)} := S_i \ (1 \leqslant i \leqslant n), \text{ and} \tag{3.1}$$

$$T_{k+1}^{(i)} := \frac{x_i T_k^{(i+1)} - x_{i+k+1} T_k^{(i)}}{x_i - x_{i+k+1}} \quad (1 \leqslant i \leqslant n-k-1), \tag{3.2}$$

for each $k = 0, 1, \ldots, n-2$, where $\{x_i\}_{i=1}^{n}$ are given constants. In this way, a

Table 2.4
The numbers $\{E_{n,n}(x^{1/2}; [0, 1])\}_{n=1}^{30}$ and the products $\{e^{\pi\sqrt{2n}} E_{n,n}(x^{1/2}; [0, 1])\}_{n=1}^{30}$

n	$E_{n,n}(x^{1/2}; [0, 1])$	$e^{\pi\sqrt{2n}} E_{n,n}(x^{1/2}; [0, 1])$
1	4.36890 12692 07636 15708 55971E-02	3.71442 65436 83164 13938 92631E+00
2	8.50148 47040 73829 49029 74113E-03	4.55247 41186 02959 57656 51746E+00
3	2.28210 60097 25259 48790 63105E-03	5.01604 81727 06945 03720 15671E+00
4	7.36563 61403 07030 56162 49126E-04	5.32413 85504 99584 35820 53531E+00
5	2.68957 06008 51835 09961 78760E-04	5.54906 50092 01360 99613 33338E+00
6	1.07471 16229 45128 49486 08235E-04	5.72308 60623 70144 61495 92486E+00
7	4.60365 92662 63495 95712 92708E-05	5.86316 39054 52748 12034 22807E+00
8	2.08515 86406 33032 71711 10359E-05	5.97921 97829 97610 91541 37699E+00
9	9.88933 46452 81424 38844 04320E-06	6.07751 03145 70501 70155 39294E+00
10	4.87595 75126 31913 24358 83035E-06	6.16220 95236 00211 83504 56017E+00
11	2.48559 02684 78211 11692 06258E-06	6.23622 66709 47615 95171 86439E+00
12	1.30437 75913 43073 65266 87704E-06	6.30166 18824 78634 86712 21713E+00
13	7.02231 99787 39775 69519 98002E-07	6.36007 54354 31155 68553 36475E+00
14	3.86755 77147 25902 02910 10816E-07	6.41265 47293 14846 16444 77940E+00
15	2.17398 78201 69794 32053 20496E-07	6.46032 20136 32057 12747 12311E+00
16	1.24477 08820 89507 19282 14596E-07	6.50380 62614 76199 86766 48135E+00
17	7.24786 33767 55536 96985 57389E-08	6.54369 25164 84556 95273 52868E+00
18	4.28546 45582 73508 21569 77870E-08	6.58045 66245 60485 10758 85491E+00
19	2.56989 67632 18081 61490 49674E-08	6.61449 02150 91157 33238 81633E+00
20	1.56132 88569 94866 81639 44414E-08	6.64611 90161 27510 21410 43688E+00
21	9.60112 26128 42236 48089 87184E-09	6.67561 65126 49122 88565 64179E+00
22	5.97082 33987 05558 05529 86137E-09	6.70321 42882 24997 72564 24257E+00
23	3.75238 13816 41316 36908 64502E-09	6.72910 99634 76020 91105 20998E+00
24	2.38149 96907 21783 08922 79694E-09	6.75347 33658 51186 98619 64983E+00
25	1.52547 32895 10979 37481 47207E-09	6.77645 13791 85256 90333 45348E+00
26	9.85676 33494 96352 99581 37413E-10	6.79817 17950 31113 66957 70741E+00
27	6.42135 80507 26624 69236 53248E-10	6.81874 64002 91279 67507 96788E+00
28	4.21588 48429 92714 57582 85061E-10	6.83827 34742 22969 81803 71436E+00
29	2.78832 41651 33927 54110 60214E-10	6.85683 98240 93862 32677 02634E+00
30	1.85707 20011 62821 79531 25707E-10	6.87452 24571 33671 11724 75540E+00

triangular table, consisting of $\frac{1}{2}n(n+1)$ entries, is created. As in [14], we chose here $x_i := 1/\sqrt{i}$ for $1 \leqslant i \leqslant n$, for all six values of α considered.

For completeness, for the case $\alpha = \frac{3}{8}$ we give in Table 3.1 the sixth and seventh columns of the Richardson extrapolation applied to the products $\{e^{\pi\sqrt{3n/2}} E_{n,n}(x^{3/8}; [0, 1])\}_{n=16}^{30}$, truncated to ten decimal places. We selected the sixth and seventh columns of this Richardson extrapolation because the entries in the sixth column are *strictly increasing*, while those in the seventh column are *strictly decreasing*. Thus (cf. (1.8)), it would appear from these columns that

$$6.21509\,57857\ldots \leqslant \lambda\left(\tfrac{3}{8}\right) \leqslant 6.21509\,59077\ldots, \tag{3.3}$$

Table 2.5
The numbers $\{E_{n,n}(x^{5/8}; [0, 1])\}_{n=1}^{30}$ and the products $\{e^{\pi\sqrt{5n/2}}E_{n,n}(x^{5/8}; [0, 1])\}_{n=1}^{30}$

n	$E_{n,n}(x^{5/8}; [0, 1])$	$e^{\pi\sqrt{5n/2}}E_{n,n}(x^{5/8}; [0, 1])$
1	2.81563 40178 89796 69134 80629E-02	4.04431 18463 32397 64491 10697E+00
2	4.39884 59937 94298 84308 75493E-03	4.94512 22387 98411 49837 59886E+00
3	9.99299 13600 45844 37747 16615E-04	5.44745 98944 76172 08064 63262E+00
4	2.80323 35026 68798 17914 83589E-04	5.78357 18970 72993 15340 39202E+00
5	9.04758 43608 12317 26004 86942E-05	6.03019 70052 62200 28209 76238E+00
6	3.23374 36696 99001 25688 22419E-05	6.22174 90184 08442 23590 76079E+00
7	1.25020 41633 32274 03541 52535E-05	6.37641 71745 62648 58461 68088E+00
8	5.14710 77182 94197 62615 38745E-06	6.50488 77309 73759 18870 53242E+00
9	2.23179 33665 44651 62794 51177E-06	6.61392 51874 24003 52392 90149E+00
10	1.01091 57740 00349 53766 30355E-06	6.70850 69712 69237 60396 57739E+00
11	4.75388 24008 90972 37318 37097E-07	6.79044 74851 39499 81452 00681E+00
12	2.30962 46433 62569 67592 97376E-07	6.86338 66244 82989 91243 78835E+00
13	1.15478 79619 66582 68178 61922E-07	6.92857 93343 28039 53285 67702E+00
14	5.92313 48269 84741 08435 91013E-08	6.98732 56118 68973 49784 01685E+00
15	3.10847 39921 66458 31784 78662E-08	7.04063 69116 67258 75580 17558E+00
16	1.66544 57215 83276 79470 41427E-08	7.08931 39718 47770 45574 58043E+00
17	9.09258 81216 27232 69415 43102E-09	7.13400 03501 46964 54648 01389E+00
18	5.05033 75907 32570 75399 70392E-09	7.17522 01699 24937 10117 22643E+00
19	2.84987 12925 97443 99203 70322E-09	7.21340 53130 72959 25494 25371E+00
20	1.63183 14971 83373 42321 63729E-09	7.24891 53862 74041 30666 18899E+00
21	9.47126 37317 57471 99180 21711E-10	7.28205 26309 89841 52353 71941E+00
22	5.56690 83437 77470 91671 65153E-10	7.31307 32273 09515 94692 51419E+00
23	3.31077 77107 53636 54895 10532E-10	7.34219 59813 44017 39786 53884E+00
24	1.99081 38748 34394 23189 82231E-10	7.36960 90845 98121 42531 49195E+00
25	1.20955 03157 09291 20091 30360E-10	7.39547 54325 17099 78795 24238E+00
26	7.42072 58043 74237 62774 10120E-11	7.41993 68524 26590 64029 30394E+00
27	4.59472 24867 60590 98385 51107E-11	7.44311 74962 35437 63852 09282E+00
28	2.86976 02279 32671 75616 54425E-11	7.46512 65865 59763 11948 76192E+00
29	1.80720 45140 52968 87690 73159E-11	7.48606 06573 27609 49017 30259E+00
30	1.14699 90145 12775 81199 68015E-11	7.50600 53955 10132 08044 35337E+00

whereas

$$2^{2(\alpha+1)}|\sin(\alpha\pi)|\Big|_{\alpha=\frac{3}{8}} = 6.21509\,58961\,20149\,22937\,66358\ldots, \tag{3.4}$$

which gives, in our opinion, strong numerical evidence for the conjecture in (1.11). The agreement between (3.3) and (3.4) for the case $\alpha = \frac{3}{8}$ is similar for the other values of α that we considered!

The success of the Richardson extrapolation method (with $x_i := 1/\sqrt{i}$) applied to the products $\{e^{\pi\sqrt{4\alpha n}}E_{n,n}(x^\alpha; [0, 1])\}_{n=1}^{30}$ also gives strong numerical evidence for the following new

Table 2.6
The numbers $\{E_{n,n}(x^{3/4}; [0, 1])\}_{n=1}^{30}$ and the products $\{e^{\pi\sqrt{3n}} E_{n,n}(x^{3/4}; [0, 1])\}_{n=1}^{30}$

n	$E_{n,n}(x^{3/4}; [0, 1])$	$e^{\pi\sqrt{3n}} E_{n,n}(x^{3/4}; [0, 1])$
1	1.64573 00979 36866 67238 48753E-02	3.79776 22853 48286 66761 74303E+00
2	2.07993 18985 91969 74041 55169E-03	4.57167 13223 77859 23185 55445E+00
3	4.04080 54008 43569 84709 29667E-04	5.00722 37387 58117 43799 92215E+00
4	9.95398 91774 98346 79827 76799E-05	5.30072 77031 77788 85273 50487E+00
5	2.86755 20825 93508 91777 22850E-05	5.51719 34381 46541 98517 21330E+00
6	9.25221 76379 86409 33197 58118E-06	5.68596 87880 36028 33521 69793E+00
7	3.25659 17752 62948 36060 66367E-06	5.82265 74299 34634 45817 47988E+00
8	1.22878 70122 09549 24707 37458E-06	5.93647 14082 15934 66413 39464E+00
9	4.90958 68093 97044 93813 61687E-07	6.03326 55367 79887 79572 13533E+00
10	2.05844 56555 42563 21726 29699E-07	6.11697 16127 15044 93540 86752E+00
11	8.99434 32683 01585 02725 00011E-08	6.19034 56084 16724 14868 66620E+00
12	4.07376 56882 36152 46638 78683E-08	6.25538 67954 19133 57549 76463E+00
13	1.90435 78794 40295 90898 67147E-08	6.31358 69513 82633 99204 46110E+00
14	9.15594 37731 32759 83337 03693E-09	6.36608 57074 21045 53923 00665E+00
15	4.51438 55241 25788 73109 23375E-09	6.41377 12550 07524 51924 31183E+00
16	2.27707 59596 07888 58990 60866E-09	6.45734 78237 87079 47627 10037E+00
17	1.17257 95969 16423 64148 22131E-09	6.49738 21903 31684 79588 45259E+00
18	6.15352 32440 68344 67313 55949E-10	6.53433 65279 01871 56370 01103E+00
19	3.28591 79546 59078 37535 26981E-10	6.56859 21104 01510 69512 44642E+00
20	1.78303 90487 48956 10620 07828E-10	6.60046 67427 51918 47242 27956E+00
21	9.82034 77931 88970 88464 10822E-11	6.63022 77949 79389 20410 58388E+00
22	5.48408 97473 39301 99629 46629E-11	6.65810 20963 57222 71106 25295E+00
23	3.10237 06679 81053 04393 27580E-11	6.68428 35483 06420 01467 38526E+00
24	1.77638 29002 84184 94145 47403E-11	6.70893 90542 05182 26872 93133E+00
25	1.02875 51750 21828 14096 15955E-11	6.73221 31899 58937 66435 68081E+00
26	6.02185 48086 32076 17834 21902E-12	6.75423 19203 81701 90022 11785E+00
27	3.56064 66962 98407 28719 96660E-12	6.77510 55840 74885 65419 16526E+00
28	2.12553 97693 16307 35693 62758E-12	6.79493 13114 90959 89064 03625E+00
29	1.28036 45461 75429 20742 62177E-12	6.81379 49994 41500 55988 83155E+00
30	7.77898 31723 45459 56241 82309E-13	6.83177 29353 26251 75596 24058E+00

CONJECTURE:

For $\alpha \in (0, 1)$, $e^{\pi\sqrt{4\alpha n}} E_l n,n(x^\alpha; [0, 1])$ *admits an asymptotic series expansion of the form*

$$e^{\pi\sqrt{4\alpha n}} E_{n,n}(x^\alpha; [0, 1]) \approx \lambda(\alpha) + \frac{K_1(\alpha)}{n^{\frac{1}{2}}} + \frac{K_2(\alpha)}{n} + \frac{K_3(\alpha)}{n^{\frac{3}{2}}} + \dots,$$

$$\text{as } n \to \infty, \tag{3.5}$$

where the constants $K_j(\alpha)$ $(j = 1, 2, 3, \dots)$ *are independent of* n, *and where* $\lambda(\alpha)$ *is given by* (1.11).

The special case $\alpha = 1/2$ of this conjecture also appears in [14].

Table 2.7
The numbers $\{E_{n,n}(x^{7/8}; [0, 1])\}_{n=1}^{30}$ and the products $\{e^{\pi\sqrt{7n/2}} E_{n,n}(x^{7/8}; [0, 1])\}_{n=1}^{30}$

n	$E_{n,n}(x^{7/8}; [0, 1])$	$e^{\pi\sqrt{7n/2}} E_{n,n}(x^{7/8}; [0, 1])$
1	7.32620 52883 13424 45762 67968E-03	2.61452 93126 00047 45977 79413E+00
2	7.50678 61351 85264 75619 12560E-04	3.05671 23395 85287 52230 08357E+00
3	1.25604 46206 38789 12046 50548E-04	3.31199 55913 04795 37533 45294E+00
4	2.73734 84514 54840 45345 87694E-05	3.48625 33712 75975 09331 21754E+00
5	7.08939 06706 52461 24013 82116E-06	3.61580 54831 25817 41482 87648E+00
6	2.07911 69496 19330 00620 92191E-06	3.71737 89623 74412 50137 53917E+00
7	6.70599 61378 35626 50438 37991E-07	3.79998 34243 63208 29678 70104E+00
8	2.33343 78398 69812 77405 70091E-07	3.86898 71756 61024 39354 14079E+00
9	8.64189 39460 19764 81578 71713E-08	3.92782 58711 92227 47215 66247E+00
10	3.37283 31494 98510 48359 57473E-08	3.97881 93479 37890 38912 11581E+00
11	1.37683 61357 39451 92623 44761E-08	4.02360 08987 33021 55581 44499E+00
12	5.84407 67205 02816 64169 51861E-09	4.06335 97371 61809 49600 87936E+00
13	2.56717 08308 62615 71408 08500E-09	4.09898 59819 40672 15155 99978E+00
14	1.16261 97398 33359 57065 55673E-09	4.13116 14630 82730 70939 97138E+00
15	5.41116 98897 66543 11353 16372E-10	4.16041 88256 71919 75483 31467E+00
16	2.58145 18994 11657 34172 23494E-10	4.18718 12694 48100 02206 97002E+00
17	1.25944 75910 42103 36642 76658E-10	4.21179 00223 23807 17852 39911E+00
18	6.27195 59138 40680 57060 23689E-11	4.23452 37973 82542 84551 71052E+00
19	3.18280 22404 32781 53649 64085E-11	4.25561 28663 93489 90748 09552E+00
20	1.64350 07014 06777 63822 33673E-11	4.27524 94308 34747 98005 44731E+00
21	8.62443 23929 47496 08245 21521E-12	4.29359 53922 34025 88315 82471E+00
22	4.59412 47613 07638 33348 92474E-12	4.31078 82610 67156 48125 92356E+00
23	2.48171 70971 12215 47751 71806E-12	4.32694 57106 47483 18354 97225E+00
24	1.35828 49843 19624 42877 47025E-12	4.34216 91295 14867 93351 76379E+00
25	7.52610 35519 80063 48346 86123E-13	4.35654 64233 04287 31621 41582E+00
26	4.21867 30808 55439 02050 66122E-13	4.37015 42470 47016 44257 14299E+00
27	2.39069 04657 66483 76980 75793E-13	4.38305 98002 15752 73113 75232E+00
28	1.36884 38789 85339 55755 74023E-13	4.39532 22825 13077 16566 66516E+00
29	7.91464 75901 00204 99780 36652E-14	4.40699 40838 54138 33856 67766E+00
30	3.61891 91127 06692 88013 47478E-14	4.41812 17642 00901 01448 28743E+00

Table 3.1
The sixth and seventh columns of the extrapolation table for $\alpha = \frac{3}{8}$

Sixth column	Seventh column
6.21509 54649	6.21509 59376
6.21509 55433	6.21509 59284
6.21509 56043	6.21509 59221
6.21509 56524	6.21509 59176
6.21509 56909	6.21509 59142
6.21509 57220	6.21509 59116
6.21509 57474	6.21509 59094
6.21509 57683	6.21509 59077
6.21509 57857	

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