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Is the Optimal ω Best for the SOR Iteration Method?

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ABSTRACT

The successive-overrelaxation (SOR) iterative method for linear systems is well understood if the associated Jacobi matrix B is consistently ordered and weakly cyclic of index 2. If, in addition, B^2 has only nonnegative eigenvalues and if $\rho(B)$, the spectral radius of B, is strictly less than unity, then by D. M. Young's classical theorem, the optimal relaxation parameter for the SOR method is given by

$$\omega_b := \frac{2}{1 + \sqrt{1 - \rho^2(B)}}.$$

Young derived this result assuming that

$$\sigma(B^2) \subset [0, \beta^2] \qquad (\text{with} \quad \beta = \rho(B)) \tag{$*$}$$

is the only information available about the spectrum $\sigma(B^2)$ of B^2. It is also well known that no polynomial acceleration can improve the asymptotic rate of convergence of the SOR scheme if the optimal relaxation parameter has been selected. The recent claim by J. Dancis "that a smaller average spectral radius can be achieved by using a polynomial acceleration together with a suboptimal relaxation factor ($\omega < \omega_b$)" therefore comes as a surprise. A closer look however reveals that this improvement

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can only be achieved if more profound information on $\sigma(B^2)$, of the form

$$\sigma(B^2) \subset [0, \gamma^2] \cup \{\beta^2\} \qquad (\text{with} \quad \gamma < \beta), \tag{**}$$

is at hand. We show that no polynomial acceleration of the SOR method (for *any* real ω) is asymptotically faster than the SOR scheme with $\omega = \omega_b$ under the assumption (*), thereby answering the question in the title of this paper in the affirmative, as well as solving an old related conjecture of D. M. Young. We also carefully investigate the question of what can be gained from the additional information (**).

1. INTRODUCTION

In J. Dancis's recent paper [1] with the surprising title "The optimal ω is not best for the SOR iteration method," the author considers the convergence of the SOR iterative method in the well-known case where the Jacobi matrix B is a consistently ordered weakly cyclic of index 2 matrix, with B^2 possessing only nonnegative real eigenvalues which are less than unity. The associated SOR iteration matrix $\mathscr{L}_\omega$, defined in (2.5), is known to be convergent only for any ω satisfying $0 < \omega < 2$. Fixing an ω with $0 < \omega < 2$ and regarding the SOR iteration (cf. (2.4))

$$\mathbf{x}_{m+1} := \mathscr{L}_\omega \mathbf{x}_m + \mathbf{c}_\omega \tag{1.1}$$

as the basic iterative method, Dancis [1] applies three different *semiiterative methods* (also known as *polynomial acceleration techniques*) to the basic iterative method of (1.1), in the hopes of obtaining a more rapidly convergent iterative method. This would seem to fly in the face of conventional wisdom in this area, since it is well known (cf. [6]) that semiiterative methods *cannot* improve the convergence rate of the basic iterative method (1.1) in the particular case when $\omega = \omega_b$ (where ω_b is defined in (2.6)). Nevertheless, Dancis shows in [1] that an improvement is indeed possible, but curiously, neither the title nor the abstract of [1] mentions that this improvement strongly depends on the *explicit knowledge* of the two largest real eigenvalues of B^2.

The results of Dancis [1] have certainly served to stimulate our investigation in this paper, largely because of questions left unanswered in [1]. For example, if a particular semiiterative method applied to (1.1) gives a faster convergence rate than that of (1.1), then what is the *best* asymptotic rate of convergence which one can obtain from *any* semiiterative method applied to (1.1)? It turns out that known techniques from complex approximation theory and conformal mapping theory can be neatly applied to answer this question, but we must distinguish between two cases, according to what is assumed to be known about the spectrum of B^2. Thus, the two issues we address in this paper are:

(1) What is the fastest asymptotic rate of convergence, for a semiiterative method based on (1.1) when $0 < \omega < 2$, under the assumption that only the largest real eigenvalue of B^2 is known?

(2) What is the fastest asymptotic rate of convergence, for a semiiterative method based on (1.1) when $0 < \omega < 2$, under the additional hypothesis that the k ($k \geqslant 2$) largest real eigenvalues of B^2 are known?

Both of these questions are fully answered (in fact, for *all* real ω) in the subsequent sections of this paper. In a later paper, we also show that our analysis of the questions above is general enough to consider particular extensions, such as to the case when the spectrum of B^2 is assumed to contain nonnegative *and* nonpositive real eigenvalues.

2. BACKGROUND AND TERMINOLOGY

Consider the linear system

$$A\mathbf{x} = \mathbf{b}, \qquad \text{where} \quad A \in \mathbb{R}^{N \times N}, \quad \mathbf{b} \in \mathbb{R}^N, \tag{2.1}$$

with the standard splitting of the coefficient matrix A,

$$A = D - L - U,$$

where D is a nonsingular block diagonal matrix, and where L and U denote respectively strictly lower and strictly upper triangular matrices. We further assume that the corresponding block Jacobi matrix

$$B := D^{-1}(L + U) \tag{2.2}$$

is consistently ordered and weakly cyclic of index 2 (cf. [7, Definition 4.2]), and that the eigenvalues of B^2 are all nonnegative real numbers less than 1, i.e., the spectrum $\sigma(B^2)$ of B^2 satisfies

$$\sigma(B^2) \subset [0, \beta^2] \qquad \text{with} \quad \beta := \rho(B) < 1. \tag{2.3}$$

These assumptions imply that there is a unique solution $\mathbf{x}$ to the matrix equation (2.1).

We next review classical results for the SOR iterative method:

$$\mathbf{x}_m = \mathscr{L}_\omega \mathbf{x}_{m-1} + \mathbf{c}_\omega \qquad (m = 1, 2, \dots), \tag{2.4}$$

where $\mathscr{L}_\omega$ (the SOR matrix) and $\mathbf{c}_\omega$ are defined by

$$\begin{gathered} \mathscr{L}_\omega := (D - \omega L)^{-1}[(1-\omega)D + \omega U] \quad \text{and} \\ \mathbf{c}_\omega := \omega (D - \omega L)^{-1}\mathbf{b} \qquad (\omega \in \mathbb{R}). \end{gathered} \tag{2.5}$$

Here, ω is the associated relaxation parameter. Under the given assumptions on B and ω, the SOR iterative method of (2.4) converges (for any initial vector $\mathbf{x}_0$) to the solution of (2.1) if and only if $0 < \omega < 2$ holds (cf. Young [9, Theorem 6-2.2]). The optimal relaxation parameter ω_b which minimizes $\rho(\mathscr{L}_\omega)$ as a function of ω is given by (cf. [9, Theorem 6-2.3])

$$\omega_b = \omega_b(\beta) = \frac{2}{1 + \sqrt{1 - \beta^2}} = 1 + \left(\frac{\beta}{1 + \sqrt{1-\beta^2}}\right)^2, \tag{2.6}$$

and there also holds

$$1 > \rho(\mathscr{L}_\omega) > \rho(\mathscr{L}_{\omega_b}) = \omega_b - 1 \qquad \text{for all} \quad 0 < \omega < 2 \quad \text{with} \quad \omega \neq \omega_b. \tag{2.7}$$

As Dancis did in [1] for $0 < \omega < 2$, we now apply, for any fixed real ω, a semiiterative method to the iterates $\{\mathbf{x}_m\}_{m=0}^\infty$ which are generated from the SOR iterations of (2.4), i.e., we consider vector sequences $\{\mathbf{y}_m\}_{m=0}^\infty$ of the form

$$\mathbf{y}_m := \sum_{j=0}^{m} \pi_{m,j}\mathbf{x}_j \qquad (m = 0, 1, \dots), \tag{2.8}$$

where the coefficients $\pi_{m,j}$ are (complex) constants which satisfy the constraint $\sum_{j=0}^m \pi_{m,j} = 1$ $(m = 0,1,\dots)$. For ease of notation, we collect the coefficients $\pi_{m,j}$ of (2.8) into the infinite lower triangular matrix

$$P := \begin{bmatrix} \pi_{0,0} & & & \\ \pi_{1,0} & \pi_{1,1} & & \\ \vdots & \vdots & \ddots & \\ \pi_{m,0} & \pi_{m,1} & \cdots\ \pi_{m,m} & \\ \vdots & \vdots & & \ddots \end{bmatrix}, \tag{2.9}$$

and we call P the *generating matrix* of the semiiterative method (2.8). If $1 \notin \sigma(\mathscr{L}_\omega)$, it is well known (cf. [7, p. 134]) that the associated error vectors $\mathbf{e}_m := (I - \mathscr{L}_\omega)^{-1}\mathbf{c}_\omega - \mathbf{y}_m$, for this semiiterative method based on the basic iterative method of (2.4), satisfy

$$\mathbf{e}_m = p_m(\mathscr{L}_\omega)\mathbf{e}_0 \qquad (m = 0, 1, \dots),$$

where $p_m(z) := \sum_{j=0}^{m} \pi_{m,j} z^j \in \Pi_m$, so that $p_m(1) = 1$. (Here, Π_m denotes the collection of all complex polynomials of degree at most m.)

For a given P and for $1 \notin \sigma(\mathscr{L}_\omega)$, the quantity

$$\kappa(\mathscr{L}_\omega, P) := \limsup_{m \to \infty} \sup_{\mathbf{e}_0 \neq \mathbf{0}} \left(\frac{\|\mathbf{e}_m\|}{\|\mathbf{e}_0\|} \right)^{1/m},$$

(which depends only on the structure of the Jordan canonical form of the matrix $\mathscr{L}_\omega$, and is independent of the vector norm $\|\cdot\|$ chosen on $\mathbb{C}^N$) measures the *asymptotic decay* of the norms of the error vectors $\mathbf{e}_m$ associated with (2.8). In theory, one can always select a semiiterative scheme such that $\kappa(\mathscr{L}_\omega, P) = 0$ (e.g., from the Cayley-Hamilton theorem, this holds true if p_m is a multiple of the characteristic polynomial of $\mathscr{L}_\omega$ for all $m \geqslant N$) but this selection requires the *knowledge of all eigenvalues of* $\mathscr{L}_\omega$. Here, we merely assume we have *a priori* information of the form $\sigma(\mathscr{L}_\omega) \subseteq \Omega$, where $\Omega \subset \mathbb{C}$ is a compact set with $1 \notin \Omega$, and we call such a set a *covering domain* for $\sigma(\mathscr{L}_\omega)$. In this setting, we measure the performance of (2.8) by the quantity

$$\kappa(\Omega, P) := \max\{\kappa(\mathscr{L}_\omega, P) : \mathscr{L}_\omega \in \mathbb{R}^{N \times N}, N \text{ arbitrary, with } \sigma(\mathscr{L}_\omega) \subseteq \Omega\}.$$

The best, i.e., *smallest*, convergence factor we can hope to achieve by any semiiterative method in this worst-case philosophy is the *asymptotic conver-*

gence factor of Ω, defined by

$$\kappa(\Omega) := \inf\{\kappa(\Omega, P) : P \text{ generates a semiiterative method}\}. \quad (2.10)$$

The infimum in (2.10) is actually a minimum, i.e., there is always a generating matrix $\tilde{P}$ with $\kappa(\Omega) = \kappa(\Omega, \tilde{P})$ (cf. [2, §5]); alternatively, $\kappa(\Omega)$ could have been equivalently defined (cf. [2, §5]) by

$$\kappa(\Omega) = \lim_{m \to \infty} \left[\min\left\{ \max_{z \in \Omega} |p_m(z)| : p_m \in \Pi_m, \ p_m(1) = 1 \right\} \right]^{1/m}, \quad (2.11)$$

which couples this asymptotic convergence factor $\kappa(\Omega)$ with *complex approximation theory*. Note that (2.11) can also be used to extend the definition of the asymptotic convergence factor $\kappa(\Omega)$ to all compact sets $\Omega \subset \mathbb{C}$. This leads to $\kappa(\Omega) = 1$ for every compact set $\Omega \subset \mathbb{C}$ with $1 \in \Omega$.

With respect to the information $\sigma(\mathscr{L}_\omega) \subset \Omega$, the rate of convergence of the SOR iterative method (2.4) can therefore be improved by the application of a semiiterative scheme of the form (2.8) *only if*

$$\kappa(\Omega) < \min\{1, \rho(\mathscr{L}_\omega)\}.$$

(As we shall see (cf. (2.16)), there are indeed cases where $\kappa(\Omega) < 1$ while $\rho(\mathscr{L}_\omega) > 1$.)

Next, we list some properties of the convergence factor $\kappa(\Omega)$ [cf. (2.10)] which we will use in the subsequent sections. If Ω belongs to the class $\mathbb{M}$ (defined to consist of any compact subset of $\mathbb{C}$ which consists of more than one point, which does not contain the point $z = 1$, and whose complement (with respect to the extended complex plane $\mathbb{C}_\infty$) is simply connected), then

$$\kappa(\Omega) = \frac{1}{|\Phi(1)|} \quad (2.12)$$

(cf. [2, Theorem 11]), where Φ is a conformal map from $\mathbb{C}_\infty \setminus \Omega$ onto the exterior of the unit circle with $\Phi(\infty) = \infty$. (We note, by the Riemann mapping theorem, that Φ exists and is unique, up to a constant factor of modulus 1.) Thus, if $\Omega \in \mathbb{M}$, the problem of determining its asymptotic convergence factor $\kappa(\Omega)$ is reduced to a problem in *conformal-mapping theory*.

It turns out from the classical SOR theory, for a fixed $\beta := \rho(B)$ with $\sigma(B^2) \subset [0, \beta^2]$ (cf. (2.3)) and with $0 < \beta < 1$ (the case $\beta = 0$ being uninteresting) and for *any* real $\omega \neq 0$, that there are only *three* different types of covering domains $\Omega = \Omega_{\omega,\beta}$, which need to be considered, and these will be described in detail in Section 3. But, with the new quantity ω_e, defined by

$$\omega_e := \frac{2}{1-\sqrt{1-\beta^2}} \left(= \frac{\omega_b}{\omega_b - 1} \right), \tag{2.13}$$

so that $\omega_e > 2$, we have the necessary notation to state the first of our main results.

THEOREM 1. *Assume that the Jacobi matrix B of* (2.2) *is a consistently ordered weakly cyclic of index 2 matrix, and that the eigenvalues of B^2 are all nonnegative and lie in* $[0, \beta^2]$, *where* $0 < \beta = \rho(B) < 1$. *Then the asymptotic convergence factor $\kappa(\Omega_{\omega,\beta})$ satisfies the following properties*:

(i) *For* $-\infty < \omega < 1$ *and* $\omega \neq 0$, $\kappa(\Omega_{\omega,\beta})$ *is a strictly monotonically decreasing function of ω which satisfies*

$$\begin{aligned} &\rho(\mathscr{L}_\omega) > 1 > \kappa(\Omega_{\omega,\beta}) > \omega_b - 1 \qquad (-\infty < \omega < 0), \\ &1 > \rho(\mathscr{L}_\omega) > \kappa(\Omega_{\omega,\beta}) > \omega_b - 1 \qquad (0 < \omega < 1), \end{aligned} \tag{2.14}$$

with $\lim_{\omega \uparrow 0} \kappa(\Omega_{\omega,\beta}) = \lim_{\omega \downarrow 0} \kappa(\Omega_{\omega,\beta}) = \sqrt{\omega_b - 1}$.

(ii) *For* $1 \leqslant \omega \leqslant \omega_b$, $\kappa(\Omega_{\omega,\beta})$ *is a constant function of ω which satisfies*

$$\begin{aligned} &1 > \rho(\mathscr{L}_\omega) > \kappa(\Omega_{\omega,\beta}) = \omega_b - 1 \qquad (1 \leqslant \omega < \omega_b), \\ &1 > \rho(\mathscr{L}_\omega) = \kappa(\Omega_{\omega,\beta}) = \omega_b - 1 \qquad (\omega = \omega_b). \end{aligned} \tag{2.15}$$

(iii) *For* $\omega_b < \omega < \omega_e$, $\kappa(\Omega_{\omega,\beta})$ *is a strictly monotonically increasing function of ω which satisfies*

$$\begin{aligned} &1 > \rho(\mathscr{L}_\omega) > \kappa(\Omega_{\omega,\beta}) > \omega_b - 1 \qquad (\omega_b < \omega < 2), \\ &1 = \rho(\mathscr{L}_\omega) > \kappa(\Omega_{\omega,\beta}) > \omega_b - 1 \qquad (\omega = 2), \\ &\rho(\mathscr{L}_\omega) > 1 > \kappa(\Omega_{\omega,\beta}) > \omega_b - 1 \qquad (2 < \omega < \omega_e). \end{aligned} \tag{2.16}$$

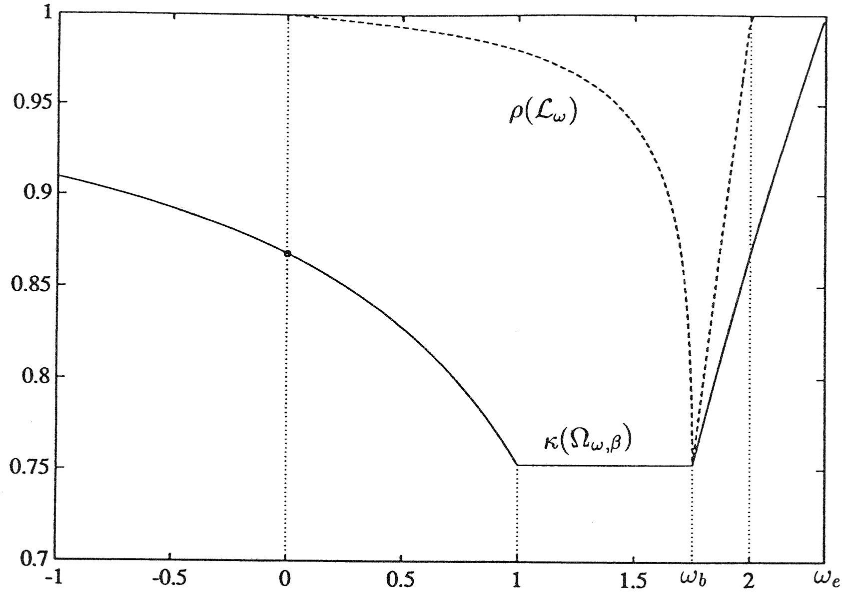

FIG. 1. $\rho(\mathscr{L}_\omega)$ and $\kappa(\Omega_{\omega,\beta})$ as functions of the real variable ω [for $\beta = \rho(B) = 0.99$, $\omega_b = 1.75274\ldots$, and $\omega_e = 2.32847\ldots$].

(iv) *For* $\omega = 0$ *and for* $\omega_e \leqslant \omega < \infty$,

$$\rho(\mathscr{L}_\omega) \geqslant 1 = \kappa(\Omega_{\omega,\beta}) > \omega_b - 1, \tag{2.17}$$

where ω_b *and* ω_e *are defined, respectively, in* (2.6) *and* (2.13).

In particular, for any real ω,

$$\kappa(\Omega_{\omega,\beta}) < 1 \quad \textit{if and only if} \quad \omega \in (-\infty, \omega_e) \setminus \{0\}. \tag{2.18}$$

The results of Theorem 1 can be seen in Figure 1. The first new and startling result of Theorem 1 for us was that $\kappa(\Omega_{\omega,\beta}) = \rho(\mathscr{L}_{\omega_b})$ for *all* $1 \leqslant \omega \leqslant \omega_b$ in (2.15). On reflection, we can say that this result was anticipated by the old result of [6, Theorem 2], which showed that the semiiterative method obtained by applying Chebyshev polynomials to the Gauss-Seidel method (i.e., SOR with $\omega = 1$) gives the same asymptotic rate of convergence as that of $\mathcal{L}_{\omega_b}$, i.e.,

$$\kappa(\Omega_{1,\beta}) = \omega_b - 1. \tag{2.19}$$

On the other hand, a well-known consequence of the same paper [6, Theorem 4] is that no semiiterative method, applied to $\mathcal{L}_{\omega_b}$, can improve the asymptotic rate of convergence of $\mathcal{L}_{\omega_b}$, i.e.,

$$\kappa(\Omega_{\omega_b,\beta}) = \omega_b - 1, \tag{2.20}$$

and intuitively, it would be difficult to imagine that semiiteration, applied to $\mathcal{L}_{\omega}$ (where $1 < \omega < \omega_b$), could improve on *both* (2.19) and (2.20)!

A further surprise for us was the appearance of the constant ω_e of (2.13), which also plays a role in the theory of Markov chains (cf. Kontovasilis, Plemmons, and Stewart [3]), and that the SOR iterative method can actually be forced to converge by suitable semiiteration, precisely for any real $\omega \in (-\infty, \omega_e) \setminus \{0\}$.

As a final comment in this section, we note from Theorem 1 that

$$\kappa(\Omega_{\omega,\beta}) \geqslant \omega_b - 1 = \rho(\mathcal{L}_{\omega_b}) \qquad (\omega \in \mathbb{R}), \tag{2.21}$$

which affirmatively solves a conjecture of Young [9, p. 379] that (2.21) holds for the interval (0, 2).

3. THE COVERING DOMAINS $\Omega_{\omega,\beta}$ AND THEIR ASYMPTOTIC CONVERGENCE FACTORS

Using Young's fundamental relationship

$$(\lambda + \omega - 1)^2 = \lambda\omega^2\mu^2 \tag{3.1}$$

between the eigenvalues λ of $\mathcal{L}_{\omega}$ and the eigenvalues μ of B (cf. [9, Theorem 5-2.2]), sharp covering domains $\Omega_{\omega,\beta}$ for the eigenvalues of $\mathcal{L}_{\omega}$ can be derived. To this end, we examine the following three different cases, which are also treated in [9, pp. 203–206].

(i) $-\infty < \omega \leqslant 1$ and $\omega \neq 0$. In this case, all eigenvalues of $\mathcal{L}_{\omega}$ are real. More precisely, the covering domains $\Omega_{\omega,\beta}$ are real intervals which exclude $z = 1$, i.e.,

$$\sigma(\mathscr{L}_\omega) \subset \Omega_{\omega,\beta} := \begin{cases} [\lambda_2, \lambda_1] \subset (1, \infty) & (-\infty < \omega < 0), \\ [\lambda_1, \lambda_2] \subset [0, 1) & (0 < \omega < 1), \end{cases} \tag{3.2}$$

where

$$\begin{aligned} \lambda_1 = \lambda_1(\omega, \beta) &:= \left[\frac{\omega\beta - \sqrt{\omega^2\beta^2 - 4(\omega - 1)}}{2}\right]^2, \\ \lambda_2 = \lambda_2(\omega, \beta) &:= \left[\frac{\omega\beta + \sqrt{\omega^2\beta^2 - 4(\omega - 1)}}{2}\right]^2. \end{aligned} \tag{3.3}$$

We note that the inclusions of (3.2) are *sharp*, i.e., $\lambda_1, \lambda_2 \in \sigma(\mathscr{L}_\omega)$. Moreover, we see that $1 \notin \Omega_{\omega,\beta}$ and that $\Omega_{\omega,\beta} \in \mathbb{M}$. For $\omega = 1$, the SOR scheme reduces to the Gauss-Seidel method and $\Omega_{1,\beta} = [0, \beta^2]$ holds.

(ii) $1 < \omega \leqslant \omega_b$ or $\omega \geqslant \omega_e$. In these cases, $\sigma(\mathscr{L}_\omega)$ is contained, in the terminology of Dancis [1], in a banjo-shaped set (cf. Figure 2(a))

$$\sigma(\mathscr{L}_\omega) \subset \Omega_{\omega,\beta} := \partial\mathbb{D}(0; \omega - 1) \cup [\lambda_1, \lambda_2], \tag{3.4}$$

where $\partial\mathbb{D}(c; r)$ denotes the circle with center c and radius r, and where the end points of the interval $[\lambda_1, \lambda_2]$ in (3.4), namely λ_1 and λ_2, are in $\sigma(\mathscr{L}_\omega)$ and are again given by (3.3). Here, $1 \notin \Omega_{\omega,\beta}$ (for $1 < \omega \leqslant \omega_b$, there holds $\lambda_1 \leqslant \lambda_2 < 1$, whereas $1 < \lambda_1 \leqslant \lambda_2$ for $\omega \geqslant \omega_e$) but $\Omega_{\omega,\beta} \notin \mathbb{M}$, since the complement of $\Omega_{\omega,\beta}$ is not simply connected; more precisely, $\mathbb{C}_\infty \setminus \Omega_{\omega,\beta}$ consists of two connected components. The main distinction between these two subcases is that the critical point $z = 1$ lies in the unbounded component of $\mathbb{C}_\infty \setminus \Omega_{\omega,\beta}$ if $1 < \omega \leqslant \omega_b$, and in the bounded component of $\mathbb{C}_\infty \setminus \Omega_{\omega,\beta}$ if $\omega \geqslant \omega_e$. As we shall see (cf. the proof of Theorem 1), this implies that, for $\omega \geqslant \omega_e$, *no* semiiterative method converges when applied to the SOR iteration with relaxation parameter ω. For $\omega = \omega_b$ and $\omega = \omega_e$, there holds $\lambda_1 = \lambda_2$ and $\Omega_{\omega,\beta}$ degenerates to circles with radii $\omega_b - 1$ and $\omega_e - 1$, respectively, which are centered at the origin.

(iii) $\omega_b < \omega < \omega_e$. In this case, all eigenvalues of $\mathscr{L}_\omega$ are located on a

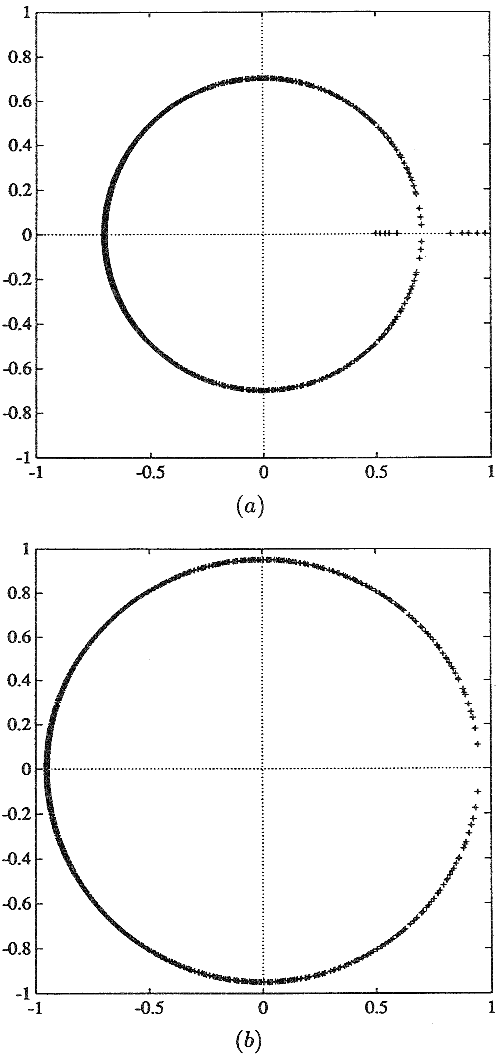

FIG. 2. Eigenvalues of $\mathscr{L}_\omega$, (a) $\omega = 1.7$ and (b) $\omega = 1.95$, for the two-dimensional model problem with 2500 unknowns [here, $\rho(B) = \cos(\pi/51) = 0.99810\ldots$ and $\omega_b = 1.88401\ldots$].

circular arc (cf. Figure 2(b))

$$\sigma(\mathcal{L}_\omega) \subset \Omega_{\omega,\beta} := \{(\omega-1)e^{i\theta} : \arg\lambda_2 \leqslant \theta \leqslant \arg\lambda_1 \pmod{2\pi}\}, \quad (3.5)$$

where the branch of the square root in (3.3) has to be selected such that $\operatorname{Im}\lambda_2 > 0$. Note that $\lambda_1 = \bar{\lambda}_2$, and $\Omega_{\omega,\beta}$ is therefore symmetric with respect to the real axis. In this case, $1 \notin \Omega_{\omega,\beta}$ and $\Omega_{\omega,\beta} \in \mathbb{M}$.

With our given information, namely, that $\sigma(B^2) \subset [0, \beta^2]$ for the spectrum of the Jacobi matrix B (cf. (2.2)), the above covering domains $\Omega_{\omega,\beta}$ of $\sigma(\mathcal{L}_\omega)$ are *optimal* in the following sense: For each $\lambda \in \Omega_{\omega,\beta}$, there exists a Jacobi matrix B, which is a consistently ordered weakly cyclic of index 2 matrix, with $\rho(B) = \beta$ and $\sigma(B^2) \subset [0, \beta^2]$, such that λ is an eigenvalue of the corresponding SOR iteration matrix $\mathcal{L}_\omega$.

We next compute the asymptotic convergence factors of the covering domains $\Omega_{\omega,\beta}$ for $\sigma(\mathcal{L}_\omega)$. First, for $-\infty < \omega \leqslant 1$ and $\omega \neq 0$, $\sigma(\mathcal{L}_\omega)$ is contained in an interval (cf. (3.2)) whose asymptotic convergence factor is well known (cf. [2, §6]).

PROPOSITION 2. *The asymptotic convergence factor of the real interval* $[\zeta, \eta]$ *is given by*

$$\kappa([\zeta,\eta]) = \begin{cases} \dfrac{\eta-\zeta}{\left(\sqrt{1-\zeta}+\sqrt{1-\eta}\right)^2} & \text{if } \zeta < \eta < 1, \\[2ex] \dfrac{\eta-\zeta}{\left(\sqrt{\zeta-1}+\sqrt{\eta-1}\right)^2} & \text{if } 1 < \zeta < \eta. \end{cases}$$

Next, for $1 < \omega \leqslant \omega_b$ and for $\omega \geqslant \omega_e$, $\sigma(\mathcal{L}_\omega)$ is contained in a banjo-shaped set of the form (3.4). It can be seen from (2.11), using the maximum modulus principle, that the asymptotic convergence factor of $\Omega_{\omega,\beta} = \partial\mathbb{D}(0;\tau) \cup [\gamma, \lambda]$ (cf. (3.4)) equals the asymptotic convergence factor of

$$B_{\tau,\lambda} := \bar{\mathbb{D}}(0;\tau) \cup [\tau,\lambda] \qquad \text{with } -\tau \leqslant \gamma \leqslant \tau \leqslant \lambda; \qquad (3.6)$$

i.e., the eigenvalues of $\mathcal{L}_\omega$ *interior* to the circle $\partial\mathbb{D}(0;\tau)$, as shown in Figure 2(a), have no effect on the resulting asymptotic convergence factor. The complement of the set $B_{\tau,\lambda}$ is simply connected, so that $B_{\tau,\lambda} \in \mathbb{M}$, provided that $1 \notin B_{\tau,\lambda}$, which is equivalent to $\lambda < 1$. Then $\kappa(B_{\tau,\lambda})$ can be calculated from (2.12). On the other hand, if $1 \in B_{\tau,\lambda}$ (i.e., if $\lambda \geqslant 1$), then (2.11) and the maximum modulus principle imply that $\kappa(B_{\tau,\lambda}) = 1$.

PROPOSITION 3. *The asymptotic convergence factor of the set* $B_{\tau,\lambda}$ (*cf.* (3.6)) *is given by*

$$\kappa(B_{\tau,\lambda}) = t - \sqrt{t^2 - 1}, \qquad \textit{where} \quad t := \frac{2\lambda(\tau^2+1) - (\lambda-\tau)^2}{(\lambda+\tau)^2} \quad (>1),$$

provided that $0 < \tau \leqslant \lambda < 1$.

Proof. We explicitly construct a conformal mapping function Φ which maps $\mathbb{C}_\infty \setminus B_{\tau,\lambda}$ onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}(0;1)$, where Φ is normalized by $\Phi(\infty) = \infty$. The mapping Φ can be expressed as the composition of three elementary mappings. The Joukowski-like transformation

$$u = \Phi_1(z) := \frac{z}{\tau} + \frac{\tau}{z}$$

maps $\mathbb{C}_\infty \setminus B_{\tau,\lambda}$ conformally onto $\mathbb{C}_\infty \setminus [-2, \Phi_1(\lambda)]$, the linear transformation

$$v = \Phi_2(u) := \frac{2u + 2 - \Phi_1(\lambda)}{2 + \Phi_1(\lambda)}$$

maps $\mathbb{C}_\infty \setminus [-2, \Phi_1(\lambda)]$ conformally onto $\mathbb{C}_\infty \setminus [-1, 1]$, and, finally, the inverse Joukowski transformation

$$\omega = \Phi_3(v) := v + \sqrt{v^2 - 1}$$

maps $\mathbb{C}_\infty \setminus [-1, 1]$ conformally onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}(0;1)$, where the branch of the square root has to be chosen so that $|\Phi_3(v)| > 1$ for all $v \notin [-1, 1]$ (cf. Krzyż [4, Exercise 2.5.9]).

Putting these pieces together, the composition of these mappings, namely $\Phi = \Phi_3 \circ \Phi_2 \circ \Phi_1$, then conformally maps $\mathbb{C}_\infty \setminus B_{\tau,\lambda}$ onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}(0;1)$, and as $B_{\tau,\lambda} \in \mathbb{M}$, it follows from (2.12) that

$$\kappa(B_{\tau,\lambda}) = \frac{1}{|\Phi(1)|} = \frac{1}{|(\Phi_3 \circ \Phi_2 \circ \Phi_1)(1)|}$$

has the form given in the statement of Proposition 3. ∎

Note that Proposition 3 remains valid in the limiting cases $\lambda \downarrow \tau$ and $\tau \downarrow 0$, i.e.,

$$\lim_{\lambda \downarrow \tau} \kappa(B_{\tau,\lambda}) = \kappa(\overline{\mathbb{D}}(0;\tau)) \quad \text{and} \quad \lim_{\tau \downarrow 0} \kappa(B_{\tau,\lambda}) = \kappa([0,\lambda]).$$

Finally, for $\omega_b < \omega < 2$, we have to investigate the "snap-ring"-shaped circular arcs (cf. (3.5)) of the form

$$C_{\tau,\alpha} := \{\tau e^{i\theta} : \alpha \leqslant \theta \leqslant 2\tau - \alpha\}, \tag{3.7}$$

where $\tau > 0$ and $0 < \alpha < \pi$, and thus $C_{\tau,\alpha} \in \mathbb{M}$.

PROPOSITION 4. *The asymptotic convergence factor of the set* $C_{\tau,\alpha}$ *(cf.* (3.7)*) is given by*

$$\kappa(C_{\tau,\alpha}) = \frac{\sin \alpha}{\sin(\alpha/2)}\left(\frac{\tau}{1+\tau+\sqrt{1-2\tau\cos\alpha+\tau^2}}\right).$$

Proof. Again, we explicitly construct, in three steps, a conformal mapping Φ of $\mathbb{C}_\infty \setminus C_{\tau,\alpha}$ onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}(0;1)$, with $\Phi(\infty) = \infty$. First,

$$u = \Phi_1(z) := i\frac{z - \tau\cos\alpha}{\tau\sin\alpha}$$

maps $\mathbb{C}_\infty \setminus C_{\tau,\alpha}$ conformally onto $\mathbb{C}_\infty \setminus \Gamma$, where $\Gamma \subset \partial\mathbb{D}(-i\cot\alpha; 1/\sin\alpha)$ is the circular arc joining $+1$ with -1 in the counterclockwise direction. Next,

$$v = \Phi_2(u) := u + \sqrt{u^2 - 1}$$

maps $\mathbb{C}_\infty \setminus \Gamma$ conformally onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}(-i\cot(\alpha/2); [1 - \cos(\alpha/2)]/\sin(\alpha/2))$, provided that we chose the branch of the square root which guarantees $|v|^2 > 1 + 2\tan\alpha \operatorname{Im} v$ (cf. Magnus [5, §6]). Finally, the linear transformation

$$w = \Phi_3(v) := \sin\left(\frac{\alpha}{2}\right)v + i\cos\left(\frac{\alpha}{2}\right)$$

maps $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}(-i\cot(\alpha/2);\ [1-\cos(\alpha/2)]/\sin(\alpha/2))$ conformally onto $\mathbb{C}_\infty \setminus \overline{\mathbb{D}}(0;1)$.

From (2.12), we conclude again that

$$\kappa(C_{\tau,\alpha}) = \frac{1}{|\Phi(1)|} = \frac{1}{|(\Phi_3 \circ \Phi_2 \circ \Phi_1)(1)|}$$
$$= \frac{\sin\alpha}{\sin(\alpha/2)}\left(\frac{\tau}{1+\tau+\sqrt{1-2\tau\cos\alpha+\tau^2}}\right),$$

which has the desired form given in the statement of Proposition 4. ■

We are now in a position to prove Theorem 1.

Proof of Theorem 1. Let us first assume that $-\infty < \omega < 0$ or $0 < \omega \leqslant 1$. Then $\Omega_{\omega,\beta}$ is a real interval whose end points λ_1 and λ_2 are given by (3.3). From Proposition 2, we obtain

$$\kappa(\Omega_{\omega,\beta}) = \frac{\beta\sqrt{\omega^2\beta^2+4(1-\omega)}}{2-\omega\beta^2+2\sqrt{1-\beta^2}},$$

and consequently,

$$\lim_{\omega\uparrow 0}\kappa(\Omega_{\omega,\beta}) = \lim_{\omega\downarrow 0}\kappa(\Omega_{\omega,\beta}) = \frac{\beta}{1+\sqrt{1-\beta^2}} = \sqrt{\omega_b-1}.$$

Moreover, differentiating the above expression for $\kappa(\Omega_{\omega,\beta})$ with respect to $\omega \in (-\infty,1)\setminus\{0\}$ shows that $\kappa(\Omega_{\omega,\beta})$ is a strictly monotonically decreasing function of ω, so that for each $\omega \neq 0$ with $-\infty < \omega < 1$,

$$\kappa(\Omega_{\omega,\beta}) > \kappa(\Omega_{1,\beta}) = \left(\frac{\beta}{1+\sqrt{1-\beta^2}}\right)^2 = \omega_b - 1,$$

the last equality following from (2.6). This establishes the last inequalities of (2.14) of Theorem 1. The remaining inequalities of (2.14) follow from (2.7).

We next consider the omitted value $\omega = 0$. In this case, $\mathscr{L}_0 = I_N$ (so the iterative method (2.4) is not consistent with the linear system (2.1)) and the associated covering domain is $\Omega_{0,\beta} = \{1\} \notin \mathbb{M}$. We use (2.11) to define $\kappa(\Omega_{0,\beta})$ to be unity. Hence, $\kappa(\Omega_{0,\beta}) := 1 = \rho(\mathscr{L}_0)$.

Next, we suppose that $1 \leqslant \omega \leqslant \omega_b$ (cf. (2.6)). From the discussion prior to Proposition 3, we can choose $\Omega_{\omega,\beta}$ to be the banjo-shaped set $B_{\tau,\lambda}$ of (3.6) with $\tau := \omega - 1$ and $\lambda := \lambda_2$, where λ_2 is given in (3.3). For the extreme cases $\omega = 1$ (i.e., $\tau = 0$) and $\omega = \omega_b$ (i.e., $\tau = \lambda$), $B_{\tau,\lambda}$ degenerates to an interval and a disk, respectively. Inserting $\tau = \omega - 1$ into the expression for t of Proposition 3 leads to

$$t = \frac{2\lambda(\omega^2 - 2\omega + 2) - (\lambda - \omega + 1)^2}{(\lambda + \omega - 1)^2} = \frac{2\lambda\omega^2 - (\lambda + \omega - 1)^2}{(\lambda + \omega - 1)^2}.$$

Since $(\lambda + \omega - 1)^2 = \lambda\omega^2\beta^2$ from (3.1), it follows that

$$t = \frac{2\lambda\omega^2 - \lambda\omega^2\beta^2}{\lambda\omega^2\beta^2} = \frac{2 - \beta^2}{\beta^2},$$

and thus, by Proposition 3,

$$\kappa(\Omega_{\omega,\beta}) = t - \sqrt{t^2 - 1} = \left(\frac{\beta}{1 + \sqrt{1 - \beta^2}}\right)^2 = \omega_b - 1,$$

for all $1 \leqslant \omega \leqslant \omega_b$, which establishes, from (2.6), the result of (2.15) of Theorem 1.

Next, we consider the case $\omega_b < \omega < \omega_e$. With $\lambda_2 = [\omega\beta + i\sqrt{4(\omega - 1) - \omega^2\beta^2}\,]^2/4$ from (3.3), it follows that $\Omega_{\omega,\beta}$ is the circular arc $C_{\omega - 1, \arg \lambda_2} \in \mathbb{M}$. We conclude from Proposition 4 that

$$\kappa(\Omega_{\omega,\beta}) = \frac{\beta\sqrt{\omega - 1}}{1 + \sqrt{1 - \beta^2}} = \sqrt{(\omega_b - 1)(\omega - 1)} \quad \text{and} \quad \rho(\mathscr{L}_\omega) = \omega - 1,$$

from which all relations (2.16) of Theorem 1 directly follow.

To complete the proof of Theorem 1, it remains to consider the case $\omega_e \leqslant \omega < \infty$. In this case, the eigenvalues of $\mathscr{L}_\omega$ lie in the banjo-shaped set

$$\Omega_{\omega,\beta} = \partial\mathbb{D}(0; \omega - 1) \cup [\lambda_1, \lambda_2]$$

with $\omega - 1 > 1$. But $z = 1$ is then an *interior* point of this set, and the maximum modulus principle, together with (2.11), therefore implies that $\kappa(\Omega_{\omega,\beta}) = 1$. ■

4. ADDITIONAL HYPOTHESIS

So far, we have assumed (cf. (2.3)) that

$$\sigma(B^2) \subset [0, \beta^2] \qquad \text{with} \quad 0 < \beta = \rho(\mathrm{B})$$

is the only information available to us for the spectrum of the Jacobi matrix B (cf. (2.2)). Following Dancis [1], we now assume that we have *additional information* of the form

$$\begin{gathered} \sigma(B^2) \subset [0, \gamma^2] \cup \{\beta^2\} \qquad \text{with} \\ 0 < \gamma := \max\{|\mu| : \mu \in \sigma(B) \text{ and } |\mu| < \beta\}. \end{gathered} \tag{4.1}$$

This corresponds to the case $k = 2$ of question (2) in Section 1, where we assume now that the *two* largest eigenvalues, β^2 and γ^2, of B^2 are explicitly known.

Using Young's fundamental relationship (3.1), this leads to sharper inclusions for $\sigma(\mathscr{L}_\omega)$:

$$\sigma(\mathscr{L}_\omega) \subset \Lambda_{\omega,\beta,\gamma}, \qquad \text{where} \quad \Lambda_{\omega,\beta,\gamma} := \Omega_{\omega,\gamma} \cup \{\lambda_1, \lambda_2\}, \tag{4.2}$$

where $\Omega_{\omega,\gamma}$ is defined in (3.2)–(3.5) by simply replacing β with γ throughout in these equations, and where λ_1 and λ_2 are given by (3.3), *without* replacing β with γ. More precisely, with

$$\begin{aligned} \lambda_3 = \lambda_3(\omega, \gamma) &:= \left[\frac{\omega\gamma - \sqrt{\omega^2\gamma^2 - 4(\omega - 1)}}{2}\right]^2, \\ \lambda_4 = \lambda_e(\omega, \gamma) &:= \left[\frac{\omega\gamma + \sqrt{\omega^2\gamma^2 - 4(\omega - 1)}}{2}\right]^2, \end{aligned} \tag{4.3}$$

and with the "suboptimal" relaxation parameters (cf. (2.6) and (2.13))

$$\omega_s := \frac{2}{1+\sqrt{1-\gamma^2}} < \omega_b \quad \text{and} \quad \omega_f := \frac{2}{1-\sqrt{1-\gamma^2}} > \omega_e, \quad (4.4)$$

we obtain

$$\sigma(\mathscr{L}_\omega) \subset \begin{cases} \{\lambda_1, \lambda_2\} \cup [\lambda_4, \lambda_3] & (-\infty < \omega < 0), \\ \{\lambda_1, \lambda_2\} \cup [\lambda_3, \lambda_4] & (0 < \omega < 1), \\ \{\lambda_1, \lambda_2\} \cup \partial\mathbb{D}(0; \omega - 1) \cup [\lambda_3, \lambda_4] & (1 \leqslant \omega \leqslant \omega_s), \\ \{\lambda_1, \lambda_2\} \cup \{(\omega - 1)e^{i\theta}: & \\ \quad \arg \lambda_4 \leqslant \theta \leqslant \arg \lambda_3 \ (\mathrm{mod}\ 2\pi)\} & (\omega_s < \omega < \omega_f), \\ \{\lambda_1, \lambda_2\} \cup \partial\mathbb{D}(0; \omega - 1) \cup [\lambda_3, \lambda_4] & (\omega_f \leqslant \omega < \infty), \end{cases}$$

where, for $\omega_s < \omega < \omega_f$, the branch of the square root in (4.3) has to be selected such than Im $\lambda_4 > 0$.

Since $\Lambda_{\omega,\beta,\gamma}$ of (4.2) differs from $\Omega_{\omega,\gamma}$ only by a discrete set, it follows from (2.11) that $\kappa(\Lambda_{\omega,\beta,\gamma}) = \kappa(\Omega_{\omega,\gamma})$. The following result is therefore an immediate consequence of Theorem 1.

THEOREM 5. *Assume that the Jacobi matrix B of* (2.2) *is a consistently ordered and weakly cyclic of index* 2 *matrix, and that the eigenvalues of B^2 are all nonnegative and lie in* $[0, \beta^2]$, *where* $0 < \beta = \rho(B) < 1$. *Assume further (cf.* (4.1)*) that the two largest eigenvalues β^2 and γ^2 of B^2 are known, where $0 < \gamma < \beta$. Then the asymptotic convergence factor $\kappa(\Lambda_{\omega,\beta,\gamma})$, considered as a function of the real parameter ω, satisfies the following properties*:

(i) *For $-\infty < \omega < 1$ and $\omega \neq 0$, $\kappa(\Lambda_{\omega,\beta,\gamma})$ is a strictly monotonically decreasing function of ω which satisfies*

$$\rho(\mathscr{L}_\omega) > 1 > \kappa(\Omega_{\omega,\beta}) > \kappa(\Lambda_{\omega,\beta,\gamma}) > \omega_s - 1 \qquad (-\infty < \omega < 0),$$

$$1 > \rho(\mathscr{L}_\omega) > \kappa(\Omega_{\omega,\beta}) > \kappa(\Lambda_{\omega,\beta,\gamma}) > w_s - 1 \qquad (0 < \omega < 1). \quad (4.5)$$

(ii) *For* $1 \leqslant \omega \leqslant \omega_s$, $\kappa(\Lambda_{\omega,\beta,\gamma})$ *is a constant function of* ω *which satisfies*

$$1 > \rho(\mathscr{L}_\omega) > \kappa(\Omega_{\omega,\beta}) > \kappa(\Lambda_{\omega,\beta,\gamma}) = \omega_s - 1. \tag{4.6}$$

(iii) *For* $\omega_s < w < \omega_f$, $\kappa(\Lambda_{\omega,\beta,\gamma})$ *is a strictly monotonically increasing function of* ω *which satisfies*

$$\begin{aligned} &1 > \rho(\mathscr{L}_\omega) \geqslant \kappa(\Omega_{\omega,\beta}) > \kappa(\Lambda_{\omega,\beta,\gamma}) > \omega_s - 1 \quad (\omega_s < \omega < 2),\\ &\rho(\mathscr{L}_\omega) \geqslant 1 > \kappa(\Omega_{\omega,\beta}) > \kappa(\Lambda_{\omega,\beta,\gamma}) > \omega_s - 1 \quad (2 \leqslant \omega < \omega_e),\\ &\rho(\mathscr{L}_\omega) > \kappa(\Omega_{\omega,\beta}) \geqslant 1 > \kappa(\Lambda_{\omega,\beta,\gamma}) > \omega_s - 1(\omega_e \leqslant \omega < \omega_f). \end{aligned} \tag{4.7}$$

(iv) *For* $\omega = 0$ *and for* $\omega_f \leqslant \omega < \infty$,

$$\rho(\mathscr{L}_\omega) \geqslant 1 = \kappa(\Omega_{\omega,\beta}) = \kappa(\Lambda_{\omega;\beta,\gamma}) > \omega_s - 1, \tag{4.8}$$

where ω_s *and* ω_f *are defined in* (4.4).

In particular, for any real ω,

$$\kappa(\Lambda_{\omega,\beta,\gamma}) < 1 \quad \textit{if and only if} \quad \omega \in (-\infty, \omega_f) \setminus \{0\}. \tag{4.9}$$

The results of Theorem 5 are illustrated in Figure 3.

Assuming $\sigma(B^2) \subset [0, \gamma^2] \cup \{\beta^2\}$ (cf. (4.1)), the question remains as to how one constructs semiiterative methods which achieve the best asymptotic rate of convergence $\omega_s - 1$. Dancis [1] suggests three different sequences $\{p_m\}_{m=0}^\infty$ of polynomials, each generating a semiiterative method, which he then applies to SOR iterations (2.4) with $\omega = \omega_s$:

$$p_m^{(1)}(z) := z^{m-1}\frac{z-\lambda_2}{1-\lambda_2},$$

$$p_m^{(2)}(z) := z^{m-k}\frac{z^k-\lambda_2^k}{1-\lambda_2^k},$$

$$p_m^{(3)}(z) := z^{m-k}\frac{z-\lambda_2}{1-\lambda_2}\,\frac{z^k-(\omega_s-1)^k}{1-(\omega_s-1)^k}\,\frac{1-(\omega_s-1)}{z-(\omega_s-1)},$$

where k is chosen such that $[(\omega_s - 1)/(\omega_b - 1)]^k \leqslant 0.1$. If $P^{(j)}$ denotes the

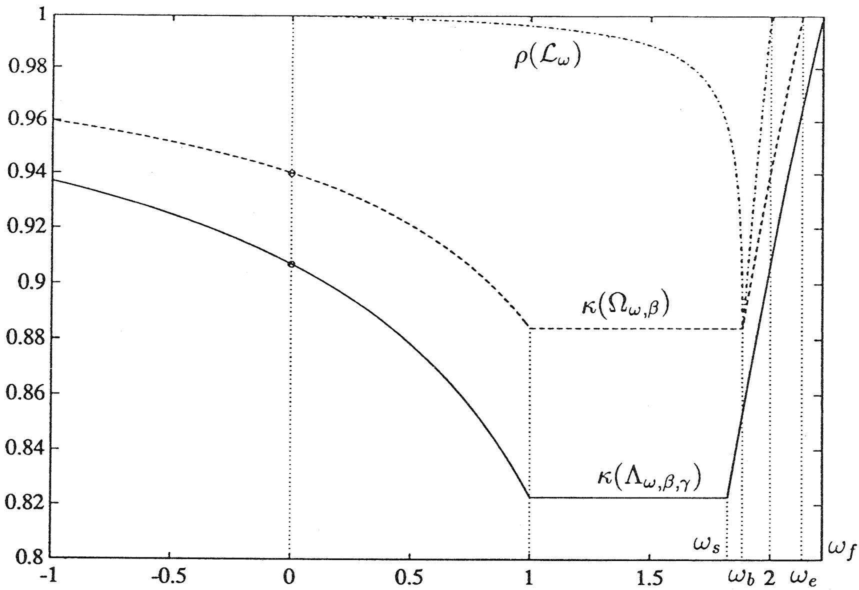

FIG. 3. $\rho(\mathscr{L}_\omega)$, $\kappa(\Omega_{\omega,\beta})$, and $\kappa(\Lambda_{\omega,\beta,\gamma})$ as functions of ω for the two-dimensional model problem with 2500 unknowns ($\beta = \rho(B) = \cos(\pi/51) = 0.99810\ldots$, $\omega_b = 1.88401\ldots$, and $\omega_e = 2.13119\ldots$; $\gamma = [\cos(\pi/51) + \cos(2\pi/51]/2 = 0.99526\ldots$, $\omega_s = 1.82277\ldots$, and $\omega_f = 2.21540\ldots$).

generating matrix (cf. (2.9)) associated with the above $\{p_m^{(j)}\}_{m=0}^{\infty}$, then

$$\kappa\left(\mathscr{L}_{\omega_s}, P^{(j)}\right) = \omega_s - 1 \qquad (j = 1, 2, 3).$$

While the asymptotic behavior of these three methods is the same, their performance for a finite number of iteration steps may differ considerably (cf. [1, §9]). In addition, certain orderings of the unknowns, even if they do not affect the asymptotic convergence factor, can lead to quite different convergence histories for the above schemes. We intend to investigate these and related questions in a forthcoming paper.

It is clear from the example considered in Figure 3 that a *significant improvement* can be achieved, as Dancis indicated in [1], in the asymptotic convergence rates of asymptotically optimal semiiterative methods applied to the SOR iterations (2.4), if one has additional explicit information concerning the largest eigenvalues of the matrix B^2. Moreover, while Theorem 5 specifically treats the case of $k = 2$ explicitly known largest eigenvalues of

B^2, it is clear that the techniques that we have developed here extend without change to the case when $k > 2$.

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REFERENCES

1 J. Dancis, The optimal ω is not best for the SOR iteration method, *Linear Algebra Appl.* 154–156:819–845 (1991).

2 M. Eiermann, W. Niethammer, and R. S. Varga, A study of semiiterative methods for nonsymmetric systems of linear equations, *Numer. Math.* 47:505–533 (1985).

3 K. Kontovasilis, R. J. Plemmons, and W. J. Stewart, Block cyclic SOR for Markov chains with p-cyclic infinitesimal generators, *Linear Algebra Appl.* 154–156:145–223 (1991).

4 J. G. Krzyż, *Problems in Complex Variable Theory*, Elsevier, New York, 1971.

5 A. Magnus, Rate of convergence of sequences of Padé-type approximants and pole detection in the complex plane, in *Padé Approximation and its Applications*, Lecture Notes in Math. 888 (M. G. de Bruin and H. van Rossum, Eds.), Springer-Verlag, Berlin, 1981, pp. 300–308.

6 R. S. Varga, A comparison of the successive overrelaxation method and semi-iterative methods using Chebyshev polynomials, *J. Soc. Industr. Appl. Math.* 5:39–46 (1957).

7 R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N.J., 1962.

8 D. M. Young, Iterative methods for solving partial differential equations of elliptic type, *Trans. Amer. Math. Soc.* 76:92–111 (1954).

9 D. M. Young, *Iterative Solution of Large Linear Systems*, Academic, New York, 1971.

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