Analysis 12, 319 – 333 (1992)

Analysis

ON A GENERALIZATION OF MAHLER'S INEQUALITY

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Received: **January 13, 1992**

ABSTRACT: We give here a generalization of an inequality, due to K. Mahler, which arises from an application of Jensen's formula to polynomials.

AMS 1980 Subject Classification: Primary 30A10; Secondary 30C10.

1. Introduction. For any function $f(z) = \sum_{j=0}^{\infty} a_j z^j \not\equiv 0$ which is analytic in $|z| \leq 1$, let a_N be its first nonzero Taylor coefficient. Then, the well-known *Jensen's formula* (cf. Ahlfors [1, p. 208]) for $f(z)$ is

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|d\theta = \log|a_N| + \sum_{z_j \in Z_\Delta(f)} \log\left(\frac{1}{|z_j|}\right), \tag{1.1}$$

where $Z_\Delta(f)$ denotes the set of all zeros (counting multiplicities) of $f(z)$ in $0 < |z| < 1$. For any complex polynomial $p_m(z) = \sum_{j=0}^{m} a_j z^j \not\equiv 0$, consider the functional $J(p_m)$ defined by

$$J(p_m) := \frac{1}{2\pi}\int_0^{2\pi} \log|p_m(e^{i\theta})|d\theta - \log\left(\sum_{j=0}^{m}|a_j|\right) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{p_m(e^{i\theta})}{\sum_{j=0}^{m}|a_j|}\right| d\theta. \tag{1.2}$$

If a_N is the first nonzero Taylor coefficient of $p_m(z)$ and if $\{\zeta_j\}_{j=1}^{r}$ denotes the zeros (counting

[1] Research supported by the National Science Foundation.

multiplicities) of $p_m(z)$ in $0 < |z| < 1$, Jensen's formula, applied to $p_m(z)$, gives

$$J(p_m) = \log \left\{ \frac{|a_N|}{\prod_{j=1}^{r} |\zeta_j| \cdot \sum_{j=0}^{m} |a_j|} \right\}, \tag{1.3}$$

where, as usual, the product above is defined to be unity if $r = 0$. Because of the form of the equation above, it is convenient for us to define the associated functional $K(p_m)$ as

$$K(p_m) := \frac{|a_N|}{\prod_{j=1}^{r} |\zeta_j| \cdot \sum_{j=0}^{m} |a_j|}, \tag{1.4}$$

so that

$$\exp J(p_m) = K(p_m). \tag{1.5}$$

Recalling that the *Mahler measure* in the literature is defined, for any $f(z)$ analytic in $|z| \leq 1$, as

$$M(f) := \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{i\theta})| d\theta \right\}, \tag{1.6}$$

it is interesting to note from the equations above that $K(p_m)$ of (1.4) can be equivalently expressed, in terms of the Mahler measure, as

$$M\left(\frac{p_m}{\sum_{j=0}^{m} |a_j|} \right) = K(p_m). \tag{1.7}$$

Some years ago, K. Mahler [4] obtained a lower bound for $K(p_m)$ which is dependent only on the degree of $p_m(z)$. We state his result in a slightly sharpened form:

THEOREM A. *For any complex polynomial $p_m(z) \not\equiv 0$ of degree at most n,*

$$K(p_m) \geq \frac{1}{2^n}, \tag{1.8}$$

with equality holding in (1.8) *only if $p_m(z) = \gamma(e^{i\psi} + z)^n$, where $\gamma \neq 0$ and ψ is real.*

We remark that the treatment of the case of equality in (1.8) appears in Rigler, Trimble, and Varga [5, Lemma 1]. With (1.7), we note that (1.8) can be expressed in terms of Mahler's measure as

$$M\left(\frac{p_m(z)}{\sum_{j=0}^{m} |a_j|} \right) \geq \frac{1}{2^n} \qquad (p_m(z) \not\equiv 0, p_m \text{ of degree at most } n). \tag{1.9}$$

Inequality (1.8) or (1.9) is known in the literature as *Mahler's inequality*, and it has important applications in the theory of transcendental numbers (cf. [4]). More recently, it has been used in developing the theory for polynomials possessing the Beauzamy-Enflo property (cf. [2] and [3]) of having *concentration at low degrees.* For a recent survey of this topic, see [6, Ch. 6].

2. Statement of Results. For our generalization of Mahler's inequality (1.8), we need the following notation. For any $p_m(z) = \sum_{j=0}^{m} a_j z^j \not\equiv 0$, define the nonnegative numbers $\delta_k(p_m)$ by

$$\delta_k(p_m) := \frac{\sum_{j=0}^{k} |a_j|}{\sum_{j=0}^{m} |a_j|} \qquad (k = 0, 1, \cdots), \tag{2.1}$$

so that $\delta_\ell(p_m) = 1$ for all $\ell \geq m$. We also define the ratios

$$L_k(p_m) := \frac{K(p_m)}{\delta_k(p_m)} \qquad (k = 0, 1, \cdots), \tag{2.2}$$

where $L_k(p_m) := +\infty$ if $\delta_k(p_m) = 0$.

It is natural to ask, on fixing a nonnegative integer k with $k \leq n$, if there is a lower bound, as in (1.8) of Theorem A, for the ratio $L_k(p_m)$ of (2.2) when $p_m(z) \not\equiv 0$ is a polynomial of degree at most n. Our main result, which gives such a lower bound, is

THEOREM 1. *For any complex polynomial $p_m(z) \not\equiv 0$ of degree at most n,*

$$L_k(p_m) \geq \frac{1}{\sum_{i=0}^{k} \binom{n}{i}} \qquad (k = 0, 1, \cdots, n), \tag{2.3}$$

where equality holds in (2.3) *for some k with $k \geq 1$ only if $p_m(z) = \gamma(e^{i\psi} + z)^n$ where $\gamma \neq 0$ and ψ is real, and for $k = 0$ only if $p_m(z)$ has no zeros in $0 \leq |z| < 1$.*

Since $\delta_m(p_m) = 1$ from (2.1) so that $L_m(p_m) = K(p_m)$, we see that the case $k = n$ of inequality (2.3) reduces exactly to Mahler's result of Theorem A. Actually, we shall prove a more general form of the inequality (2.3), in (3.39) of Theorem 2, from which Theorem 1 is obtained. The necessary constructions are given in §3 below.

3. The Three R's: Rotation, Reflection, and Reduction. For a any complex polynomial $p_m(z) \not\equiv 0$, it follows from (1.4), (2.1), and (2.2) that $L_k(\gamma p_m) = L_k(p_m)$ for any scalar

$\gamma \neq 0$; hence, we may assume that $p_m(z)$ is a monic polynomial. In addition, it follows from (2.2) that

$$L_k(p_m) = L_{k+s}(z^s p_m) \qquad (k = 0, 1, \cdots; s \text{ a nonnegative integer}). \tag{3.1}$$

But, if the desired result (2.3) of Theorem 1 is derived for complex monic polynomials $p_m(z)$ with the added hypothesis that $p_m(0) \neq 0$, then, because the right side of (2.3) is strictly *decreasing* in k $(0 \leq k < n)$, we see from (3.1) that the inequality of (2.3) would necessarily also hold for *any* complex polynomial $p_m(z) \not\equiv 0$ of degree at most n. Thus, in what follows, we assume that $p_m(z)$ is a monic polynomial of degree m with $p_m(0) \neq 0$ (where the restriction that $p_m(z)$ is of degree at most n will come later), and we write

$$p_m(z) = \prod_{j=1}^{m} (\zeta_j + z), \tag{3.2}$$

where the complex numbers ζ_j are ordered by modulus so that

$$0 < |\zeta_1| \leq \cdots \leq |\zeta_r| < 1 \leq |\zeta_{r+1}| \leq \cdots \leq |\zeta_m| \qquad (\text{where } 0 \leq r \leq m). \tag{3.3}$$

<u>**LEMMA 1.**</u> (Rotation). *For any complex monic polynomial* $p_m(z) = \sum_{j=0}^{m} a_j z^j = \prod_{j=1}^{m} (\zeta_j + z)$ *with* $p_m(0) \neq 0$ *and with* $m \geq 1$, *its unique associated rotated polynomial* $\overset{o}{p}_m(z)$, *defined by*

$$\overset{o}{p}_m(z) := \prod_{j=1}^{m} (|\zeta_j| + z), \tag{3.4}$$

satisfies

$$L_k(p_m) \geq L_k(\overset{o}{p}_m) \quad (k = 1, 2, \cdots, m) \quad \text{while} \quad L_0(p_m) = L_0(\overset{o}{p}_m). \tag{3.5}$$

More precisely,

$$L_k(p_m) > L_k(\overset{o}{p}_m) \qquad (k = 1, 2, \cdots, m), \tag{3.6}$$

unless the ζ_j*'s* $(1 \leq j \leq m)$ *all lie on a ray, i.e., there exists a real* ψ *such that* $\zeta_j = |\zeta_j| e^{i\psi}$ $(1 \leq j \leq m)$, *in which case equality holds throughout in* k *in* (3.5).

<u>**Proof.**</u> Let $\zeta := (\zeta_1, \zeta_2, \cdots, \zeta_m)$ denote the row vector, in $\mathbb{C}^m$, which is derived from the product representation for $p_m(z)$ in (3.2), and let $\sigma_i(\zeta)$ denote the i-th symmetric function $(i = 0, 1, \cdots, m)$ of the m components of ζ. From the definition of $p_m(z)$,

$$a_j = \sigma_{m-j}(\zeta) \qquad (j = 0, 1, \cdots, m),$$

and, with the ordering of (3.3), it follows from (1.4), (2.1), and (2.2) that

$$L_k(p_m) = \frac{|\sigma_m(\zeta)|}{\prod_{j=1}^{r} |\zeta_j| \cdot \sum_{i=0}^{k} |\sigma_{m-i}(\zeta)|} \qquad (k = 0, 1, \cdots, m). \tag{3.7}$$

Since $|\zeta_j| > 0$ for all $1 \le j \le m$ from (3.3), then $|\sigma_m(\zeta)| = \prod_{j=1}^{m} |\zeta_j| > 0$, and on dividing, (3.7) becomes

$$L_k(p_m) = \frac{1}{\prod_{j=1}^{r} |\zeta_j| \cdot \sum_{i=0}^{k} \left| \frac{\sigma_{m-i}(\zeta)}{\sigma_m(\zeta)} \right|} \qquad (k = 0, 1, \cdots, m).$$

On defining the vector $1/\zeta := (1/\zeta_1, 1/\zeta_2, \cdots, 1/\zeta_m)$, it is evident that

$$\frac{\sigma_{m-i}(\zeta)}{\sigma_m(\zeta)} = \sigma_i(\frac{1}{\zeta}) \qquad (i = 0, 1, \cdots, m),$$

so that

$$L_k(p_m) = \frac{1}{\prod_{j=1}^{r} |\zeta_j| \cdot \sum_{i=0}^{k} |\sigma_i(\frac{1}{\zeta})|} \qquad (k = 0, 1, \cdots, m). \tag{3.8}$$

In a completely analogous way, we see from (3.4) (where each ζ_j is *rotated* to $|\zeta_j|$ on the positive real axis) that

$$L_k(\overset{o}{p}_m) = \frac{1}{\prod_{j=1}^{r} |\zeta_j| \cdot \sum_{i=0}^{k} \sigma_i(|\frac{1}{\zeta}|)} \qquad (k = 0, 1, \cdots, m), \tag{3.9}$$

where $|1/\zeta| := (1/|\zeta_1|, 1/|\zeta_2|, \cdots, 1/|\zeta_m|)$. But, since $\sigma_0(\cdot) \equiv 1$ and since the triangle inequality clearly gives

$$|\sigma_i(\frac{1}{\zeta})| \le \sigma_i(\frac{1}{|\zeta|}) \qquad (i = 1, 2, \cdots, m), \tag{3.10}$$

it is evident from (3.8) and (3.9) that

$$L_k(p_m) \ge L_k(\overset{o}{p}_m) \qquad (k = 1, 2, \cdots, m) \text{ while } L_0(p_m) = L_0(\overset{o}{p}_m), \tag{3.11}$$

which gives the desired result of (3.5) of Lemma 1.

To complete the proof of Lemma 1, consider the case $i = 1$ of (3.10). The triangle inequality again gives

$$|\sigma_1(\frac{1}{\zeta})| = |\frac{1}{\zeta_1} + \cdots + \frac{1}{\zeta_m}| \le \frac{1}{|\zeta_1|} + \cdots + \frac{1}{|\zeta_m|} = \sigma_1(|\frac{1}{\zeta}|),$$

with equality holding above only if all the components $1/\zeta_j$ lie on a ray, i.e., there is a real number ψ such that

$$\zeta_j = |\zeta_j| e^{i\psi} \qquad (j = 1, 2, \cdots, m). \tag{3.12}$$

If (3.12) holds, then $|\sigma_j(\frac{1}{\zeta})| = \sigma_j(|\frac{1}{\zeta}|)$ for all $0 \le j \le m$, and equality holds throughout in (3.11). On the other hand, if $|\sigma_1(\frac{1}{\zeta})| < |\sigma_1(|\frac{1}{\zeta}|)|$, then because $|\sigma_1(\frac{1}{\zeta})|$ and $\sigma_1(|\frac{1}{\zeta}|)$ occur, respectively, in all the sums in (3.8) and (3.9) when $1 \le k \le m$, then (3.6) is valid. □

LEMMA 2. (Reflection). *For the rotated polynomial* $\overset{o}{p}_m(z) = \prod_{j=1}^{m}(|\zeta_j| + z)$ *of* (3.4), *define its unique associated reflected polynomial* $\tilde{p}_m(z)$ *by*

$$\tilde{p}_m(z) := \begin{cases} \prod_{j=1}^{r}(\frac{1}{|\zeta_j|} + z) \cdot \prod_{j=r+1}^{m}(|\zeta_j| + z) \text{ when } r \ge 1; \\ \overset{o}{p}_m(z) \text{ when } r = 0, \end{cases} \tag{3.13}$$

where r *is defined in* (3.3). *If* $r \ge 1$, *then*

$$K(\tilde{p}_m) = K(\overset{o}{p}_m), \tag{3.14}$$

and

$$\delta_k(\tilde{p}_m) > \delta_k(\overset{o}{p}_m) \quad (k = 0, 1, \cdots, m-1) \text{ while } \delta_m(\tilde{p}_m) = \delta_m(\overset{o}{p}_m) = 1, \tag{3.15}$$

so that

$$L_k(\overset{o}{p}_m) > L_k(\tilde{p}_m) \quad (k = 0, 1, \cdots, m-1) \text{ while } L_m(\overset{o}{p}_m) = L_m(\tilde{p}_m). \tag{3.16}$$

Proof. If $r = 0$, there is nothing to prove, so assume that $r \ge 1$. Set $|\zeta_1| = R$ where (cf. (3.3)) $0 < R < 1$, and set

$$Q(z) := \prod_{i=2}^{m}(|\zeta_j| + z) =: \sum_{j=0}^{m-1} b_j z^j \text{ with } b_{-1} := 0 =: b_m,$$

where $Q(z) := 1$ if $m = 1$. By definition, $b_j > 0$ for all $0 \le j \le m-1$, and

$$\overset{o}{p}_m(z) = (R + z)Q(z) = \sum_{j=0}^{m}(Rb_j + b_{j-1})z^j.$$

We then define the new polynomial $g(z)$ by

$$g(z) := (\frac{1}{R} + z)Q(z) = \sum_{j=0}^{m}(\frac{b_j}{R} + b_{j-1})z^j,$$

(where R is *reflected* in the unit circle to $1/R$). As is readily seen from the definition in (1.4),

$$K(\overset{o}{p}_m) = \frac{Rb_0}{\prod_{j=1}^{r}|\zeta_j| \cdot \sum_{j=0}^{m}(Rb_j + b_{j-1})} = \frac{b_0}{\prod_{j=2}^{r}|\zeta_j| \cdot \sum_{j=0}^{m}(Rb_j + b_{j-1})}, \tag{3.17}$$

while

$$K(g) = \frac{b_0/R}{\prod_{j=2}^{r}|\zeta_j| \cdot \sum_{j=0}^{m}(\frac{b_j}{R} + b_{j-1})} = \frac{b_0}{\prod_{j=2}^{r}|\zeta_j| \cdot \sum_{j=0}^{m}(b_j + Rb_{j-1})}. \tag{3.18}$$

Now, because $b_{-1} := 0 =: b_m$, it can be verified that

$$\sum_{j=0}^{m}(Rb_j + b_{j-1}) = \sum_{j=0}^{m}(b_j + Rb_{j-1}); \tag{3.19}$$

whence from (3.17) and (3.18),

$$K(g) = K(\overset{o}{p}_m). \tag{3.20}$$

Next, from the definitions of $\overset{o}{p}_m(z)$ and $g(z)$, we have (cf. (2.1))

$$\delta_k(\overset{o}{p}_m) = \frac{\sum_{j=0}^{k}(Rb_j + b_{j-1})}{\sum_{j=0}^{m}(Rb_j + b_{j-1})} \text{ and } \delta_k(g) = \frac{\sum_{j=0}^{k}(b_j + Rb_{j-1})}{\sum_{j=0}^{m}(b_j + Rb_{j-1})},$$

where, from (3.19), the above two fractions have the same denominators. It can be verified that the numerators of the fractions above satisfy the inequality

$$\sum_{j=0}^{k}(Rb_j + b_{j-1}) < \sum_{j=0}^{k}(b_j + Rb_{j-1}) \qquad (k = 0, 1, \cdots, m-1),$$

if and only if $b_k > Rb_k$. But this last inequality is certainly true since $b_k > 0$ for $0 \le k \le m-1$ and since $0 < R < 1$. Thus,

$$\delta_k(g) > \delta_k(\overset{o}{p}_m) \qquad (k = 0, 1, \cdots, m-1), \text{ while } \delta_m(\overset{o}{p}_m) = \delta_m(g) = 1. \tag{3.21}$$

Now, the above mapping of the factor $(|\delta_1| + z)$ into $(1/|\delta_1| + z)$ produces the polynomial $g(z)$ for which (3.20) and (3.21) are valid. But this procedure can be successively applied to each of any remaining factors $\{(|\zeta_j| + z)\}_{j=2}^{r}$ of $\overset{o}{p}_m(z)$, and the above analysis shows that the final resulting reflected polynomial, $\tilde{p}_m(z)$, then similarly satisfies (3.20) and (3.21), which gives the desired results of (3.14) and (3.15). □

LEMMA 3. (Reduction). *For the reflected polynomial* $\tilde{p}_m(z) := \prod_{j=1}^{m}(t_j + z)$ *of* (3.13), *assume that the* t_j's, *which satisfy* $t_j \ge 1$ $(j = 1, 2, \cdots, m)$, *are ordered so that*

$$t_1 \ge t_2 \ge \cdots \ge t_s > 1 = t_{s+1} = t_{s+2} = \cdots = t_m \qquad (\textit{where } 0 \le s \le m). \tag{3.22}$$

Then, there is a unique associated reduced polynomial $\hat{p}_m(z)$ *of the form*

$$\hat{p}_m(z) = (\rho' + z)(1 + z)^{m'-1} \textit{ with } 1 \le \rho' < \infty \textit{ and with } 1 \le m' \le m, \tag{3.23}$$

such that if $s = 0$ *or* $s = 1$, *then* $\hat{p}_m(z) \equiv \tilde{p}_m(z)$, *so that*

$$L_k(\hat{p}_m) = L_k(\tilde{p}_m) \qquad (k = 0, 1, \cdots, m), \tag{3.24}$$

and if $s \geq 2$, then

$$K(\hat{p}_m) = K(\tilde{p}_m), \tag{3.25}$$

and

$$\delta_k(\hat{p}_m) > \delta_k(\tilde{p}_m) \quad (k = 1, 2, \cdots, m-1) \ \ while \ \ \delta_k(\hat{p}_m) = \delta_k(\tilde{p}_m) \quad (k = 0, m), \tag{3.26}$$

so that

$$L_k(\tilde{p}_m) > L_k(\hat{p}_m) \quad (k = 1, 2, \cdots, m-1) \ \ while \ \ L_k(\tilde{p}) = L_k(\hat{p}_m) \quad (k = 0, m). \tag{3.27}$$

Proof. If $s = 0$ or $s = 1$, it is immediately evident from (3.22) that $\tilde{p}_m(z)$ is of the form (3.23) with $m' = m$.

For the case $s \geq 2$, as the constructions are similar, but not identical, to that of the last part of the proof in [5, Lemma 2], we merely sketch the remainder of the proof.

If $s \geq 2$, then $t_1 \geq t_2 > 1$ from (3.22), and we write

$$\tilde{p}_m(z) = (t_1 + z)(t_2 + z)Q_1(z), \text{ where } Q_1(z) := \prod_{j=3}^{m}(t_j + z) =: \sum_{j=0}^{m-2} q_j z^j. \tag{3.28}$$

It follows that $q_j > 0$ for all $0 \leq j \leq m-2$, and we set $q_{-2} := q_{-1} := 0 =: q_{m-1} =: q_m$. Next, with $t_1 > 1$ and $t_2 > 1$, assume that

$$1 + t_1 + t_2 - t_1 t_2 \leq 0, \tag{3.29}$$

and define $\tilde{t}_3$ by

$$1 + \frac{1}{\tilde{t}_3} = (1 + \frac{1}{t_1})(1 + \frac{1}{t_2}). \tag{3.30}$$

A calculation using (3.29) and (3.30) shows that $\tilde{t}_3 \geq 1$. With $\tilde{t}_3$, define

$$g_1(z) := (\tilde{t}_3 + z)Q_1(z) \qquad \text{(a polynomial of degree } m-1\text{)},$$

so that $g_1(z)$ and $\tilde{p}_m(z)$ both have positive Taylor coefficients, and all zeros of $g_1(z)$ and $\tilde{p}_m(z)$ lie in $|z| \geq 1$. Consequently (cf. (1.4)),

$$K(g_1) = \frac{\tilde{t}_3 Q_1(0)}{(\tilde{t}_3 + 1)Q_1(1)} \text{ and } K(\tilde{p}_m) = \frac{t_1 t_2 Q_1(0)}{(t_1 + 1)(t_2 + 1)Q_1(1)},$$

and, with the definition of $\tilde{t}_3$ in (3.30), it can be verified that

$$K(g_1) = K(\tilde{p}_m). \tag{3.31}$$

Next, we similarly have (cf. (2.1)) that

$$\delta_k(g_1) = \frac{\sum_{j=0}^{k}(\tilde{t}_3 q_j + q_{j-1})}{(\tilde{t}_3 + 1)Q_1(1)} \text{ and } \delta_k(\tilde{p}_m) = \frac{\sum_{j=0}^{k}[t_1 t_2 q_j + (t_1 + t_2)q_{j-1} + q_{j-2}]}{(t_1 + 1)(t_2 + 1)Q_1(1)},$$

for any $k = 0, 1, \cdots, m$, and a short calculation, using (3.30), shows that

$$\delta_k(g_1) > \delta_k(\tilde{p}_m) \quad (k = 1, 2, \cdots, m-1) \text{ while } \delta_k(g_1) = \delta_k(\tilde{p}_m) \quad (k = 0, m). \tag{3.32}$$

Next, assume that $t_1 > 1$ and $t_2 > 1$ do not satisfy (3.29), i.e.,

$$1 + t_1 + t_2 - t_1 t_2 > 0. \tag{3.33}$$

In this case, the construction above is modified in the following way. Define $\tilde{t}_4$ by

$$2(1 + \frac{1}{\tilde{t}_4}) = (1 + \frac{1}{t_1})(1 + \frac{1}{t_2}). \tag{3.34}$$

A similar calculation using (3.33) and (3.34) shows that $\tilde{t}_4 > 1$ if and only if $2t_1t_2/[1 + t_1 + t_2 - t_1t_2] > 1$. To show that $\tilde{t}_4 > 1$, consider the function

$$w(x, y) := \frac{2xy}{1 + x + y - xy} \text{ for all } x \geq 1 \text{ and all } y \geq 1 \text{ for which } 1 + x + y - xy > 0.$$

Note that $x = 1$ and any $y \geq 1$, and similarly $y = 1$ and any $x \geq 1$, satisfy these constraints. Since $w_x(x, y) > 0$ and $w_y(x, y) > 0$ for all admissible pairs (x, y) which satisfy these constraints, then

$$\min\{w(x, y) : x \geq 1, y \geq 1 + x + y - xy > 0\} = 1 = w(1, 1),$$

and, moreover, for any admissible pair (x, y),

$$w(x, y) > 1, \text{ unless } x = y = 1. \tag{3.35}$$

Consequently, as $\tilde{t}_4 > 1$ if and only if $w(t_1, t_2) > 1$, where (by hypothesis) $t_1 > 1$ and $t_2 > 1$, it follows from (3.35) that $\tilde{t}_4 > 1$.

With the definition of $\tilde{t}_4$ in (3.34), we define in this case the polynomial

$$g_2(z) := (\tilde{t}_4 + z)(1 + z)Q_1(z) \qquad \text{(a polynomial of degree } m),$$

where $Q_1(z)$ is given in (3.28). In a completely similar fashion to the construction involving $g_1(z)$, we deduce that

$$K(g_2) = K(\tilde{p}_m), \tag{3.36}$$

and

$$\delta_k(g_2) > \delta_k(\tilde{p}_m) \quad (k = 1, 2, \cdots, m-1) \text{ and } \delta_k(g_2) = \delta_k(\tilde{p}_m) \quad (k = 0, m). \tag{3.37}$$

The above construction shows that the quadratic factor $(t_1 + z)(t_2 + z)$ of $\tilde{p}_m(z)$, with $t_1 > 1$ and $t_2 > 1$, is mapped either into $(\tilde{t}_3 + z)$ with $\tilde{t}_3 \geq 1$ when (3.29) is valid, or into

$(\tilde{t}_4 + z)(1+z)$ with $\tilde{t}_4 > 1$ when (3.33) is valid, thereby producing the polynomial $g_1(z)$ or $g_2(z)$ which satisfies, respectively, either (3.31)-(3.32) or (3.36)-(3.37). Note that the total number (namely, s from (3.22)) of zeros of modulus greater than unity of $\tilde{p}_m(z)$, has been *reduced* by at least one in passing from $\tilde{p}_m(z)$ to $g_1(z)$ or $g_2(z)$. Further, if we call the result of the above construction simply $g(z)$ (i.e., $g(z) = g_1(z)$ if (3.29) is valid, or $g(z) = g_2(z)$ if (3.33) is valid), it may be the case that the resultant polynomial $g(z)$ also has two zeros, each of modulus greater than unity. In this case, the construction above can be *repeated*, thereby producing the polynomial $h(z)$ which similarly satisfies (cf. (3.31)-(3.32) or (3.36)-(3.37))

$$K(h) = K(g) = K(\tilde{p}_m) \tag{3.38}$$

and

$$\delta_k(h) \geq \delta_k(g) > \delta_k(\tilde{p}_m) \quad (k = 1, 2, \cdots, m-1) \text{ while } \delta_k(h) = \delta_k(\tilde{p}_m) \quad (k = 0, m). \tag{3.39}$$

In this way, after a finite number of steps, we arrive at the unique *reduced* polynomial $\hat{p}_m(z) = (\rho' + z)(1+z)^{m'-1}$ of (3.23), where $1 \leq \rho' < \infty$ and where $1 \leq m' \leq m$, and this polynomial satisfies the desired results of (3.25)-(3.27). □

For the particular polynomial

$$\hat{p}_m(z) := g_{m',\rho'}(z) := (\rho' + z)(1+z)^{m'-1} \qquad (1 \leq \rho' < \infty \text{ and } 1 \leq m' \leq m), \tag{3.40}$$

derived above in Lemmas 1-3 from the monic polynomial $p_m(z)$ with $p_m(0) \neq 0$, a calculation shows that

$$L_k(\hat{p}_m) = L_k(g_{m',\rho'}) = \begin{cases} \dfrac{1}{\sum_{j=0}^{k} \binom{m'-1}{j} + \dfrac{1}{\rho'} \sum_{j=1}^{k} \binom{m'-1}{j-1}} & (k = 0, 1, \cdots, m'), \\[2ex] \dfrac{1}{2^{m'-1}(1 + \frac{1}{\rho'})} = K(g_{m',\rho'}) & (k \geq m'). \end{cases} \tag{3.41}$$

Thus, from the results of Lemmas 1-3, we immediately have

<u>**THEOREM 2.**</u> *Given any complex monic polynomial* $p_m(z) = \sum_{j=0}^{m} a_j z^j = \prod_{j=1}^{m} (\zeta_j + z)$ *with* $p_m(0) \neq 0$ *and with* $m \geq 1$, *there is a unique associated reduced polynomial* $\hat{p}_m(z) = (\rho' + z)(1+z)^{m'-1}$ *with* $1 \leq \rho' < \infty$ *and* $1 \leq m' \leq m$, *such that*

$$L_k(p_m) \geq L_k(\hat{p}_m) \qquad (k = 0, 1, \cdots, m), \tag{3.42}$$

where $L_k(\hat{p}_m)$ *is given by* (3.41). *In addition, with the definitions of* (2.2), *the inequalities of*

(3.42) *imply that*

$$K(p_m) \geq \max_{0\leq k\leq m} \{\delta_k(p_m)\cdot L_k(\hat{p}_m)\}. \tag{3.43}$$

In essence, (3.42) shows that to minimize any $L_k(p_m)$ $(k=0,1,\cdots,m)$ over the set of all complex monic polynomials $p_m(z)$, with $p_m(0)\neq 0$, of degree at most n, it suffices to consider only the special monic polynomials

$$g_{m',\rho'}(z) := (\rho'+z)(1+z)^{m'-1} \text{ where } 1\leq \rho' < \infty \text{ and } 1\leq m' \leq n.$$

In particular, we see from (3.41) that each $L_k(g_{m',\rho'})$ is, for fixed m', a *strictly increasing function* of ρ' on the interval $[1,+\infty)$, provided that $1\leq k\leq n$, while $L_0(g_{m',\rho'})\equiv 1$, and that $L_k(g_{m',\rho'})$ is, for fixed ρ', a *strictly decreasing function* of $m'\geq 1$, provided that $k\geq 1$. From these observations, we come to the

Proof of Theorem 1. From the strictly increasing and decreasing nature of $L_k(g_{m',\rho'})$, respectively, as a function of $\rho'\geq 1$ and $m'\geq 1$, it follows that $g_{n,1}(z)=(1+z)^n$, as a monic polynomial of degree n, minimizes *each* $L_k(g_{m',\rho'})$, where $1\leq m'\leq n$ and $1\leq\rho'<\infty$, i.e., from (3.41) and (3.42),

$$L_k(p_m) \geq L_k(g_{n,1}) = \frac{1}{\sum_{j=0}^{k}\binom{n-1}{j}+\sum_{j=1}^{k}\binom{n-1}{j-1}} = \frac{1}{\sum_{j=0}^{k}\binom{n}{j}} \qquad (k=0,1,\cdots,n) \tag{3.44}$$

for any complex monic polynomial $p_m(z)$ with $p_m(0)\neq 0$ and $m\geq 1$, the last equality in (3.44) following from the identity $\binom{n-1}{j}+\binom{n-1}{j-1}=\binom{n}{j}$. But as observed in the introduction of §3, (3.44) then actually holds for any complex polynomial $p_m(z)\not\equiv 0$ of degree at most n, which establishes (2.3) of Theorem 1. Also, from Lemmas 1-3, we see that equality holds in (3.44) for some $1\leq k\leq n$ only if $p_m(z)=\gamma(e^{i\psi}+z)^n$ with $\gamma\neq 0$ and ψ real. For the case $k=0$ of (3.44), we have from (1.4) and (2.2) that

$$L_0(p_m) = \frac{|a_N|}{\prod_{j=1}^{r}|\zeta_j|\cdot|a_0|}, \tag{3.45}$$

where $\{\zeta_j\}_{j=1}^{r}$ denotes the zeros of $p_m(z)$ in $0<|z|<1$. The ratio in (3.45) is bounded only if $a_0\neq 0$, in which case

$$L_0(p_m) = \frac{1}{\prod_{j=1}^{r}|\zeta_j|} \geq 1, \tag{3.46}$$

with equality holding above only if $p_m(z)$ has no zeros in $0\leq|z|<1$. □

4. Another Lower Bound for $K(p_m)$. While (3.42) of Theorem 2 gives a lower bound for $K(p_m)$, it is possible to derive a lower bound for $K(p_m)$ which does not depend upon the calculation of an associated reduced polynomial $\hat{p}_m(z)$. This is given in

THEOREM 3. *For any complex polynomial $p_m(z) \not\equiv 0$ of degree at most n,*

$$K(p_m) \geq \max_{0 \leq k \leq n} \left\{ \frac{\delta_k(p_m)}{\sum_{j=0}^{k} \binom{n}{j}} \right\}, \tag{4.1}$$

with equality holding in (4.1) for some k with $k \geq 1$ only if $p_m(z) = \gamma(e^{i\phi} + z)^n$ where $\gamma \neq 0$ and ψ is real, and for $k = 0$ only if $p_m(z)$ has no zeros in $0 \leq |z| < 1$.

Proof. From (2.2) and (3.44),

$$\frac{K(p_m)}{\delta_k(p_m)} = L_k(p_m) \geq \frac{1}{\sum_{j=0}^{k} \binom{n}{j}} \qquad (k = 0, 1, \cdots, n), \tag{4.2}$$

so that

$$K(p_m) \geq \frac{\delta_k(p_m)}{\sum_{j=0}^{k} \binom{n}{j}} \text{ for each } \delta_k(p_m) > 0, \text{ where } 0 \leq k \leq n,$$

from which (4.1) directly follows. If equality holds in (4.2),let ℓ be any nonnegative integer with $0 \leq \ell \leq n$ which gives equality in (4.2), i.e.,

$$L_\ell(p_m) = \frac{1}{\sum_{j=0}^{\ell} \binom{n}{j}}. \tag{4.3}$$

If (4.3) holds with $\ell \geq 1$, it necessarily follows from Theorem 1 that $p_m(z) = \gamma(e^{i\psi} + z)^n$, with $\gamma \neq 0$ and ψ real, so that in fact

$$L_k(p_m) = \frac{1}{\sum_{j=0}^{k} \binom{n}{j}} \qquad \text{for } all \quad k = 0, 1, \cdots, n.$$

If (4.3) holds with $\ell = 0$, then $L_0(p_m) = 1$. But from (3.46), this implies that $p_m(z)$ that has no zeros in $0 \leq |z| < 1$. □

We first remark that the lower bound for $K(p_m)$ of (3.43) is always at least as good as the lower bound of (4.1). However, the computation of the right side of inequality (4.1) is easy, since it only depends on the moduli of the Taylor coefficients of $p_m(z)$ and does not require finding the zeros of $p_m(z)$. In addition, we are assured that the lower bound of $K(p_m)$ in (4.1) is *at least as good* as Mahler's lower bound (1.8), which corresponds to the single term $k = n$ in the braces of (4.1).

5. Some Examples. To illustrate the above results, consider the monic complex polynomial

$$p_5(z) := (1+z)^3(-4+z^2) = -4 - 12z - 11z^2 - z^3 + 3z^4 + z^5, \tag{5.1}$$

for which $p_5(0) \neq 0$. For this polynomial, we readily determine that

$$\begin{cases} \overset{o}{p}_5(z) & \equiv & \tilde{p}_5(z) = (1+z)^3(2+z)^2 = 4 + 16z + 25z^2 + 19z^3 + 7z^4 + z^5, \\ \hat{p}_5(z) & = & (8+z)(1+z)^4 = 8 + 33z + 52z^2 + 38z^3 + 12z^4 + z^5. \end{cases} \tag{5.2}$$

In Table 1, we give the associated values of $K(\cdot)$ and $\delta_k(\cdot)$ for $k = 0, 1, \cdots, 5$ for the three polynomials $p_5(z), \overset{o}{p}_5(z)$, and $\hat{p}_5(z)$. The entries in all tables below have been truncated to six decimal digits.

TABLE 1

p	$\delta_0(p)$	$\delta_1(p)$	$\delta_2(p)$	$\delta_3(p)$	$\delta_4(p)$	$\delta_5(p)$	$K(p)$
p_5	0.125 000	0.500 000	0.843 750	0.875 000	0.968 750	1	0.125 000
$\overset{o}{p}_5$	0.055 555	0.277 777	0.625 000	0.888 888	0.986 111	1	0.055 555
$\hat{p}_5$	0.055 555	0.284 722	0.645 833	0.909 722	0.993 055	1	0.055 555

In Table 2, we similarly give the values of $L_k(\cdot)$ for $k = 0, 1, \cdots, 5$ for these 3 polynomials.

TABLE 2

p	$L_0(p)$	$L_1(p)$	$L_2(p)$	$L_3(p)$	$L_4(p)$	$L_5(p)$
p_5	1	0.250 000	0.148 148	0.142 857	0.129 032	0.125 000
$\overset{o}{p}_5$	1	0.200 000	0.088 888	0.062 500	0.056 338	0.055 555
$\hat{p}_5$	1	0.195 121	0.086 021	0.061 068	0.055 944	0.055 555

These tables have been included to reflect the contents of Lemmas 1-3.

For the polynomial $p_5(z)$ of (5.1), the lower bounds for $K(p)$ of (3.43) and (4.1) for $p_5(z)$ are both *sharp*, giving the $K(p_5) = 0.125\ 000$, while Mahler's inequality of (1.8) gives $K(p_5) \geq \frac{1}{32} = 0.031\ 250$.

Consider next the monic complex polynomial

$$h_5(z) := (1+z)^2(\frac{1}{2}+z)(\frac{1}{3}+z)(-4+z), \tag{5.3}$$

for which $h_5(0) \neq 0$. For this polynomial, we determine that

$$(5.4)\qquad \begin{cases} \overset{o}{h}_5(z) &= (1+z)^2(\frac{1}{2}+z)(\frac{1}{3}+z)(4+z), \\ \tilde{h}_5(z) &= (1+z)^2(2+z)(3+z)(4+z), \\ \hat{h}_5(z) &= (4+z)(1+z)^3. \end{cases}$$

In Tables 3 and 4, we give the corresponding values of $K(\cdot), \delta_k(\cdot)$, and $L_k(\cdot)$.

TABLE 3

p	$\delta_0(p)$	$\delta_1(p)$	$\delta_2(p)$	$\delta_3(p)$	$\delta_4(p)$	$\delta_5(p)$	$K(p)$
h_5	0.025 641	0.198 717	0.589 743	0.916 666	0.961 538	1	0.153 846
$\overset{o}{h}_5$	0.016 666	0.137 500	0.450 000	0.804 166	0.975 000	1	0.100 000
$\tilde{h}_5$	0.100 000	0.408 333	0.762 500	0.950 000	0.995 833	1	0.100 000
$\hat{h}_5$	0.100 000	0.425 000	0.800 000	0.975 000	1	1	0.100 000

TABLE 4

p	$L_0(p)$	$L_1(p)$	$L_2(p)$	$L_3(p)$	$L_4(p)$	$L_5(p)$
h_5	6	0.774 193	0.260 869	0.167 832	0.160 000	0.153 846
$\overset{o}{h}_5$	6	0.727 272	0.222 222	0.124 352	0.102 564	0.100 000
$\tilde{h}_5$	1	0.244 897	0.131 147	0.105 263	0.100 418	0.100 000
$\hat{h}_5$	1	0.235 294	0.125 000	0.102 564	0.100 000	0.100 000

For the polynomial $h_5(z)$ of (5.3), we have that $K(h_5) = 0.153\ 846$, and the lower bounds for $K(h_5)$ from (3.43) and (4.1) are, respectively, 0.100 000 and 0.036 858, while Mahler's inequality of (1.6) gives $K(h_5) \geq \frac{1}{32} = 0.031\ 250$.

References

1. Ahlfors, L.V.: **Complex Analysis**, Third edition, New York:McGraw-Hill 1979.
2. Beauzamy, B., Enflo, P.: Estimations de produits de polynômes. J. Number Theory **21**, 390-412 (1985).
3. Beauzamy, B.: Jensen's inequality for polynomials with concentration at low degree. Numer. Math. **49**, 221-225(1986).

4. Mahler, K.: An application of Jensen's formula to polynomials. Mathematika **7**, 98-100 (1960).

5. Rigler, A.K., Trimble, S.Y., Varga, R.S.: Sharp lower bounds for a generalized Jensen inequality. Rocky Mountain J. Math **19**, 353-373 (1989).

6. Varga, R.S.: **Scientific Computation on Mathematical Problems and Conjectures**, CBMS-NSF Regional Conference Series in Applied Math., #60. Philadelphia: Society for Industrial and Applied Mathematics 1990.

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