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A Lower Bound for the de Bruijn-Newman Constant Λ. II

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ABSTRACT A new constructive method is given here for determining lower bounds for the de Bruijn-Newman constant Λ, which is related to the Riemann Hypothesis. This method depends on directly tracking real and nonreal zeros of an entire function $F_\lambda(z)$, where $\lambda < 0$, instead of finding, as was previously done, nonreal zeros of associated Jensen polynomials. We apply this new method to obtain the new lower bound for Λ,

$$-0.385 < \Lambda,$$

which improves previous published lower bounds of -50 and -5.

1 Introduction

The purposes of this paper are i) to give a new constructive method for finding lower bounds for the *de Bruijn-Newman constant* Λ, which is related to the Riemann Hypothesis, and ii) to apply this method to obtain a new lower bound for Λ. This new lower bound (to be given below) is the best constructive lower bound for Λ known to us at this time.

By way of background, in Csordas, Norfolk, and Varga [4], the entire function $H_\lambda(x)$ was defined by

$$H_\lambda(x) := \int_0^\infty e^{\lambda t^2}\Phi(t)\cos(xt)dt \qquad (\lambda \in \mathbb{R}), \tag{1.1}$$

where

$$\Phi(t) := \sum_{n=1}^{\infty}(2n^4\pi^2e^{9t} - 3n^2\pi e^{5t})\exp(-n^2\pi e^{4t}) \qquad (0 \le t < \infty). \tag{1.2}$$

It is known (cf. Pólya [12] or Csordas, Norfolk, and Varga [3, Theorem A])

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that Φ satisfies the following properties:

$$\begin{cases} \text{i)} & \Phi(z) \text{ is analytic in the strip } -\pi/8 < Im\ z < \pi/8; \\ \text{ii)} & \Phi(t) = \Phi(-t) \text{ and } \Phi(t) > 0 \quad (t \in \mathbf{R}); \\ \text{iii)} & \text{for any } \varepsilon > 0, \lim_{t\to\infty} \Phi^{(n)}(t) \exp[(\pi - \varepsilon)e^{4t}] = 0 \quad (n = 0, 1, \ldots). \end{cases} \tag{1.3}$$

It was also shown in [4, Appendix A] that, for *each* $\lambda \in \mathbf{R}$, $H_\lambda(x)$, as defined in (1.1), is an entire function of order 1 and of maximal type (i.e., its type, σ_λ, satisfies $\sigma_\lambda = +\infty$).

For the choice $\lambda = 0$, the function $H_0(x)$ is related to the Riemann ξ-function through the following identity:

$$\xi\left(\frac{x}{2}\right)/8 = H_0(x), \tag{1.4}$$

where the Riemann ξ-function, in turn, is related to the Riemann ζ-function through

$$\xi(iz) = \frac{1}{2}\left(z^2 - \frac{1}{4}\right)\pi^{-z/2-1/4}\,\Gamma\left(\frac{z}{2} + \frac{1}{4}\right)\zeta\left(z + \frac{1}{2}\right). \tag{1.5}$$

It is known (cf. Henrici [6, p. 305]) that the Riemann Hypothesis is equivalent to the statement that all the zeros of $\xi(z)$ are real, which implies from (1.4) that the *Riemann Hypothesis is equivalent to the statement that all zeros of $H_0(x)$ are real.*

Next, two results of de Bruijn [2] in 1950 established that

$$\begin{cases} \text{i)} & H_\lambda(x) \text{ has only real zeros for } \lambda \geq 1/2, \text{ and} \\ \text{ii)} & \text{if } H_\lambda(x) \text{ has only real zeros for some real } \lambda, \text{ then} \\ & H_{\lambda'}(x) \text{ also has only real zeros for any } \lambda' \geq \lambda. \end{cases} \tag{1.6}$$

In particular, it follows from (1.6ii) that if the Riemann Hypothesis is true, then $H_\lambda(x)$ must possess only real zeros for any $\lambda \geq 0$. In 1976, C.M. Newman [10] showed that there exists a real number Λ, satisfying $-\infty < \Lambda \leq 1/2$, such that

$$\begin{cases} \text{i)} & H_\lambda(x) \text{ has only real zeros when } \lambda \geq \Lambda, \text{ and} \\ \text{ii)} & H_\lambda(x) \text{ has some nonreal zeros when } \lambda < \Lambda. \end{cases} \tag{1.7}$$

This constant Λ has been called in [4] the *de Bruijn-Newman constant.* Since the Riemann Hypothesis is equivalent to $H_0(x)$ having all its zeros real, then from (1.7i), the truth of the Riemann Hypothesis would imply that $\Lambda \leq 0$. (Interestingly, Newman [10] makes the complementary conjecture that $\Lambda \geq 0$.) Because of the connection of this constant Λ to the Riemann Hypothesis, there is an obvious interest in determining upper and lower bounds for Λ. A constructive lower bound, $-50 < \Lambda$, was first given in [4] in 1988. Subsequently, te Riele [14] has given strong numerical evidence that $-5 < \Lambda$. Our object here is to report on recent research activity in finding improved lower bounds for Λ.

Returning to $H_\lambda(x)$ of (1.1), we see, on expanding $\cos(xt)$ and integrating termwise, that the Maclauren expansion for $H_\lambda(x)$ is given by

$$H_\lambda(x) = \sum_{m=0}^{\infty} \frac{\hat{b}_m(\lambda)(-x^2)^m}{(2m)!} \qquad (\lambda \in \mathbf{R}), \tag{1.8}$$

where

$$\hat{b}_m(\lambda) := \int_0^{\infty} t^{2m} e^{\lambda t^2} \Phi(t) dt \qquad (m = 0, 1, \ldots). \tag{1.9}$$

On setting $z = -x^2$ in (1.8), the function $F_\lambda(z)$ is then defined by

$$F_\lambda(z) := \sum_{m=0}^{\infty} \frac{\hat{b}_m(\lambda) z^m}{(2m)!} \qquad (\lambda \in \mathbf{R}), \tag{1.10}$$

so that

$$F_\lambda(-x^2) = H_\lambda(x) \qquad (\lambda \in \mathbf{R}). \tag{1.11}$$

Since $H_\lambda(x)$ is an entire function of order one, it follows from (1.11) that $F_\lambda(z)$ is an entire function of order 1/2. Hence, for each real $\lambda, F_\lambda(z)$ necessarily has (cf. Boas [1, p. 24]) infinitely many zeros. Moreover, it follows from (1.7) that

$$\begin{cases} \text{i)} \quad F_\lambda(z) \text{ has only real zeros when } \lambda \geq \Lambda, \text{ and} \\ \text{ii)} \quad F_\lambda(z) \text{ has some nonreal zeros when } \lambda < \Lambda. \end{cases} \tag{1.12}$$

The constructive method used in [4], for finding lower bounds for the de Bruijn-Newman constant Λ, can be described as follows. With the moments of (1.9), define the m-th *Jensen polynomial* for $F_\lambda(z)$ by

$$G_m(t;\lambda) := \sum_{k=0}^{m} \binom{m}{k} \frac{\hat{b}_k(\lambda) \cdot k!}{(2k)!} t^k \qquad (m = 1, 2, \ldots). \tag{1.13}$$

It was shown in Proposition 1 of [4] that if, for some real $\hat{\lambda}$ and some positive integer m, $G_m(t;\hat{\lambda})$ possessed a nonreal zero, then

$$\hat{\lambda} < \Lambda. \tag{1.14}$$

In [4], each of the exact moments $\{\hat{b}_m(-50)\}_{m=0}^{16}$ was approximated by the Romberg integration method with a relative accuracy of at least 60 significant digits, thereby producing the approximate moments $\{\hat{\beta}_m(-50)\}_{m=0}^{16}$, and the associated approximate Jensen polynomial (cf. (1.13)), namely

$$g_{16}(t;-50) := \sum_{k=0}^{16} \binom{16}{k} \frac{\hat{\beta}_k(-50)k!}{(2k)!} t^k,$$

was shown to possess a nonreal zero. Then, using a perturbation argument of Ostrowski (cf. [4, Proposition 2]), it was rigorously shown that $G_{16}(t;-50)$ also possessed a nonreal zero, so that from (1.14), $-50 < \Lambda$.

Further use of this Jensen polynomial method subsequently produced for us the (unpublished) lower bounds for Λ of Table 1. (All entries in the tables which follow are truncated to 3 decimal digits.)

TABLE 1

λ	degree m	digits required	complex zero of $G_m(t;\lambda)$
-100	10	12	$-453.840 + i\ 9.703$
-50	16	12	$-220.919 + i\ 7.092$
-20	41	18	$-111.065 + i\ 1.322$
-15	56	20	$-79.834 + i\ 0.282$
-12	75	20	$-59.204 + i\ 0.536$
-10	97	21	$-45.530 + i\ 0.156$
-8	142	21	$-30.993 + i\ 0.124$

By means of an improved perturbation argument, far fewer total significant digits (than that suggested in [4]) were actually required, in the computation of the moments $\{b_m(\lambda)\}_{m=0}^{N}$, to produce guaranteed lower bounds for Λ. This is indicated in column 3 of Table 1. The second column of Table 1 gives the *smallest* degree m for which the Jensen polynomial $G_m(t;\lambda)$, considered as a function of m, possessed nonreal zeros. The entries in this second column of Table 1 show an *alarming* increase in this smallest degree as λ increases to 0. To underscore this, te Riele [14], using this Jensen polynomial method but with a modification involving Sturm sequences, has recently reported strong numerical evidence for the lower bound:

$$-5 < \Lambda, \tag{1.15}$$

based on a Jensen polynomial of degree 406, where 250 significant digits were used in the associated computations! The results of te Riele and Table 1 seem to indicate that further improvements in lower bounds for Λ, using this Jensen polynomial method, would require lengthy calculations involving great precision.

2 Tracking Zeros of $F_\lambda(z)$

We propose here a new method for determining lower bounds for Λ, based on directly tracking particular pairs of zeros of $F_\lambda(z)$, as a function of λ. We begin by noting that $F_\lambda(z)$ of (1.10) can be expressed, in analogy with (1.1), in integral form as

$$F_\lambda(z) = \int_0^\infty e^{\lambda t^2} \Phi(t) \cosh(t\sqrt{z}\,)dt \qquad (\lambda \in \mathbf{R}). \tag{2.1}$$

Now suppose, for λ_0 real, that $z(\lambda_0)$ is some simple zero of $F_{\lambda_0}(z)$, so that $z(\lambda)$ remains a simple zero of $F_\lambda(z)$ in some small real interval in λ containing λ_0 in its interior. In this interval, $F_\lambda(z(\lambda)) \equiv 0$ so that, with the definition of $\hat{b}_m(\lambda)$ of (1.9),

$$\begin{aligned} F_\lambda(z(\lambda)) \equiv 0 &= \int_0^\infty e^{\lambda t^2} \Phi(t) \cosh\left(t\sqrt{z(\lambda)}\,\right) dt \\ &= \sum_{m=0}^\infty \frac{\hat{b}_m(\lambda)}{(2m)!} (z(\lambda))^m. \end{aligned} \tag{2.2}$$

On differentiating (2.2) with respect to λ, we obtain

$$0 \equiv \sum_{m=0}^\infty \frac{\hat{b}_{m+1}(\lambda)(z(\lambda))^m}{(2m)!} + \frac{dz(\lambda)}{d\lambda} \sum_{m=0}^\infty \frac{(m+1)\hat{b}_{m+1}(\lambda)(z(\lambda))^m}{(2m+2)!}.$$

Because the final sum above is nonzero (as $z(\lambda)$ is assumed to be a simple zero), then solving for $dz(\lambda)/d\lambda$ yields

$$\frac{dz(\lambda)}{d\lambda} = -\frac{\sum_{m=0}^\infty \hat{b}_{m+1}(\lambda)(z(\lambda))^m/(2m)!}{\sum_{m=0}^\infty (m+1)\hat{b}_{m+1}(\lambda)(z(\lambda))^m/(2m+2)!}. \tag{2.3}$$

Thus, with accurate estimates of $\{\hat{b}_m(\lambda)\}_{m=0}^N$ and with asymptotic estimates for $\{\hat{b}_m(\lambda)\}_{m=N+1}^\infty$, accurate estimates of $dz(\lambda)/d\lambda$ can be obtained from (2.3).

It is also important to note that replacing $e^{\lambda t^2}$ by its Maclauren expansion and integrating termwise in (1.9), gives

$$\hat{b}_m(\lambda) = \sum_{j=0}^{\infty} \frac{\hat{b}_{m+j}(0)\lambda^j}{j!} \qquad (m = 0, 1, \ldots; \ \lambda \in \mathbf{R}). \qquad (2.4)$$

Hence, for λ small and negative, one needs from (2.4) to compute *only one extended table* of high-precision moments $\{\hat{b}_m(0)\}_{m=0}^{N}$, from which the moments $\{\hat{b}_m(\lambda)\}_{m=0}^{N'}$ can be directly estimated from (2.4), where $N' < N$. (We remark that the choice of N' depends on m, λ, N, and the desired accuracy; cf. [11].)

In our applications described below, our extended table of high-precision moments was $\{\hat{b}_m(0)\}_{m=0}^{1000}$, where each moment was computed (on a SUN 3/80 computer in the Department of Mathematics and Computer Science at Kent State University) to an accuracy of 360 significant digits, using basically the trapezoidal rule with a sufficiently fine mesh. (This will be discussed in detail in §3.)

It is well known that considerable numerical effort has been given to the problem of studying the (nontrivial) nonreal zeros of the Riemann ζ-function in the critical strip $0 < Re\ z < 1$. In 1986, van de Lune, te Riele, and Winter [8] impressively showed that all $1,500,000,001(=: T)$ nonreal zeros of $\zeta(z)$, in the subset of the critical strip defined by

$$0 < Im\ z < 545,439,823.215\ldots,$$

lie on $Re\ z = 1/2$ and are simple. Expressing these zeros as

$$\left\{\rho_n := \frac{1}{2} + i\gamma_n\right\}_{n=1}^{T} \qquad (\text{where } 0 < \gamma_1 < \cdots < \gamma_T), \qquad (2.5)$$

it follows from (1.4), (1.5), and (1.11) that

$$z_n(0) := -4\gamma_n^2 \qquad (n = 1, 2, \ldots, T) \qquad (2.6)$$

are then the consecutive T largest (negative and simple) zeros of $F_0(z)$ of (1.10). From the tabulation of $\{\gamma_n\}_{n=1}^{15,000}$, accurate to 28 significant digits, given in te Riele [13], one can easily determine from (2.6) accurate estimates of $\{z_n(0)\}_{n=1}^{15,000}$.

In Table 2, we give the values $\{z_n(0)\}_{n=1}^{14}$, along with their differences and the derivative $dz_n(0)/d\lambda$, determined from (2.3).

It turns out, as is indicated in Table 2, that certain *pairs* of these known consecutive real zeros $z_n(0)$ and $z_{n+1}(0)$ of $F_0(z)$ are quite close and give promise of producing, as λ decreases from 0, nonreal conjugate complex zeros $z_n(\lambda)$ and $z_{n+1}(\lambda)$ of $F_\lambda(z)$. (We recall from (1.12) that if a real $\hat{\lambda}$ is such that $z_n(\hat{\lambda})$ and $z_{n+1}(\hat{\lambda})$ are nonreal zeros, then $\hat{\lambda} < \Lambda$.) In Table 3, we give the associated pairs, $z_n(0)$ and $z_{n+1}(0)$, on which we concentrated.

TABLE 2

n	$z_n(0) := -4\gamma_n^2$	$dz_n(0)/d\lambda$	$z_n(0) - z_{n+1}(0)$
1	−799.161	+32.771	+968.542
2	−1,767.704	+63.486	+734.467
3	−2,502.171	+58.608	+1,200.520
4	−3,702.692	+116.274	+636.180
5	−4,338.873	+61.317	+1,312.010
6	−5,650.883	+126.089	+1,046.483
7	−6,697.366	+140.344	+811.574
8	−7,508.941	+77.875	+1,709.036
9	−9,217.978	+230.133	+691.759
10	−9,909.737	+84.666	+1,313.682
11	−11,223.419	+138.447	+1,521.295
12	−12,744.715	+199.889	+1,343.571
13	−14,088.286	+267.284	+713.734
14	−14,802.021	+35.196	+2,156.552

In column 3 of Table 3, we again give $dz_n(0)/d\lambda$, determined from (2.3). We note, because of the difference in *signs* of $dz_{34}(0)/d\lambda$ and $dz_{35}(0)/d\lambda$ in Table 3, that the last pair of zeros, $z_{34}(0)$ and $z_{35}(0)$, are *tending toward one another* as λ decreases from 0, i.e., these two zeros are *attracted* to each other. In Table 4, we show how $z_{34}(\lambda), dz_{34}(\lambda)/d\lambda, z_{35}(\lambda)$, and $dz_{35}(\lambda)/d\lambda$ change with decreasing values of λ. Table 4 suggests that not only are $z_{34}(\lambda)$ and $z_{35}(\lambda)$ tending toward one another, but also that $dz_{34}(\lambda)/d\lambda$ and $dz_{35}(\lambda)/d\lambda$ are respectively tending to $+\infty$ and $-\infty$.

The actual tracking of the pair of zeros $\{z_{34}(\lambda)$ and $z_{35}(\lambda)\}$ generates interesting geometrical results! In Figure 1, we have graphed the 21 pairs of zeros

$$\{z_{34}(-[0.04]j) \text{ and } z_{35}(-[0.04]j)\}_{j=0}^{20}. \tag{2.7}$$

TABLE 3

n	$z_n(0)$	$dz_n(0)/d\lambda$	$z_n(0) - z_{n+1}(0)$
4	-3,702.692	+116.274	+636.180
5	-4,338.873	+61.317	—
9	9,217.978	+230.133	+691.759
10	-9,909.737	+84.666	—
13	-14,088.286	+267.284	+713.734
14	-14,802.021	+35.196	—
19	-22,924.800	+414.348	+880.504
20	-23,805.305	+140.940	—
24	-30,572.714	+392.063	+975.518
25	-31,548.232	+44.267	—
27	-35,835.507	+465.401	+929.206
28	-36,764.714	+26.826	—
34	-49,310.231	+877.835	+753.526
35	-50,063.757	-26.626	—

We see from Figure 1 that the pair of zeros $z_{34}(\lambda)$ and $z_{35}(\lambda)$ of (2.7) start out as real distinct zeros which move toward one another. These zeros then *meet*, forming a real double zero of $F_\lambda(z)$ when $\lambda \doteq -0.38$, and then this pair of zeros bifurcates into two nonreal conjugate complex numbers which follow, as λ decreases, a parabolic-like trajectory in the complex plane when $\lambda \leq -0.40$. Because $F_\lambda(z)$ apparently has, from Figure 1, nonreal zeros when $\lambda \leq -0.40$, it would appear from (1.12) that -0.40 is a lower bound for Λ, i.e.,

$$-0.40 \overset{?}{<} \Lambda. \tag{2.8}$$

TABLE 4

λ	$z_{34}(\lambda)$	$\frac{dz_{34}(\lambda)}{d\lambda}$	$z_{35}(\lambda)$	$\frac{dz_{35}(\lambda)}{d\lambda}$
-0.30	-49,633.457	+1,489.525	-49,997.614	-626.913
-0.31	-49,648.703	+1,561.893	-49,990.996	-698.909
-0.32	-49,664.748	+1,650.191	-49,983.583	-786.835
-0.33	-49,681.783	+1,761.399	-49,975.183	-897.671
-0.34	-49,700.092	+1,907.715	-49,965.513	-1,043.617
-0.35	-49,720.131	+2,112.957	-49,954.117	-1,248.488

Our task is to *rigorously* establish in §3 the following slightly improved form of (2.8), namely

Theorem 1. *If* Λ *is the de Bruijn-Newman constant, then*

$$-0.385 < \Lambda. \tag{2.9}$$

We remark that *each* of the pairs of zeros, $z_n(0)$ and $z_{n+1}(0)$, of Table 3 did similarly give rise, via this new tracking method, to a lower bound for Λ, and the best such lower bound (coming from tracking the pair $z_{34}(\lambda)$ and $z_{35}(\lambda)$) is the result of (2.9). These results are summarized in Table 5, where the final column in Table 5 gives the largest value of $\hat{\lambda}$ (to three decimal digits) for which $z_n(\hat{\lambda})$ and $z_{n+1}(\hat{\lambda})$ were nonreal complex conjugate numbers, and for which $|Im\ z_n(\lambda)| \geq 1$.

Our primary interest here has been to introduce a new method for obtaining rigorous lower bounds for Λ, and to show, with a moderate amount of computing effort, that this method does produce improved lower bounds for Λ. We are confident that further improved lower bounds for Λ can be similarly numerically obtained for this tracking method applied to particular pairs of zeros, $z_n(\lambda)$ and $z_{n+1}(\lambda)$, with $n > 34$, as λ decreases from 0, but at the expense of more computer time.

3 Proof of Theorem 1

This section consists first of a brief discussion on how high-precision numerical approximations of the moments $\hat{b}_m(\lambda)$ of (1.9) can be determined, and this is followed by a perturbation analysis which is used to *rigorously*

TABLE 5

n	$z_n(0) := -4\gamma_n^2$	largest value of λ for which $z_n(\lambda)$ and $z_{n+1}(\lambda)$ are nonreal
4	-3,702.692	-3.955
5	-4,338.873	
9	9,217.987	-1.878
10	-9,909.737	
13	-14,088.286	-1.286
14	-14,802.021	
19	-22,924.800	-1.276
20	-23,805.305	
24	-30,572.714	-1.144
25	-31,548.232	
27	-35,835.507	-0.882
28	-36,764.714	
34	-49,310.231	-0.385
35	-50,063.757	

show that $F_\lambda(z)$ has a nonreal zero when $\lambda = -0.385$. We remark that the complete details (which are lengthy and rather tedious) for producing high-precision approximation of the moments $\hat{b}_m(\lambda)$ are given in Norfolk, Ruttan, and Varga [11].

To begin, our first step was to determine high-precision floating-point numbers $\{\beta_m(0)\}_{m=0}^{1000}$ which approximate the moments $\{\hat{b}_m(0)\}_{m=0}^{1000}$, where (cf. (1.9))

$$\hat{b}_m(0) := \int_0^\infty t^{2m}\Phi(t)dt \qquad (m = 0, 1, \ldots). \tag{3.1}$$

Fortunately, because the integrand in (3.1) is from (1.3i) an even function which is analytic in the strip $|Im\ z| < \pi/8$ for each $m \geq 0$, it follows from the work of Martensen [9] and Kress [7] that the familiar trapezoidal rule approximation (on a uniform mesh of size h) of $\hat{b}_m(0)$, defined by

$$T_m(h) := h\left\{\frac{1}{2}\left[t^{2m}\Phi(t)\right]_{t=0} + \sum_{k=1}^{\infty}(kh)^{2m}\Phi(kh)\right\} \quad (m = 0, 1, \ldots), \tag{3.2}$$

converges *exponentially rapidly* to $\hat{b}_m(0)$ as h decreases to 0, i.e., (cf. [7, Thm. 2.2 with $p = 0$]),

$$|T_m(h) - \hat{b}_m(0)| \leq \frac{\exp(-\alpha\pi/h)}{\sinh(\alpha\pi/h)} \int_0^{\infty} \left|(s + i\alpha)^{2m}\Phi(s + i\alpha)\right| ds, \tag{3.3}$$

for any α with $0 < \alpha < \pi/8 = 0.39269\ldots$, where the path of integration in (3.3) is the nonnegative real axis. From (1.2), it directly follows that the integrand in (3.3) is bounded above by

$$(s^2+\alpha^2)^m \sum_{n=1}^{\infty} \left(2n^4\pi^2 e^{9s} + 3n^2\pi e^{5s}\right) \exp\left(-n^2\pi e^{4s}\cos 4\alpha\right) \quad (s \geq 0), \tag{3.4}$$

and on specifically choosing

$$\hat{\alpha} := \frac{1}{4}\arccos\left(\frac{\ln 32}{3\pi}\right) = 0.29855\ldots \qquad (< \pi/8), \tag{3.5}$$

an easily computed upper bound, $I(\hat{\alpha}; m)$, for the integral in (3.3) can be found, so that

$$\left|T_m(h) - \hat{b}_m(0)\right| \leq \frac{\exp(-\hat{\alpha}\pi/h)}{\sinh(\hat{\alpha}\pi/h)} \tag{3.6}$$

is an upper bound for the error in the trapezoidal approximation of $\hat{b}_m(0)$. (Further details are given in [11]).

Next, we observe that the exact trapezoidal rule approximation, $T_m(h)$, involves an infinite sum in its definition in (3.2), and, in addition, there is an infinite sum in the definition of $\Phi(t)$ in (1.2), which is used in each term of $T_m(h)$. In our actual computations of approximations of $\hat{b}_m(0)$, the sum in (3.2) was summed only for $k \leq 2/h$ because of the exponential decay (cf. (1.3iii)) of $\Phi(t)$ for large $t > 0$, and only the first sixteen terms of the infinite sum defining $\Phi(t)$ in (1.2) were used to approximate $\Phi(t)$. (An upper bound of the sum of the remaining terms of $\Phi(t)$ is constructively given in [3, eq. (4.6)].) These two errors, introduced into the computation of the trapezoidal rule $T_m(h)$, can again be constructively bounded above, and the details are again given in [11]. In this way, the approximations $\{\beta_m(0)\}_{m=0}^{1000}$ to the moments $\{\hat{b}_m(0)\}_{m=0}^{1000}$ were determined, each with a computable error. Finally, from the approximate moments $\{\beta_m(0)\}_{m=0}^{1000}$, the moments (cf. (2.4))

$$\beta_m(\lambda) := \sum_{j=0}^{1000} \frac{\beta_{m+j}(0)\lambda^j}{j!} \qquad (m = 0, 1, \ldots, 550) \tag{3.7}$$

were determined. All floating-point calculations were performed with 360 significant digits of accuracy, and, based on the error estimate outlined above, the approximate moments $\{\beta_m(-0.385)\}_{m=0}^{550}$ are each accurate to 314 significant digits (cf. (3.11)).

For the perturbation analysis to show that $F_\lambda(z)$ has a nonreal zero when $\lambda = -0.385$, we begin by establishing the following known, but useful, result. (We remark that Lemma 1 is a special case of a more general result given in Henrici [5, p. 454].)

Lemma 1. *Let $p(z)$ be a complex polynomial of degree n. If $p'(z_0) \neq 0$, then the disk*

$$\{z : |z - z_0| \leq n|p(z_0)|/|p'(z_0)|\} \tag{3.8}$$

contains at least one zero of $p(z)$.

Proof. As the result of Lemma 1 is obvious if $p(z_0) = 0$, assume $p(z_0) \neq 0$ and write $p(z) = \mu\prod_{k=1}^{n}(z-\zeta_k)$, where the ζ_k's are the zeros of $p(z)$. Taking the logarithmic derivative of $p(z)$ and evaluating the result at the point z_0 gives

$$\frac{p'(z_0)}{p(z_0)} = \sum_{k=1}^{n} \frac{1}{z_0 - \zeta_k}.$$

On taking absolute values in the above expression, then

$$\frac{|p'(z_0)|}{|p(z_0)|} \leq \sum_{k=1}^{n} \frac{1}{|z_0 - \zeta_k|} \leq \frac{n}{\min_{1\leq k\leq n}|z_0 - \zeta_k|},$$

and rewriting this inequality directly gives (3.8). ∎

Our next result is also elementary in nature.

Lemma 2. *Given the complex number z_0, assume that $f(z) := \sum_{j=0}^{\infty} a_j z^j$ is analytic in the disk $|z - z_0| < R \leq \infty$. For each positive integer N, set $p_N(z) := \sum_{j=0}^{N} a_j z^j$, and write $p_N(z) =: \sum_{j=0}^{N} c_j(z - z_0)^j$, where $c_j := c_j(N; z_0)$. Assume that there exist a positive integer N and positive real numbers α_N, δ (with $0 < \delta < 1$), and τ (with $0 < \tau < R$), such that*

i) $\alpha_N \geq \sup_{j>N} |a_j|^{1/j}$,

ii) $0 \neq c_1 (:= p'_N(z_0))$,

iii) $\tau > N|c_0|/|c_1|$,

iv) $\sum_{j=0}^{N} |c_j|\tau^j \leq \frac{3}{2}|c_1|\tau$,

v) $\alpha_N(|z_0| + \tau) \leq \delta < 1$, *and*

vi) $(1/(1-\delta))\,[\alpha_N(|z_0| + \tau)]^{N+1} \leq \frac{1}{2}|c_1|\tau$, *where strict inequality holds in iv) or vi). Then, $f(z)$ also has at least one zero in $|z-z_0|<\tau$.*

Proof. To begin, assumption iii) implies, from Lemma 1, that $p_N(z)$ has at least one zero in the disk $|z-z_0|<\tau$. On the circle $|z-z_0|=\tau$, we have from i), v) and vi) that

$$\begin{aligned}
|f(z)-p_N(z)| &= |\sum_{j=N+1}^{\infty} a_j z^j| \le \sum_{j=N+1}^{\infty} [\alpha_N(|z_0|+\tau)]^j \\
&= \frac{[\alpha_N(|z_0|+\tau)]^{N+1}}{1-\alpha_N(|z_0|+\tau)} \le \frac{[\alpha_N(|z_0|+\tau)]^{N+1}}{1-\delta} \\
&\le \frac{|c_1|\tau}{2}.
\end{aligned}$$

Since $(3/2)|c_1|\tau - \sum_{j=0}^{N} |c_j|\tau^j \ge 0$ from iv), the above inequality implies that

$$\begin{aligned}
|f(z)-p_N(z)| &\le \frac{|c_1|\tau}{2} + \left\{ \frac{3|c_1|\tau}{2} - \sum_{j=0}^{N} |c_j|\tau^j \right\} = |c_1|\tau - \sum_{\substack{j=0 \\ j\neq 1}}^{N} |c_j|\tau^j \\
&\le |\sum_{j=0}^{N} c_j(z-z_0)^j| =: |p_N(z)|,
\end{aligned}$$

and, since strict inequality by assumption holds in either iv) or vi), then

$$|f(z)-p_N(z)| < |p_N(z)|.$$

But this inequality implies, on applying Rouche's theorem on $|z-z_0|=\tau$, that $f(z)$ and $p_N(z)$ have the same number of zeros in $|x-z_0|<\tau$. Consequently, $f(z)$ has at least one zero in $|z-z_0|<\tau$. ■

The next result, which reduces to the result of Lemma 2 (when $\hat{p}_N(z) \equiv p_N(z)$), is an easy consequence of the proof of Lemma 2.

Lemma 3. *Given the complex number z_0, assume that $f(z) := \sum_{j=0}^{\infty} a_j z^j$ is analytic in the disk $|z-z_0|<R\le\infty$. For each positive integer N, set $p_N(z) := \sum_{j=0}^{N} a_j z^j$, and write $p_N(z) =: \sum_{j=0}^{N} c_j(z-z_0)^j$, where $c_j := c_j(N;z_0)$. Assume that there exist a positive integer N, an approximation polynomial $\hat{p}_N(z) := \sum_{j=0}^{N} \hat{c}_j(z-z_0)^j$ to $p_N(z)$, positive real numbers α_N, ε, δ (with $0<\delta<1$), and τ (with $0<\tau<R$) such that*

i) $\alpha_N \geq \sup_{j>N} |a_j|^{1/j}$

ii) $|c_j - \hat{c}_j| < \varepsilon \quad (j = 0, 1, \ldots, N),$

ii') $|\hat{c}_1| > \varepsilon,$

iii) $\tau > N \dfrac{|\hat{c}_0| + \varepsilon}{|\hat{c}_1| - \varepsilon},$

iv) $\displaystyle\sum_{j=0}^{N} (|\hat{c}_j| + \varepsilon)\,\tau^j \leq \frac{3}{2}(|\hat{c}_1| - \varepsilon)\,\tau,$

v) $\alpha_N\,(|z_0| + \tau) \leq \delta < 1,$ *and*

vi) $\frac{1}{1-\delta}\,[\alpha_N(|z_0| + \tau)]^{N+1} \leq \frac{1}{2}(|\hat{c}_1| - \varepsilon)\,\tau,$

with strict inequality holding in iv) or vi). *Then, $f(z)$ has at least one zero in $|z - z_0| < \tau$.*

Proof. With the hypothesis above, it is elementary to verify (by the triangle inequality) that $p_N(z)$ and $f(z)$ satisfy all the hypotheses of Lemma 2; hence, $f(z)$ has at least one zero in $|z - z_0| < \tau$. ∎

With Lemma 3, we come to the

Proof of Theorem 1. Let $f(z)$ be the entire function

$$F_{\hat{\lambda}}(z) := \sum_{m=0}^{\infty} \hat{b}_m(\hat{\lambda})/(2m)!\; z^m,$$

where $\hat{\lambda}$ is defined by

$$\hat{\lambda} := -0.385, \tag{3.9}$$

and define the complex number z_0 by

$$\begin{aligned} \text{Re } z_0 \;=\; & -4.98522639992936705445742890880882513 7\\ & 178354299435912229506745988662825 10463\\ & 32837018282760459119241484170263372 2E4 \end{aligned} \tag{3.10}$$

$$\begin{aligned} \text{Im } z_0 \;=\; & 1.3230628522744934395842979614314738 67\\ & 46981439309032787945567788558243812285\\ & 14005953099708756696839999091997765 0E1 \end{aligned}$$

Now, with $N := 550$, the numbers $\beta_m(\hat{\lambda})$, which approximate the moments $\hat{b}_m(\hat{\lambda})$, were determined so that

$$|\beta_m(\hat{\lambda}) - \hat{b}_m(\hat{\lambda})| \leq 5 \cdot 10^{-315} \qquad (m = 0, 1, \ldots, 550). \tag{3.11}$$

From this, the polynomials $p_{550}(z) := \sum_{m=0}^{550} \hat{b}_m(\hat{\lambda}) z^m/(2m)!$ and $\hat{p}_{550}(z) := \sum_{m=0}^{550} \beta_m(\hat{\lambda}) z^m/(2m)!$, re-expanded as $p_{550}(z) = \sum_{m=0}^{550} c_m (z - z_0)^m$ and $\hat{p}_{550}(z) = \sum_{m=0}^{550} \hat{c}_m (z - z_0)^m$, can be verified to satisfy

$$|c_m - \hat{c}_m| \leq \epsilon = 1 \cdot 10^{-217} \qquad (m = 0, 1, \ldots, 550). \tag{3.12}$$

Then, with $\delta := 9/10$, and with $\tau := 1 \cdot 10^{-5}$, an application of Lemma 3 to $F_{\hat{\lambda}}(z)$ shows that $F_{\hat{\lambda}}(z)$ has at least one zero in $|z - z_0| < \tau$. But since (cf. (3.10)) $\mathrm{Im}\, z_0 = 13.2306\ldots$ and since $\tau = 1 \cdot 10^{-5}$, it is geometrically evident that this zero of $F_{\hat{\lambda}}(z)$ in the disk $|z - z_0| < \tau$ is necessarily *nonreal*. Thus, from (1.7), $\hat{\lambda} = 0.385 < \Lambda$, the desired result of (2.9). ■

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Plot the zeros $z_{34}(\lambda)$, $z_{35}(\lambda)$

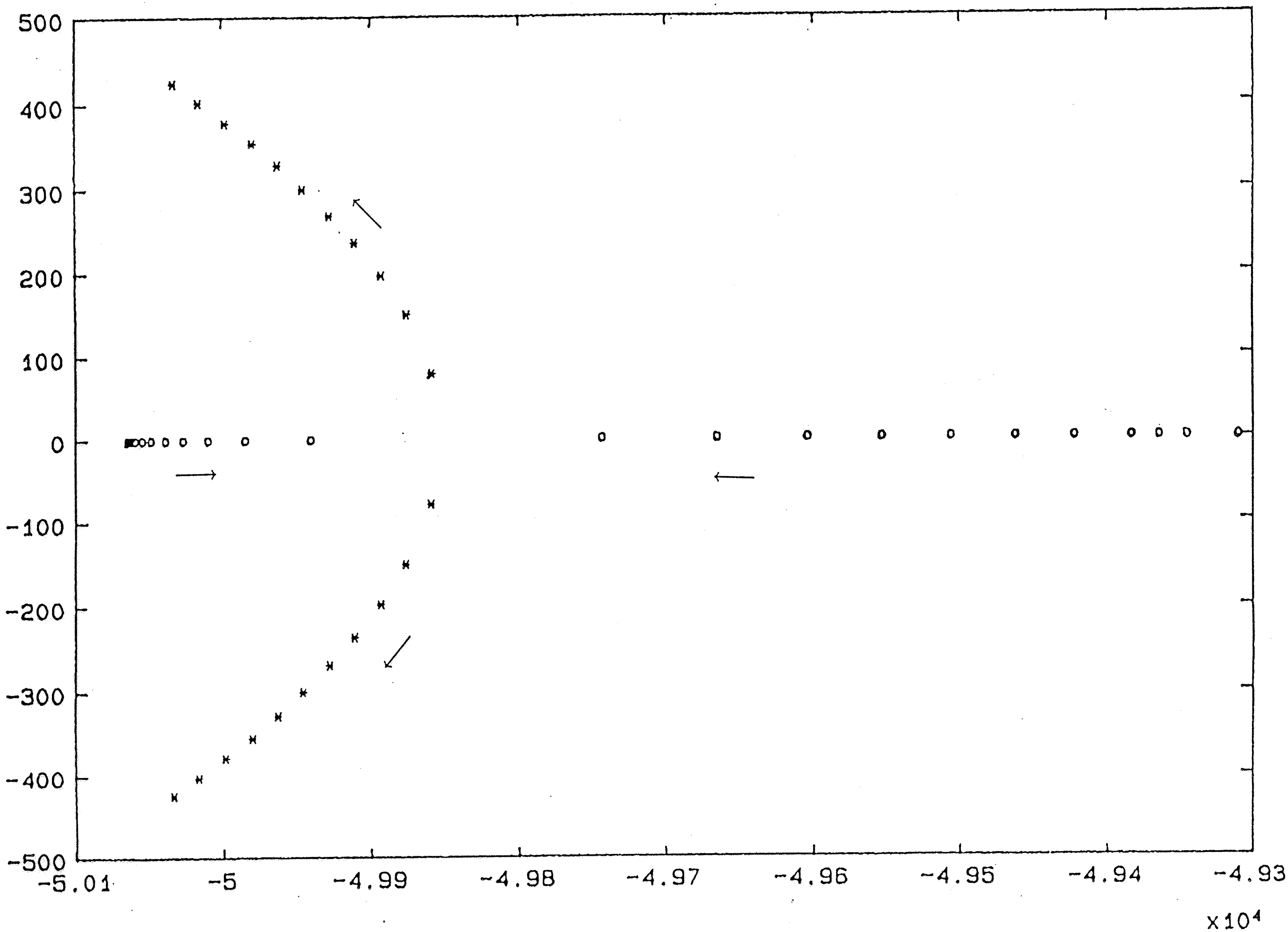

Figure 1