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# NUMERICAL RESULTS ON BEST UNIFORM RATIONAL APPROXIMATION OF $|x|$ ON $[-1, +1]$

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ABSTRACT. With $E_{n,n}(|x|; [-1, +1])$ denoting the error of best uniform rational approximation from $\pi_{n,n}$ to $|x|$ on $[-1, +1]$, we determine the numbers $\{E_{2n,2n}(|x|; [-1, +1])\}_{n=1}^{40}$, where each of these numbers was calculated with a precision of at least 200 significant digits. With these numbers, the Richardson extrapolation method was applied to the products $\{e^{\pi\sqrt{2n}}E_{2n,2n}(|x|; [-1, +1])\}_{n=1}^{40}$, and it appears, to at least 10 significant digits, that

$$8 \stackrel{?}{=} \lim_{n\to\infty} e^{\pi\sqrt{2n}}E_{2n,2n}(|x|; [-1, +1]),$$

which gives rise to an interesting new conjecture in the theory of rational approximation.

## §1. INTRODUCTION

The function $|x|$, which is continuous but not differentiable on $[-1, +1]$, has been the focus of much research in approximation theory over the years. To indicate the history of the research in this area, first let $E_n(|x|; [-1, +1])$ denote the error of best uniform (polynomial) approximation from $\pi_n$ to $|x|$ on $[-1, +1]$ (where $\pi_n$ denotes the set of all real polynomials of degree at most $n$ $(n = 0, 1, \ldots)$). Because $|x|$ is an even continuous function on $[-1, +1]$, it is easily seen (cf. [21], p. 2) that

$$E_{2n}(|x|; [-1, +1]) = E_{2n+1}(|x|; [-1, +1]) \qquad (n = 0, 1, \ldots). \tag{1.1}$$

In 1913, S. Bernstein proved in [1] that there exists a positive constant $\beta$ for which

$$\beta = \lim_{n\to\infty} 2nE_{2n}(|x|; [-1, +1]), \tag{1.2}$$

and, moreover, that $\beta$ satisfies

$$0.278 < \beta < 0.286. \tag{1.3}$$

Bernstein remarked in [1] "as a curious coincidence" that $1/(2\sqrt{\pi}) = 0.282094\ldots$, this number being nearly the average, namely 0.282, of the upper and lower bounds in (1.3). This latter remark became known in the literature as the *Bernstein Conjecture*: $\beta \stackrel{?}{=} 1/(2\sqrt{\pi})$. Recently in 1985, Varga and Carpenter [22] showed, via high-precision computations, that the Bernstein Conjecture was *false*; in particular, it was shown in [22] that

$$0.2801685\ldots \le \beta \le 0.2801733\ldots. \tag{1.4}$$

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(Estimates derived from numerical extrapolations, which give $\beta$ to fifty significant digits, are also given in [22].) It is also conjectured in [22] that $2nE_{2n}(|x|;[-1,+1])$ admits the asymptotic series expansion

$$2nE_{2n}(|x|;[-1,+1]) \approx \beta - \frac{K_1}{n^2} + \frac{K_2}{n^4} - \frac{K_3}{n^6} + \cdots \qquad (n\to\infty), \tag{1.5}$$

where the constants $K_j$ (which are independent of $n$) are all *positive.*

Then, more than fifty years after the work of Bernstein [1], D.J. Newman [15] in 1964 showed how decisively *different* best uniform *rational* approximation from $\pi_{n,n}$ to $|x|$ on $[-1,+1]$ is, in that if

$$E_{n,n}(|x|;[-1,+1]) := \inf\{\||x|-r(x)\|_{L_\infty[-1,+1]} : r(x)\in\pi_{n,n}\}, \tag{1.6}$$

then

$$\frac{1}{2e^{9\sqrt{n}}} \le E_{n,n}(|x|;[-1,+1]) \le \frac{3}{e^{\sqrt{n}}} \qquad (n=4,5,\ldots). \tag{1.7}$$

(Here, $\pi_{m,n}$ denotes the set of all real rational functions $r(x)=p(x)/q(x)$ with $p\in\pi_m$ and $q\in\pi_n$, where it is assumed that $p$ and $q$ have no common factors, that $q$ does not vanish on $[-1,+1]$, and that $q$ is normalized by $q(0)=1$.)

Since Newman's pathfinding result of [15], it has been shown by Bulanov [7] in 1968 that

$$E_{n,n}(|x|;[-1,+1]) \ge e^{-\pi\sqrt{n+1}} \qquad (n=0,1,\ldots), \tag{1.8}$$

and by Vyacheslavov [23] in 1975 that there exist positive constants $M_1$ and $M_2$ such that

$$M_1 \le e^{\pi\sqrt{n}}E_{n,n}(|x|;[-1,+1]) \le M_2 \qquad (n=1,2,\ldots). \tag{1.9}$$

From (1.8) and (1.9), it is elementary to verify that

$$\begin{aligned} e^{\pi(1-\sqrt{2})} &= 0.272180\ldots \\ &\le e^{\pi\sqrt{n}}E_{n,n}(|x|;[-1,+1]) \le M_2 \qquad (n=1,2,\ldots), \end{aligned} \tag{1.10}$$

and, if

$$\begin{aligned} \underline{M} &:= \varliminf_{n\to\infty} e^{\pi\sqrt{n}}E_{n,n}(|x|;[-1,+1]) \quad \text{and} \\ \overline{M} &:= \varlimsup_{n\to\infty} e^{\pi\sqrt{n}}E_{n,n}(|x|;[-1,+1]), \end{aligned} \tag{1.11}$$

that also

$$1 \le \underline{M} \le \overline{M}. \tag{1.12}$$

Clearly, Vyacheslavov's result in (1.9) gives the asymptotically *sharp* multiplier, namely $\pi$, for $\sqrt{n}$ in the asymptotic behavior of $E_{n,n}(|x|;[-1,+1])$ as $n\to\infty$. What only remains then is the determination of the best *asymptotic* constants $\underline{M}$ and $\overline{M}$ of (1.11).

Next, it is important to stress that deep theoretical results for the asymptotic behavior of the best uniform rational approximation $E_{n,n}(f;[-1,+1])$, as $n\to\infty$, have also been derived for more general continuous functions $f$ on $[-1,+1]$, which include $|x|$ as a special case. Specifically, it has been shown in 1966 by Szüsz and Turán [18], [19], for any continuous piecewise analytic function $f$ on $[-1,+1]$, that

$$E_{n,n}(f;[-1,+1]) = O(e^{-c_1(f)\sqrt{n}}) \qquad (n\to\infty), \tag{1.13}$$

and in 1967 by Gonchar [10], for any piecewise infinitely differentiable function $f$ on $[-1, +1]$, that

$$(1.14) \qquad e^{-c_2(f)\sqrt{n}} \le E_{n,n}(f; [-1, +1]) \le c_3(f) e^{-c_4(f) n^{\gamma}} \qquad (n = 1, 2, \ldots),$$

where, in (1.13) and (1.14),

$$c_j(f) > 0 \ (j = 1, 2, 3, 4) \quad \text{and} \quad \gamma := \gamma(f) \le 1/2.$$

In addition, Gonchar and Rakhmanov [11] recently obtained elegant results using potential-theoretic techniques in the complex plane, which, in a special case, established the particularly sharp asymptotic result of

$$(1.15) \qquad \lim_{n\to\infty} \{E_{n,n}(e^{-x}; [0, +\infty))\}^{1/n} = \Lambda = \frac{1}{9.2890254919\ldots},$$

thereby completely solving the "1/9" Conjecture. (For the history and mathematical literature associated with the "1/9" Conjecture, see [20], Chapter 7, and [21], Chapter 2.) There is optimism that the results of [11] can be eventually extended to give the sharp asymptotic behavior of $E_{n,n}(f; [0, +\infty))$ for classes of functions $f$ which are piecewise analytic (or piecewise infinitely differentiable) on $[0, +\infty)$. It is our hope that the results derived from our numerical results here on the asymptotic behavior of $E_{2n,2n}(|x|; [-1, +1])$ as $n \to \infty$ will hasten such theoretical extensions of [11]. We remark that for a recent complete treatment of Vyacheslavov's result (1.9) and results pertaining to the "1/9" Conjecture, see Petrushev and Popov [16], Chapter 4, which also contains interesting additional references, historical comments, and related results.

To outline the remainder of this paper, in §2 we give the theoretical background and numerical description for how the quantities $\{E_{2n,2n}(|x|; [-1, +1])\}_{n=1}^{40}$ were numerically determined. In §3,the Richardson extrapolation method is applied to the products

$$\{e^{\pi\sqrt{2n}} E_{2n,2n}(|x|; [-1, +1])\}_{n=1}^{40},$$

which gives strong numerical indications for two new conjectures concerning the asymptotic behavior of $e^{\pi\sqrt{2n}} E_{2n,2n}(|x|; [-1, +1])$ as $n \to \infty$. Then, in §4 we discuss the location of the zeros and poles of the unique best uniform approximation, $r^*_{n,n}(t)$, from $\pi_{n,n}$ to $\sqrt{t}$ on $[0, 1]$, and we graph the error function, $-\sqrt{t} + r^*_{n,n}(t)$, on $[0, 1]$ for the case $n = 32$, showing its extreme points. The same is done for the unique best uniform approximation, $s_{2n,2n}(x)$, from $\pi_{2n,2n}$ to $|x|$ on $[-1, +1]$. Finally, to come full circle, we also show in §4 how Newman's method of proof in [15] is related to the determination of the unique best uniform approximation from $\pi_{2n,2n}$ to $|x|$ on $[-1, +1]$.

## §2. The numbers $\{E_{2n,2n}(|x|; [-1, +1])\}_{n=1}^{40}$

We first consider the best uniform approximation from $\pi_{m,n}$ to $\sqrt{t}$ on $[0, 1]$ for each pair $(m, n)$ of nonnegative integers. In general, it is known (cf. Rivlin [17], p. 125) that there is a *unique* $r^*_{m,n} \in \pi_{m,n}$ such that

$$(2.1) \qquad E_{m,n}(\sqrt{t}; [0, 1]) = \|\sqrt{t} - r^*_{m,n}(t)\|_{L_\infty[0,1]}.$$

Moreover, for each pair $(m, n)$ of nonnegative integers, it is easily verified that

$$(2.2) \qquad W_{m,n} := \operatorname{span}\{1, t, \ldots, t^m; t^{1/2}, t^{3/2}, \ldots, t^{n+1/2}\}$$

is a *Haar space* of dimension $m + n + 2$ on the interval $[0, 1]$, i.e., any function, not identically zero, in $W_{m,n}$ has at most $m + n + 1$ distinct zeros in $[0, 1]$. Thus, in the terminology of Loeb [13], $\sqrt{t}$ is *hypernormal* on $[0, 1]$. Consequently (cf. [13] or Meinardus [14], p. 165), for any pair $(m, n)$ of nonnegative integers, the unique best uniform approximation $r^*_{m,n} := p^*/q^*$ in $\pi_{m,n}$ for which (2.1) is valid has the property that $\partial p^* = m$, $\partial q^* = n$ (where $\partial s$ denotes the exact degree of a polynomial $s$), and, moreover, that the largest alternation set for $\sqrt{t} - r^*_{m,n}(t)$ on $[0, 1]$ consists of $m + n + 2$ points.

For our applications, we restrict attention to the cases when $m = n$ $(n = 1, 2, \ldots)$, and we write $r^*_{n,n}$ in $\pi_{n,n}$ as $r^*_{n,n} := p^*_n/q^*_n$, where

$$p^*_n(t) := \sum_{j=0}^{n} a^*_j(n)t^j \quad \text{and} \quad q^*_n(t) := 1 + \sum_{j=1}^{n} b^*_j(n)t^j \qquad (n = 1, 2, \ldots). \tag{2.3}$$

With the hypernormality of $\sqrt{t}$ on $[0, 1]$, we next easily establish the result of

**Proposition 1.** *For each positive integer* $n$, *let* $r^*_{n,n}(t) := p^*_n(t)/q^*_n(t)$ *be the best uniform approximation from* $\pi_{n,n}$ *to* $\sqrt{t}$ *on* $[0, 1]$. *Then, the coefficients of* $p^*_n(t)$ *and* $q^*_n(t)$, *as given in* (2.3), *satisfy*

$$a^*_j(n) > 0 \quad (j = 0, 1, \ldots, n) \quad \text{and} \quad b^*_j(n) > 0 \quad (j = 1, 2, \ldots, n). \tag{2.4}$$

*Proof*. Fix $n$ to be any positive integer. Since the largest alternation set for $\sqrt{t} - r^*_{n,n}(t)$ in $[0, 1]$ has length $2n + 2$, it follows that there are $2n + 1$ points $\{t_k\}_{k=1}^{2n+1}$ in $(0, 1)$, satisfying

$$0 < t_1 < \cdots < t_{2n+1} < 1, \tag{2.5}$$

for which

$$-\sqrt{t_k} + r^*_{n,n}(t_k) = 0 \qquad (k = 1, 2, \ldots, 2n + 1). \tag{2.6}$$

Hence, on writing $r^*_{n,n} = p^*_n/q^*_n$, and on recalling the convention (since $r^*_{n,n} \in \pi_{n,n}$) that $q^*_n$ does not vanish on $[0, 1]$, this implies (on dropping the dependence on $n$ of the coefficients in (2.3)) that

$$\sum_{j=0}^{n} a^*_j t^j_k - \sum_{j=1}^{n} b^*_j t_k^{j+1/2} = \sqrt{t_k} \qquad (k = 1, 2, \ldots, 2n + 1), \tag{2.7}$$

which represents $2n + 1$ linear equations in the $2n + 1$ unknowns $\{a^*_j\}_{j=0}^n$ and $\{b^*_j\}_{j=1}^n$. On setting

$$a^*_{j+n} := -b^*_j \qquad (j = 1, 2, \cdots, n) \tag{2.8}$$

and reordering these unknowns as

$$(a^*_0, \overbrace{a^*_1, a^*_{n+1}}, \overbrace{a^*_2, a^*_{n+2}}, \ldots, \overbrace{a^*_n, a^*_{2n}})^T,$$

the associated coefficient matrix $A$, of order $2n + 1$, for the reordered unknowns in (2.7) can be expressed succinctly as

$$A = [(t_i)^{\alpha_j}], \tag{2.9}$$

where $\alpha_1 := 0$, $\alpha_{2l} := l$ $(l = 1, 2, \ldots, n)$, and $\alpha_{2l+1} := l + \frac{1}{2}$ $(l = 1, 2, \ldots, n)$, so that

$$0 = \alpha_1 < \alpha_2 < \cdots < \alpha_{2n+1}. \tag{2.10}$$

As such, $A$ is (cf. Gantmakher [9], p. 99) a nonsingular Vandermonde matrix with

$$\det A > 0. \tag{2.11}$$

Next, for convenience set

$$(c_1, c_2, \dots, c_{2n+1})^T := (a_0^*, \overbrace{a_1^*, a_{n+1}^*}, \overbrace{a_2^*, a_{n+2}^*}, \dots, \overbrace{a_n^*, a_{2n}^*})^T. \tag{2.12}$$

From Cramer's rule, it is well known that

$$c_j = \det A_j / \det A \qquad (j = 1, 2, \dots, 2n+1), \tag{2.13}$$

where $A_j$ denotes the matrix obtained from the matrix $A$ when the $j$th column of $A$ is replaced by the column vector of the right side of (2.7), namely $(\sqrt{t_1}, \sqrt{t_2}, \dots, \sqrt{t_{2n+1}})^T$. From this definition, it can be verified that $A_1$ can be represented as

$$A_1 = [(t_i)^{\tilde{\alpha}_j}], \tag{2.14}$$

where $\tilde{\alpha}_1 = 1/2$, $\tilde{\alpha}_{2l} = l$ $(l = 1, 2, \dots, n)$, and $\tilde{\alpha}_{2l+1} := l + 1/2$ $(l = 1, 2, \dots, n)$, so that

$$0 < \tilde{\alpha}_1 < \tilde{\alpha}_2 < \dots < \tilde{\alpha}_{2n+1}. \tag{2.15}$$

As is the case for $A$, $A_1$ is then a nonsingular Vandermonde matrix with $\det A_1 > 0$. Thus, from (2.11) and (2.13), $c_1 = a_0^* > 0$. The same argument similarly shows that $\det A_2 > 0$; whence, $c_2 = a_1^* > 0$. However, on considering $A_l$ (where $3 \le l \le 2n+1$), then $A_l$ becomes a Vandermonde matrix only after a suitable permutation of its columns so that its associated exponents $\{\alpha_j(l)\}_{j=1}^{2n+1}$ form, as in (2.10) and (2.15), an *increasing* sequence of nonnegative numbers. As is readily verified, any permutation of the columns of $A_l$ which brings its associated exponents $\{\alpha_j(l)\}_{j=1}^{2n+1}$ into an increasing sequence of nonnegative numbers is an *even* permutation for $l$ even and an *odd* permutation for $l$ odd $(3 \le l \le 2n+1)$. Thus,

$$\operatorname{sgn} \det A_l = (-1)^l \qquad (3 \le l \le 2n+1),$$

so that, from (2.11) and (2.13),

$$\operatorname{sgn} c_l = (-1)^l \qquad (3 \le l \le 2n+1). \tag{2.16}$$

With $a_0^* > 0$ and $a_1^* > 0$ from the discussion above, then (2.16), along with the definitions of (2.8) and (2.12), gives the desired conclusion that $a_j^* > 0$ $(j = 0, 1, \dots, n)$ and that $b_j^* > 0$ $(j = 1, 2, \dots, n)$. □

We remark that a stronger result for $\sqrt{t}$ than Proposition 1, based on best uniform rational approximations of Stieltjes functions, can be found in Blatt, Iserles, and Saff ([2], Lemma 3.2). Our reason for including Proposition 1 and its proof (based on Vandermonde matrices) is that it leads to the following numerically useful result.

**Proposition 2.** *For each positive integer $n$, let $\{t_k\}_{k=1}^{2n+1}$ be any points in $(0, 1)$ satisfying*

$$0 < t_1 < t_2 < \dots < t_{2n+1} < 1. \tag{2.17}$$

*Then, there exist unique positive numbers $\{a_j\}_{j=0}^{n}$ and $\{b_j\}_{j=1}^{n}$ such that*

$$g(t) := -\sqrt{t} + \left(\sum_{j=0}^{n} a_j t^j\right) \Big/ \left(1 + \sum_{j=1}^{n} b_j t^j\right) \tag{2.18}$$

*vanishes at the points $\{t_k\}_{k=1}^{2n+1}$. Moreover, with $t_0 := 0$ and $t_{2n+2} := 1$, then*

$$\operatorname{sgn} g(t) = (-1)^l \text{ on the interval } (t_l, t_{l+1}) \qquad (l = 0, 1, \dots, 2n+1). \tag{2.19}$$

*Proof*. As in (2.7), consider the following system of $2n+1$ linear equations in the $2n+1$ unknowns $\{a_k\}_{k=0}^n$ and $\{b_k\}_{k=1}^n$:

$$\sum_{j=0}^{n} a_j t_k^j - \sum_{j=1}^{n} b_j t_k^{j+1/2} = \sqrt{t_k} \qquad (k = 1, 2, \dots, 2n+1). \tag{2.20}$$

But the proof of Proposition 1 can be used to similarly show that the $2n+1$ numbers $\{a_j\}_{j=0}^n$ and $\{b_j\}_{j=1}^n$ which solve (2.20) are uniquely determined and are all positive. Consequently, $1+\sum_{j=1}^n b_j t^j$ is positive on $[0, 1]$, and $(\sum_{j=0}^n a_j t^j)/(1+\sum_{j=1}^n b_j t^j)$ is an element of $\pi_{n,n}$ for which $g(t)$ vanishes in the points $\{t_k\}_{k=1}^{2n+1}$.

Next, with $b_0 := 1$, then $h(t) := \sum_{j=0}^n a_j t^j - \sum_{j=0}^n b_j t^{j+1/2}$ is a *nonzero* element (cf. (2.2)) of the linear space $W_{n,n}$ and has, from (2.20), $2n+1$ distinct zeros in the points $\{t_k\}_{k=1}^{2n+1}$ in $(0, 1)$. Since $W_{n,n}$ is a Haar space of dimension $2n+2$, then $h(t)$ can have no additional zeros in $[0, 1]$, which gives that $g(t)$ of (2.18) is of one sign on each interval $(t_l, t_{l+1})$, $l = 0, 1, \dots, 2n+1$.

Finally, suppose that (2.19) is *false*, i.e., that $g(t)$ has the *same* sign on two adjacent intervals, say $(t_l, t_{l+1})$ and $(t_{l+1}, t_{l+2})$, where $0 \le l \le 2n$. This implies that $g'(t_{l+1}) = 0$. Now with $D(t) := \sum_{j=0}^n b_j t^j$, it follows by definition that $g(t) = h(t)/D(t)$, so that

$$g'(t) = \frac{D(t)h'(t) - h(t)D'(t)}{D^2(t)}. \tag{2.21}$$

Because $D(t) > 0$ on $[0, 1]$ from the positivity of the coefficients $b_j$ and because $h(t_k) = 0$ for all $k = 1, 2, \dots, 2n+1$, then $g'(t_{l+1}) = 0$ implies from (2.21) that $h'(t_{l+1}) = 0$. Moreover, from Rolle's Theorem, $h'(t)$ has one zero in each interval $(t_j, t_{j+1})$, where $j = 1, 2, \dots, 2n$. Thus, $h'(t)$ has $2n+1$ distinct zeros on $(0, 1)$. But, $h'(t)$ is an element of the Haar space

$$\widetilde{W}_n := \operatorname{span}\{1, t, \dots, t^{n-1}; t^{-1/2}, t^{1/2}, \dots, t^{n-1/2}\},$$

which has dimension $2n+1$ on $(0, 1]$. This implies $h'(t) \equiv 0$, which contradicts the fact that $h(t) = \sum_{j=0}^n a_j t^j - \sum_{j=0}^n b_j t^{j+1/2}$ with $a_j > 0$ and $b_j > 0$ for all $j = 0, 1, \dots, n$. Thus, (2.19) is valid. □

Though our interest here is in the specific function $\sqrt{t}$ on $[0, 1]$, we remark that the function $t^\alpha$, for any real number $\alpha$ with $0 < \alpha < 1$, is also hypernormal on $[0, 1]$, and that Propositions 1 and 2 are similarly valid for $t^\alpha$, with $0 < \alpha < 1$. In this regard, see Ganelius [8] for rational approximations of $t^\alpha$ on $[0, 1]$ when $\alpha$ is a rational number with $0 < \alpha < 1$.

Proposition 2 can be used as follows in conjunction with the Remez algorithm (cf. Rivlin [17], p. 136) for numerically determining the numbers $\{E_{n,n}(\sqrt{t}; [0, 1])\}_{n=1}^{40}$. For a fixed positive integer $n$, let $\{t_k\}_{k=1}^{2n+1}$ be any points in $(0, 1)$ satisfying (2.17). Then from Proposition 2, there exist unique positive numbers $\{a_j\}_{j=0}^n$ and $\{b_j\}_{j=1}^n$ such that $g(t)$, as defined in (2.18), vanishes at the points $\{t_k\}_{k=1}^{2n+1}$, and $\operatorname{sgn} g(t) = (-1)^l$ on each interval $(t_l, t_{l+1})$, where $t_0 := 0$, $t_{2n+2} := 1$, and $l = 0, 1, \dots, 2n+1$. Setting

$$m_l := \max\{|g(t)| : t_l \le t \le t_{l+1}\} \qquad (l = 0, 1, \dots, 2n+1),$$

there exists (from the continuity of $g(t)$ on $(0, 1]$) a $u_l$ in $[t_l, t_{l+1}]$ for which

$$m_l = |g(u_l)| \qquad (l = 0, 1, \dots, 2n+1). \tag{2.22}$$

Because $g(t_k) = 0$ for $k = 1, 2, \dots, 2n+1$, we have, more precisely, that $u_0 \in [0, t_1)$, $u_1 \in (t_1, t_2), \dots, u_{2n+1} \in (t_{2n+1}, 1]$. From (2.19) of Proposition 2, we further have that

$$\operatorname{sgn} g(u_l) = (-1)^l \qquad (l = 0, 1, \ldots, 2n+1).$$

This means that starting with the interpolation points $\{t_k\}_{k=1}^{2n+1}$, with $0 < t_1 < t_2 < \cdots < t_{2n+1} < 1$, one obtains the rational function $(\sum_{j=0}^n a_j t^j)/(1 + \sum_{j=1}^n b_j t^j)$ such that $g(t)$ of (2.18) has precisely the correct number, namely $2n+2$, of sign changes in the intervals $\{(t_l, t_{l+1})\}_{l=0}^{2n+1}$. Thus, one can move directly to the *leveling procedure* of the Remez algorithm, at the distinct points $\{u_l\}_{l=0}^{2n+1}$ of (2.22) of $[0, 1]$, where one determines the new rational function $(\sum_{j=0}^n \tilde{a}_j t^j)/(1 + \sum_{j=1}^n \tilde{b}_j t^j)$ and a positive number $\lambda$ such that

$$(2.23) \qquad -\sqrt{u_l} + \frac{\sum_{j=0}^n \tilde{a}_j u_l^j}{1 + \sum_{j=1}^n \tilde{b}_j u_l^j} + (-1)^{l+1}\lambda = 0 \qquad (l = 0, 1, \ldots, 2n+1).$$

Because $\{u_l\}_{l=0}^{2n+1}$ is a subset of $[0, 1]$, it follows from (2.23) that

$$(2.24) \qquad \left\| -\sqrt{t} + \frac{\sum_{j=0}^n \tilde{a}_j t^j}{1 + \sum_{j=1}^n \tilde{b}_j t^j} \right\|_{L_\infty[0,1]} - \lambda \geq 0.$$

With a preassigned (small) $\varepsilon > 0$, if the above difference is less than $\varepsilon$, this iteration is terminated. Otherwise, a new set of local extrema $\{\tilde{u}_l\}_{l=0}^{2n+1}$ in $[0, 1]$ is determined from $-\sqrt{t} + (\sum_{j=0}^n \tilde{a}_j t^j)/(\sum_{j=0}^n \tilde{b}_j t^j)$, and, as is well known, this procedure, when repeated, is ultimately *quadratically convergent* (cf. Meinardus [14], p. 113). In this manner, the first few of the numbers $\{E_{n,n}(\sqrt{t}; [0, 1])\}_{n=1}^{40}$ were determined, each with a precision of at least 200 significant digits, using Brent's MP (multiple precision) package [5] on a SUN 3/80 at Kent State University. For larger values of $n$, the following *ad hoc* procedure was successfully applied. Assuming that the numbers $\{E_{j,j}(\sqrt{t}; [0, 1])\}_{j=n-2}^n$ with $n \geq 3$ have all been determined to high precision, let $\{t_k^{(n)}\}_{k=0}^{2n+1}$ denote the alternation set (of length $2n+2$), derived from the converged abscissas (from the Remez algorithm) of the local extrema of $-\sqrt{t} + r_{n,n}^*(t)$ in $[0, 1]$, where

$$(2.25) \qquad \begin{cases} 0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{2n+1}^{(n)} = 1, \text{ and} \\ -\sqrt{t_k^{(n)}} + r_{n,n}^*(t_k^{(n)}) = (-1)^k E_{n,n}(\sqrt{t}; [0, 1]) \qquad (k = 0, 1, \ldots, 2n+1). \end{cases}$$

Now, it was numerically observed that the following *ratios* of these abscissas, namely

$$\lambda_k^{(n)} := t_{k+1}^{(n)} / t_1^{(n)} \qquad (k = 1, 2, \ldots, 2n-1),$$

were roughly *unchanged* with $n$, i.e., $\lambda_k^{(n+1)} \doteq \lambda_k^{(n)}$ for $k = 1, 2, \ldots, 2n-1$. (These ratios were, in fact, nearly identical for smaller values of $k$.) Then, initial estimates $\{\tilde{t}_k^{(n+1)}\}_{k=0}^{2n+3}$ for the alternation set, in the case $n+1$, were defined by $\tilde{t}_0^{(n+1)} := 0$ and

$$\{\tilde{t}_{k+1}^{(n+1)} := \tilde{t}_1^{(n+1)} \lambda_k^{(n)}\}_{k=1}^{2n},$$

where $\tilde{t}_1^{(n+1)}$ was obtained from a Newton quadratic extrapolation of first interior abscissas $\{t_1^{(n-2)}, t_1^{(n-1)}, t_1^{(n)}\}$ from the last three cases. With $\tilde{t}_{2n+3}^{(n+1)} := 1$, there are two remaining initial abscissa estimates, namely, $\tilde{t}_{2n+1}^{(n+1)}$ and $\tilde{t}_{2n+2}^{(n+1)}$, which must be defined, and these were computed from simple averages so that these remaining values $\tilde{t}_{2n+1}^{(n+1)}$ and $\tilde{t}_{2n+2}^{(n+1)}$ satisfied

$$0 = \tilde{t}_0^{(n+1)} < \tilde{t}_1^{(n+1)} < \cdots < \tilde{t}_{2n}^{(n+1)} < \tilde{t}_{2n+1}^{(n+1)} < \tilde{t}_{2n+2}^{(n+1)} < \tilde{t}_{2n+3}^{(n+1)} = 1.$$

In all cases, these initial estimates of the alternation set of $2n+4$ equioscillating extrema for the case $n+1$ of (2.25) were sufficiently accurate so that the Remez algorithm, when applied with these initial estimates, converged in all cases. (We remark that we stopped for convenience at $n = 40$ in the determination of the numbers $E_{n,n}(\sqrt{t}; [0, 1])$, rather than from a breakdown in this procedure.) In this way, each of the numbers $\{E_{n,n}(\sqrt{t}; [0, 1])\}_{n=1}^{40}$ was obtained to a precision of at least 200 significant digits.

Finally, since $\sqrt{t}$ is hypernormal on $[0, 1]$, then the unique best uniform approximation $r_{n,n}^* := p_n^*/q_n^*$ to $\sqrt{t}$ from $\pi_{n,n}$ on $[0, 1]$ has its longest alternation set in $[0, 1]$ of length $2n+2$, i.e., there exist points $\{t_l^{(n)}\}_{l=0}^{2n+1}$ satisfying $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{2n+1}^{(n)} = 1$ for which

$$(2.26) \qquad -\sqrt{t_l^{(n)}} + \frac{p_n^*(t_l^{(n)})}{q_n^*(t_l^{(n)})} = (-1)^l \cdot E_{n,n}(\sqrt{t}; [0, 1]) \qquad (l = 0, 1, \ldots, 2n+1).$$

With the change of variables $x^2 = t$, set $s_{2n,2n}^*(x) := r_{n,n}^*(x^2) = \frac{p_n^*(x^2)}{q_n^*(x^2)}$, so that $s_{2n,2n}^*(x)$ is an element of $\pi_{2n,2n}$. It is clear that $s_{2n,2n}^*(x)$, like $|x|$, is an even function of $x$, and it is readily verified that $|x| - s_{2n,2n}^*(x)$ has an alternation set of length $2(2n+2) - 1 = 4n+3 =: L$ on $[-1, +1]$. But, as $\partial q_n^*(x^2) = 2n = \partial p_n^*(x^2)$ and as

$$(2.27) \qquad L = 4n+3 \geq 2 + \max\{2n + \partial q_n^*(x^2);\, 2n + \partial p_n^*(x^2)\} = 4n+2,$$

then (cf. [14], p. 162) $s_{2n,2n}^*(x)$ is the best uniform approximation to $|x|$ from $\pi_{2n,2n}$ on $[-1, +1]$, with

$$E_{2n,2n}(|x|; [-1, +1]) = E_{n,n}(\sqrt{t}; [0, 1]) \qquad (n = 1, 2, \ldots).$$

Similarly, since

$$L = 4n+3 \geq 2 + \max\{2n+1 + \partial q_n^*(x^2);\, 2n+1 + \partial p_n^*(x^2)\} = 4n+3,$$

then $s_{2n,2n}^*(x)$ is *also* the best uniform approximation to $|x|$ from $\pi_{2n+1,2n+1}$ on $[-1, +1]$, which is similar to the polynomial case in (1.1). This gives us the following result:

**Proposition 3.** *For each positive integer* $n$, *let* $r_{n,n}^*(t) := p_n^*(t)/q_n^*(t)$ *be the best uniform approximation from* $\pi_{n,n}$ *to* $\sqrt{t}$ *on* $[0, 1]$. *Then,* $s_{2n,2n}^*(x) := p_n^*(x^2)/q_n^*(x^2)$ *is the best uniform approximation to* $|x|$ *from* $\pi_{2n,2n}$ *and* $\pi_{2n+1,2n+1}$ *on* $[-1, +1]$, *and*

$$(2.28) \qquad E_{2n,2n}(|x|; [-1, +1]) = E_{2n+1,2n+1}(|x|; [-1, +1]) = E_{n,n}(\sqrt{t}; [0, 1]).$$

We remark that the outer equality, i.e., $E_{2n,2n}(|x|; [-1, +1]) = E_{n,n}(\sqrt{t}; [0, 1])$, of (2.28) is well known, and can be found in Petrushev and Popov [16], p. 74.

From our numerical determinations of $\{E_{n,n}(\sqrt{t}; [0, 1])\}_{n=1}^{40}$, we list in Table 1 below the equivalent numbers $\{E_{2n,2n}(|x|; [-1, +1])\}_{n=1}^{40}$, truncated to 25 decimal digits, as well as the products $\{e^{\pi\sqrt{2n}} E_{2n,2n}(|x|; [-1, +1])\}_{n=1}^{40}$, again truncated to 25 decimal digits. We remark that the first three numbers $(n = 1, 2, 3)$ of column 2 of Table 1 can be found, to about four significant digits, in the Appendix of Petrushev and Popov [16].

TABLE 1

| $n$ | $E_{2n,2n}(\|x\|; [-1, +1])$ | $e^{\pi\sqrt{2n}} E_{2n,2n}(\|x\|; [-1, +1])$ |
|---|---|---|
| 1 | 4.368901269207636157085597 1e-2 | 3.714426543683164139389263 1 |
| 2 | 8.501484704073829490297411 3e-3 | 4.552474118602959576565174 6 |
| 3 | 2.282106009725259487906310 5e-3 | 5.016048172706945037201567 1 |
| 4 | 7.365636140307030561624912 6e-4 | 5.324138550499584358205353 1 |
| 5 | 2.689570600851835099617876 0e-4 | 5.549065009201360996133333 8 |
| 6 | 1.074711622945128494860823 5e-4 | 5.723086062370144614959248 6 |
| 7 | 4.603659266263495957129270 8e-5 | 5.863163905452748120342280 7 |
| 8 | 2.085158640633032717111035 9e-5 | 5.979219782997610915413769 9 |
| 9 | 9.889334645281424388440432 0e-6 | 6.077510314570501701553929 4 |
| 10 | 4.875957512631913243588303 5e-6 | 6.162209523600211835045601 7 |
| 11 | 2.485590268478211116920625 8e-6 | 6.236226670947615951718643 9 |
| 12 | 1.304377591343073652668770 4e-6 | 6.301661882478634867122171 3 |
| 13 | 7.022319978739775695199800 2e-7 | 6.360075435431155685533647 5 |
| 14 | 3.867557714725902029101081 6e-7 | 6.412654729314846164447794 0 |
| 15 | 2.173987820169794320532049 6e-7 | 6.460322013632057127471231 1 |
| 16 | 1.244770882089507192821459 6e-7 | 6.503806261476199867664813 5 |
| 17 | 7.247863376755536969855738 9e-8 | 6.543692516484556952735286 8 |
| 18 | 4.285464558273508215697787 0e-8 | 6.580456624560485107588549 1 |
| 19 | 2.569896763218081614904967 4e-8 | 6.614490215091157332388163 3 |
| 20 | 1.561328856994866816394441 4e-8 | 6.646119016127510214104368 8 |
| 21 | 9.601122612842236480898718 4e-9 | 6.675616512649122885656417 9 |
| 22 | 5.970823398705558055298613 7e-9 | 6.703214288224997725642425 7 |
| 23 | 3.752381381641316369086450 2e-9 | 6.729109963476020911052099 8 |
| 24 | 2.381499690721783089227969 4e-9 | 6.753473365851186986196498 3 |
| 25 | 1.525473289510979374814720 7e-9 | 6.776451379185256903334534 8 |
| 26 | 9.856763349496352995813741 3e-10 | 6.798171795031113669577074 1 |
| 27 | 6.421358050726624692365324 8e-10 | 6.818746400291279675079678 8 |
| 28 | 4.215884842992714575828506 1e-10 | 6.838273474222969818037143 6 |
| 29 | 2.788324165133927541106021 4e-10 | 6.856839824093862326770263 4 |
| 30 | 1.857072001162821795312570 7e-10 | 6.874522457133671117247554 0 |
| 31 | 1.245078325074423591090236 0e-10 | 6.891389963299101763905461 5 |
| 32 | 8.400599755776278634321604 9e-11 | 6.907503666267325308041961 3 |
| 33 | 5.702211575728862026377444 7e-11 | 6.922918587292003040007665 6 |
| 34 | 3.892950581599345944390982 3e-11 | 6.937684256909916668184585 7 |
| 35 | 2.672443556645653736397589 4e-11 | 6.951845402139240175290985 3 |
| 36 | 1.844299509252544160250377 7e-11 | 6.965442531166209461463720 4 |
| 37 | 1.279244840924708988199301 0e-11 | 6.978512433145669705344080 0 |
| 38 | 8.916358294918686087120193 9e-12 | 6.991088607329832331986247 5 |
| 39 | 6.243828154996281262473042 4e-12 | 7.003201633058588770167246 1 |
| 40 | 4.392048409181786189839103 7e-12 | 7.014879490023366905666533 7 |

## §3. Richardson extrapolation

The products $\{e^{\pi\sqrt{2n}}E_{2n,2n}(|x|;[-1,+1])\}_{n=1}^{40}$ themselves do not give precise estimates of $\underline{M}$ and $\overline{M}$ of (1.11), but with the *Richardson extrapolation method* (cf. Brezinski [6], p. 7), numerical (but not exact) information about $\underline{M}$ and $\overline{M}$ can be determined. To describe the Richardson extrapolation method, let $\{S_n\}_{n=1}^{N}$, with $N>2$, be a given sequence of real numbers. On setting $T_0^{(n)} := S_n$ $(n=1,2,\ldots,N)$, regard $\{T_0^{(n)}\}_{n=1}^{N}$ as the zeroth column of the Richardson extrapolation table for $\{S_n\}_{n=1}^{N}$. The first column of the Richardson extrapolation table, consisting of $N-1$ numbers, is defined by

$$(3.1) \qquad T_1^{(n)} := \frac{x_n T_0^{(n+1)} - x_{n+1}T_0^{(n)}}{x_n - x_{n+1}} \qquad (n=1,2,\ldots,N-1),$$

and inductively, the $(k+1)$ st column of the Richardson extrapolation table, consisting of $N-k-1$ numbers, is defined by

$$(3.2) \qquad T_{k+1}^{(n)} := \frac{x_n T_k^{(n+1)} - x_{n+k+1}T_k^{(n)}}{x_n - x_{n+k+1}} \qquad (n=1,2,\ldots,N-k-1),$$

for each $k=0,1,\ldots,N-2$, where the $\{x_n\}_{n=1}^{N}$ are given constants. In this way, a triangular table, consisting of $N(N+1)/2$ entries, is created. In our case, the products $\{e^{\pi\sqrt{2n}}E_{2n,2n}(|x|;[-1,+1])\}_{n=10}^{40}$ generate a triangular table of 496 entries. To conserve space, we give below in Table 2 only the 9th and 10th columns of the Richardson extrapolation method, applied to the numbers $\{\tau_n\}_{n=10}^{40}$, where

$$(3.3) \qquad \tau_n := e^{\pi\sqrt{2n}}E_{2n,2n}(|x|;[-1,+1]) \qquad (n=1,2,\ldots),$$

for the particular choice $x_n := 1/\sqrt{n}$ $(n=10,11,\ldots,40)$. Again, the numbers in these columns have been truncated to 25 decimal digits.

It is indeed evident from Table 2 that the number 8, to various accuracies, appears in all the entries of the 9th and 10th columns of the Richardson extrapolation of the numbers $\{\tau_n\}_{n=10}^{40}$, but this turns out to be true *throughout* the entire associated triangular Richardson extrapolation table. We have selected the 9th and 10th columns of this Richardson extrapolation because the entries in the 9th Richardson extrapolation column of Table 2 are *strictly decreasing*, while those in the 10th Richardson extrapolation column of Table 2 are *strictly increasing*. Based on these extrapolations, we make the following numerically plausible new conjecture:

$$(3.4) \qquad \text{Conjecture 1:} \quad 8 \stackrel{?}{=} \lim_{n\to\infty} e^{\pi\sqrt{2n}}E_{2n,2n}(|x|;[-1,+1]).$$

With the apparent success of the Richardson extrapolations (with $x_n := 1/\sqrt{n}$) of the numbers $\{e^{\pi\sqrt{2n}}E_{2n,2n}\ (|x|;[-1,+1])\}_{n=10}^{40}$, it is consistent with Conjecture 1 to make the following conjecture, which is in the spirit of (1.5):

**Conjecture 2.** $e^{\pi\sqrt{2n}}E_{2n,2n}(|x|;[-1,+1])$ *admits an asymptotic series expansion of the form*

$$(3.5) \qquad e^{\pi\sqrt{2n}}E_{2n,2n}(|x|;[-1,+1]) \approx 8 + \frac{K_1}{\sqrt{n}} + \frac{K_2}{n} + \frac{K_3}{n^{3/2}} + \cdots \qquad (n\to\infty).$$

TABLE 2

| 9th Richardson extrapolation of $\{\tau_n\}_{n=10}^{40}$ | 10th Richardson extrapolation of $\{\tau_n\}_{n=10}^{40}$ |
|---|---|
| 8.0000003315671961206546696 | 7.9999998669733737396601957 |
| 8.0000001954908160426397811 | 7.9999998870332782634666396 |
| 8.0000001102784540488829795 | 7.9999999204721918189077309 |
| 8.0000000606534067095188999 | 7.9999999487893701387926514 |
| 8.0000000328898056473217775 | 7.9999999686454709047300160 |
| 8.0000000177128920597313591 | 7.9999999813856490294315845 |
| 8.0000000095246104834276813 | 7.9999999891614761167611829 |
| 8.0000000051356329619218489 | 7.9999999937644296708897119 |
| 8.0000000027873890541471473 | 7.9999999964368836553304670 |
| 8.0000000015286155343083726 | 7.9999999979695900224400100 |
| 8.0000000008503621249324779 | 7.9999999988421666287371331 |
| 8.0000000004818513852150904 | 7.9999999993370575957653022 |
| 8.0000000002792857242205205 | 7.9999999996174919169009855 |
| 8.0000000001662223537658992 | 7.9999999997766671415448461 |
| 8.0000000001018861846283786 | 7.9999999998673924596859198 |
| 8.0000000000644065954058002 | 7.9999999999194597844657179 |
| 8.0000000000419621984410583 | 7.9999999999496419688750299 |
| 8.0000000000280990775511207 | 7.9999999999673808389086599 |
| 8.0000000000192489204346099 | 7.9999999999779992400786189 |
| 8.0000000000134077625530325 | 7.9999999999845068292101649 |
| 8.0000000000094285808538428 | 7.9999999999886129550248035 |
| 8.0000000000066398157884231 | |

Assuming that (3.5) is valid, it would follow that

$$(3.6)\quad \sqrt{n}\left\{e^{\pi\sqrt{2n}}E_{2n,2n}(|x|;[-1,+1])-8\right\}\approx K_1+\frac{K_2}{\sqrt{n}}+\frac{K_3}{n}+\cdots\qquad(n\to\infty).$$

With the known high-precision approximations of the numbers $\tau_n$ (cf. (3.3)) of the second column of Table 1, we can similarly perform Richardson extrapolation (with $x_n := 1/\sqrt{n}$) on the numbers $\sqrt{n}(\tau_n - 8)$, to estimate the constant $K_1$ of (3.6). In Table 3 (see next page), we similarly give the 8th and 9th columns of the Richardson extrapolation method, applied to the numbers (cf. (3.3)) of $\{\sqrt{n}(\tau_n-8)\}_{n=10}^{40}$, for the particular choice $x_n := 1/\sqrt{n}$ $(n = 10, 11, \ldots, 40)$, these numbers again having been truncated to 25 decimal digits. Here, we similarly see strict monotonicity of the numbers in each of these two columns, and it appears that

$$(3.7)\qquad -6.66432\ 44072\ 27\ldots \le K_1 \le -6.66432\ 44071\ 90\ldots.$$

This bootstrapping procedure can be continued to produce, via Richardson extrapolation, estimates for the successive constants $K_j$ in (3.5). As might be expected, there is a progressive loss of accuracy in the successive determination of the constants $K_j$. In Table 4 (see next page), we tabulate estimates of $\{K_j\}_{j=1}^{5}$, where each number is truncated to 10 decimal digits.

Note that as $K_1$ is negative in Table 4, it would follow from (3.5) of Conjecture 2 that the product $\tau_n := e^{\pi\sqrt{2n}}E_{2n,2n}(|x|;[-1,+1])$ would be eventually *increasing* to the value 8, as $n\to\infty$, which turns out to be consistent with the behavior of the

TABLE 3

| 8th Richardson extrapolation of $\{\sqrt{n}(\tau_n - 8)\}_{n=10}^{40}$ | 9th Richardson extrapolation of $\{\sqrt{n}(\tau_n - 8)\}_{n=10}^{40}$ |
|---|---|
| −6.6643252923192288899581422 | −6.6643238470513280068931885 |
| −6.6643248955588651439071779 | −6.6643240212973578474092243 |
| −6.6643246696670446212354260 | −6.6643241643076814195198379 |
| −6.6643245463234522051500192 | −6.6643242618337574878424280 |
| −6.6643244805227254105843835 | −6.6643243227887586915298972 |
| −6.6643244458511429413907191 | −6.6643243590760481107155901 |
| −6.6643244276777840712247340 | −6.6643243800547316540863275 |
| −6.6643244181531735877970527 | −6.6643243919664809003072336 |
| −6.6643244131412380567146977 | −6.6643243986575396699820511 |
| −6.6643244104834298819376785 | −6.6643244023947567739175251 |
| −6.6643244090578374114941530 | −6.6643244044784972229876582 |
| −6.6643244082814322566680373 | −6.6643244056422235261938739 |
| −6.6643244078503439726613513 | −6.6643244062953468702991632 |
| −6.6643244076053130326767931 | −6.6643244066650174044155402 |
| −6.6643244074621918084852786 | −6.6643244068769002380708173 |
| −6.6643244073760385664278707 | −6.6643244070004868069058853 |
| −6.6643244073225197839348864 | −6.6643244070742680700833267 |
| −6.6643244072882341387675361 | −6.6643244071196396734608119 |
| −6.6643244072656467633538892 | −6.6643244071485601513855522 |
| −6.6643244072504158643384467 | −6.6643244071677648651080498 |
| −6.6643244072399678761510565 | −6.6643244071810864075910701 |
| −6.6643244072327288717801809 | −6.6643244071907349895094533 |
| −6.6643244072277039192319918 | |

TABLE 4

| $j$ | $K_j$ |
|---|---|
| 1 | −6.6643244072 |
| 2 | +2.7758262379 |
| 3 | −0.1460115270 |
| 4 | −0.3599422092 |
| 5 | +0.0728948673 |

numerical values in the second column of Table 2. Then, one might ask how large $n_0$ would have to be so that the inequality

$$\tau_n \geq 8 - 0.1 = 7.9 \quad (\text{all } n \geq n_0) \tag{3.8}$$

is valid. Surprisingly, using the constants of Table 4 in the series of (3.5), the answer to (3.8) appears to be

$$n_0 \doteq 4,386. \tag{3.9}$$

This would indicate that to *numerically* extend the second column of Table 2 to values of $\tau_n$ which satisfy (3.8) would be computationally nearly impossible!

## §4. The zeros, poles, and extreme points of $r^*_{n,n}(t)$ and $s^*_{2n,2n}(x)$

With $r^*_{n,n}(t) := p^*_n(t)/q^*_n(t)$ denoting the unique best uniform approximation from $\pi_{n,n}$ to $\sqrt{t}$ on $[0, 1]$, our numerical results have determined the rational functions $\{r^*_{n,n}(t)\}_{n=1}^{40}$ to very high precision, and we now comment on the *location* of the zeros and the poles of $r^*_{n,n}(t)$. It turns out that the zeros and poles of each $r^*_{n,n}(t)$ are *negative real numbers* which *interlace* on the negative real axis, and this has been rigorously established, for all $n$, in Blatt, Iserles, and Saff [2], Lemma 3.2. In Table 5 (next page), we list specifically the zeros and poles of $r^*_{20,20}(t)$, according to increasing absolute values. These numbers are truncated to 25 significant digits. The second column of Table 5 is now shifted downward to indicate the interlacing of these numbers.

It is also the case that the smallest (in modulus) zero and pole of $r^*_{n,n}(t)$ are rapidly *decreasing* with increasing $n$, and, more curiously, that the largest (in modulus) zero and pole of $r^*_{n,n}(t)$ are *increasing* with increasing $n$. This is indicated in Tables 6 and 7 (see pp. 285 and 286 ).

With $s^*_{2n,2n}(x) := r^*_{n,n}(x^2)$ denoting (cf. Proposition 3) the unique best uniform approximation from $\pi_{2n,2n}$ to $|x|$ on $[-1, +1]$, it is evident, from the fact that the zeros and poles of $r^*_{n,n}(t)$ are negative real numbers which interlace, that all the zeros and poles of $s^*_{2n,2n}(x)$ are then *purely imaginary*, and that they also *interlace* on the upper imaginary axis, as well as on the lower imaginary axis, as shown in Blatt, Iserles, and Saff [2], Proposition 1.6.

Next, we consider the *extreme points* $\{t_k^{(n)}\}_{k=0}^{2n+1}$, as defined in (2.25), of the best uniform approximation $r^*_{n,n}(t)$ from $\pi_{n,n}$ to $\sqrt{t}$ on $[0, 1]$. In Figure 1, we graph $-\sqrt{t} + r^*_{32,32}(t)$ on the interval $[0, 1]$, which has 66 extreme points. (We remark that the graphs of $-\sqrt{t} + r^*_{n,n}(t)$ for other values of $n$ between 1 and 40 are all similar.) In Figure 2, we graph $-|x| + s^*_{64,64}(x)$ on $[-1, +1]$, which has, from (2.27), $L = 131$ extreme points. What is interesting is that the *distribution* of the extreme points in Figure 1 is *far* from an arcsine distribution of points on $[0, 1]$, which arises naturally in best uniform *polynomial* approximation to a continuous function on $[0, 1]$ (cf. Kadec [12] and Blatt, Saff, and Totik [3]). In this regard, for a recent contribution to the distribution of extreme points in *rational* approximations to continuous functions, see Borwein, Kroó, Grothmann, and Saff [4].

Next, to come full circle to connect with Newman's method of proof in [15], again let $r^*_{n,n}(t) := p^*_n(t)/q^*_n(t)$ denote the unique best uniform approximation from $\pi_{n,n}$ to $\sqrt{t}$ on $[0, 1]$, so that (cf. Proposition 3) $s^*_{2n,2n}(x) := p^*_n(x^2)/q^*_n(x^2)$ is the best uniform approximation from $\pi_{2n,2n}$ to $|x|$ on $[-1, +1]$.

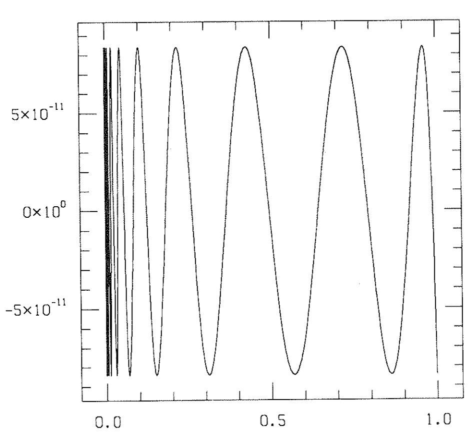


FIGURE 1. $-\sqrt{t} + r^*_{32,32}(t)$

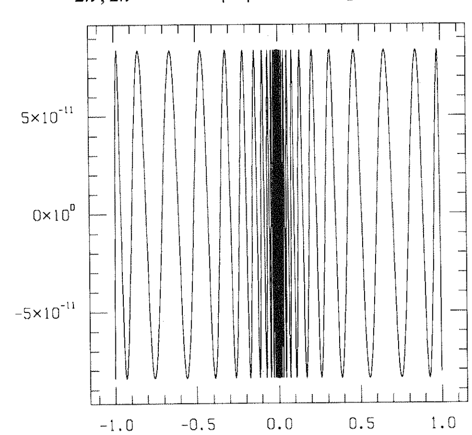


FIGURE 2. $-|x| + s^*_{64,64}(x)$

TABLE 5

| zeros of $r^*_{20,20}(t)$ | poles of $r^*_{20,20}(t)$ |
|---|---|
| $-1.71713250430383703346842 8e-15$ | $-4.12203707192468595203864 2e-14$ |
| $-3.70318290924199429120811 6e-13$ | $-2.22992144210023570224564 4e-12$ |
| $-1.05870297788772654466272 2e-11$ | $-4.27226912701816294554983 3e-11$ |
| $-1.52863604102721876835910 3e-10$ | $-4.98043989366491787700613 2e-10$ |
| $-1.50447378695597780869730 3e-9$ | $-4.26832469843709414270474 3e-9$ |
| $-1.14830375401563005328465 4e-8$ | $-2.95111853706933923521694 3e-8$ |
| $-7.28738605191084842825455 6e-8$ | $-1.73717506110572614477095 5e-7$ |
| $-4.01296590077362980565333 7e-7$ | $-9.01197304501395594246745 5e-7$ |
| $-1.97274359955448122809480 9e-6$ | $-4.21899035297235206468282 8e-6$ |
| $-8.83253642829358250044443 2e-6$ | $-1.81317776424389222962186 8e-5$ |
| $-3.65527301105415810514356 3e-5$ | $-7.24592395230814576389970 5e-5$ |
| $-1.41406427773210453334180 3e-4$ | $-2.71957624489844398094324 1e-4$ |
| $-5.15946155666325695168118 2e-4$ | $-9.66413679659364074590578 7e-4$ |
| $-1.78873079098920905930879 1e-3$ | $-3.27429887607124565912660 2e-3$ |
| $-5.93303674336788108010152 6e-3$ | $-1.06531959392965437925904 8e-2$ |
| $-1.89807395671092934694800 6e-2$ | $-3.36196364651970488269812 8e-2$ |
| $-5.93660819156761513285006 0e-2$ | $-1.04970620029201261050394 2e-1$ |
| $-1.87211850959645773925916 2e-1$ | $-3.40991176863735999395486 0e-1$ |
| $-6.48806165290118757953793 4e-1$ | $-1.34873629428694959858566 6e0$ |
| $-3.29964780109543613444286 9e0$ | $-1.45688330922195741117357 9e1$ |

TABLE 6

| $n$ | Smallest (in modulus) zero of $r^*_{n,n}(t)$ | Smallest (in modulus) pole of $r^*_{n,n}(t)$ |
|---|---|---|
| 1 | −1.368923031534783976006006e-2 | −4.196433776070805662759262e-1 |
| 2 | −5.095440950046164913548179e-4 | −1.236229093048010694616062e-2 |
| 3 | −3.668754854764301601156389e-5 | −8.814871037462490620077316e-4 |
| 4 | −3.821548098532945054547309e-6 | −9.174728550114113098656249e-5 |
| 5 | −5.095437613649789174645734e-7 | −1.223196722384140909651538e-5 |
| 6 | −8.135776767581113201857043e-8 | −1.953027123111307771363052e-6 |
| 7 | −1.492867968836581654265778e-8 | −3.583684048208389856216454e-7 |
| 8 | −3.062620917232778789030957e-9 | −7.351930256952474132939666e-8 |
| 9 | −6.888888944579748020677368e-10 | −1.653702106290587264702107e-8 |
| 10 | −1.674691676157399999015334e-10 | −4.020156396851685957714361e-9 |
| 11 | −4.351851987754064935433282e-11 | −1.044677403412738226113334e-9 |
| 12 | −1.198454897639809752244714e-11 | −2.876933205451490043826655e-10 |
| 13 | −3.473571680400674142505709e-12 | −8.338431195646377264044048e-11 |
| 14 | −1.053631249266870628966651e-12 | −2.529278933893578905213889e-11 |
| 15 | −3.329118463153412218161126e-13 | −7.991666157260343615638806e-12 |
| 16 | −1.091424949783213873155457e-13 | −2.620004043356584270891723e-12 |
| 17 | −3.700283784992045452648459e-14 | −8.882661588536826998558918e-13 |
| 18 | −1.293632302714508152801726e-14 | −3.105409917910581318633591e-13 |
| 19 | −4.652071606136310354212030e-15 | −1.116746178508780598705148e-13 |
| 20 | −1.717132504303837033468428e-15 | −4.122037071924685952038642e-14 |
| 21 | −6.493204306908037948411602e-16 | −1.558716569721306645149564e-14 |
| 22 | −2.511212637658449447811252e-16 | −6.028254407838885874745387e-15 |
| 23 | −9.918111377797253165882600e-17 | −2.380877578188369021273296e-15 |
| 24 | −3.994993665370975117059856e-17 | −9.590123039129199271092594e-16 |
| 25 | −1.639170960560711118083876e-17 | −3.934887639548997194771480e-16 |
| 26 | −6.843585470243554598866683e-18 | −1.642826802388382091074808e-16 |
| 27 | −2.904482801853040010638739e-18 | −6.972313292070952075957630e-17 |
| 28 | −1.251965943925848345316009e-18 | −3.005388355711813077403102e-17 |
| 29 | −5.476480697718971255690096e-19 | −1.314648485372852496645883e-17 |
| 30 | −2.429251729561977623797927e-19 | −5.831504360433917003348557e-18 |
| 31 | −1.091964153242014292546902e-19 | −2.621298420240920688320193e-18 |
| 32 | −4.970906825854820752088878e-20 | −1.193283696272594942092688e-18 |
| 33 | −2.290349139166647354491561e-20 | −5.498063798589801565372890e-19 |
| 34 | −1.067512127334877980146886e-20 | −2.562600470595076629004990e-19 |
| 35 | −5.030742854842496517977711e-21 | −1.207647545836081648394104e-19 |
| 36 | −2.295951030517160327206388e-21 | −5.751564859178373936699723e-20 |
| 37 | −1.152716167627621967794938e-21 | −2.767135771094096855140326e-20 |
| 38 | −5.600026212069963803150742e-22 | −1.344306021349157823041807e-20 |
| 39 | −2.746103613369048578300742e-22 | −6.592118470345778964292469e-21 |
| 40 | −1.358780396791876849841798e-22 | −3.261800212937462485588787e-21 |

## TABLE 7

| $n$ | Largest (in modulus) zero of $r^*_{n,n}(t)$ | Largest (in modulus) pole of $r^*_{n,n}(t)$ |
|---|---|---|
| 1 | −1.368923031534783976006006e-2 | −4.196433776070805662759262e-1 |
| 2 | −1.192906926906936332406069e-1 | −1.076378523043827619476925e0 |
| 3 | −2.640577400238288442074837e-1 | −1.772091188030805623721620e0 |
| 4 | −4.257119677091359195052433e-1 | −2.486693073254502304149400e0 |
| 5 | −5.962567801550604082353637e-1 | −3.212922186202416817620196e0 |
| 6 | −7.721764150282669289322387e-1 | −3.947259084964369332782131e0 |
| 7 | −9.516676149597912387743964e-1 | −4.687683028201020515194932e0 |
| 8 | −1.133698646456227804901685e0 | −5.432903907034385828756759e0 |
| 9 | −1.317629592104832120632036e0 | −6.182036649546584856400985e0 |
| 10 | −1.503038235574910374167784e0 | −6.934441759088715604439639e0 |
| 11 | −1.689632068429636444622313e0 | −7.689638751833178079740678e0 |
| 12 | −1.877200296530244741596106e0 | −8.447255416074735265532792e0 |
| 13 | −2.065586004852788950567396e0 | −9.206996247803105556130201e0 |
| 14 | −2.254669176675043063501016e0 | −9.968621863026240552203745e0 |
| 15 | −2.444355886460392526433957e0 | −1.073193504075731167876604e1 |
| 16 | −2.634571169916340058138727e0 | −1.149677095232836424715549e1 |
| 17 | −2.825254172201641785281781e0 | −1.226299013325732616994746e1 |
| 18 | −3.016354756365245681484291e0 | −1.303047330905229121333566e1 |
| 19 | −3.207831075953284449719518e0 | −1.379911750849602584068894e1 |
| 20 | −3.299647801095436134442869e0 | −1.456883309221957411173579e1 |
| 21 | −3.591774797882112527126323e0 | −1.533954144547711386170889e1 |
| 22 | −3.784186128745445910483593e0 | −3.784186128745445910483593e0 |
| 23 | −3.976859284428676641888241e0 | −1.688366659501876719663205e1 |
| 24 | −4.169774585863404828801473e0 | −4.169774585863404828801473e0 |
| 25 | −4.362914712614906328311850e0 | −1.843102404666098337231393e1 |
| 26 | −4.556264326928174860566678e0 | −1.920579411089108860745803e1 |
| 27 | −4.749809770906511170871789e0 | −1.998123655668571281061711e1 |
| 28 | −4.943538820290648575993641e0 | −4.943538820290648575993641e0 |
| 29 | −5.137440482516175886695985e0 | −2.153399520695165797435302e1 |
| 30 | −5.331504829754723302969704e0 | −2.231124739110224220585381e1 |
| 31 | −5.525722859850400261795768e0 | −2.308904320486567760595054e1 |
| 32 | −5.720086379689786072413115e0 | −2.386735680013799059570384e1 |
| 33 | −5.914587906756959226112125e0 | −2.464616431904602214388958e1 |
| 34 | −6.109220585539266195280818e0 | −2.542544368539493259229790e1 |
| 35 | −6.303978116145199158049530e0 | −2.620517442344431956795667e1 |
| 36 | −6.498854693029970540636320e0 | −2.698533749975341498805667e1 |
| 37 | −6.693844952138135601265645e0 | −2.776591518459983420305286e1 |
| 38 | −6.888943925095670306938255e0 | −2.854689093008630573631162e1 |
| 39 | −7.084146999338062764189254e0 | −2.932824926254003169258184e1 |
| 40 | −7.279449883262351921945916e0 | −3.010997568720580080176381e1 |

As in (2.3), we write $p_n^*(t) := \sum_{k=0}^{n} a_k^*(n)t^k$ and $q_n^*(t) := \sum_{k=0}^{n} b_k^*(n)t^k$ (where $b_0^*(n) = 1$ ), so that

$$
(4.1)\qquad
\begin{aligned}
-|x| + s_{2n,2n}^*(x) &= -|x| + \frac{\sum_{k=0}^{n} a_k^*(n)x^{2k}}{1 + \sum_{k=1}^{n} b_k^*(n)x^{2k}} \\
&= -|x| + \frac{\sum_{k=0}^{n} a_k^*(n)x^{2k}/b_n^*(n)}{\left[x^{2n} + \sum_{k=0}^{n-1} b_k^*(n)x^{2k}/b_n^*(n)\right]},
\end{aligned}
$$

where we have used the fact (cf. Proposition 1) that $b_n^*(n) \neq 0$. Now, because $|x|$ and $s_{2n,2n}^*(x)$ are both even functions on $[-1, +1]$, (4.1) can be written as

$$
(4.2)\qquad -|x| + s_{2n,2n}^*(x) = -x + \frac{x\left(\sum_{k=1}^{n} a_k^*(n)x^{2k-1}/b_n^*(n)\right) + a_0^*(n)/b_n^*(n)}{\left[x^{2n} + \sum_{k=0}^{n-1} b_k^*(n)x^{2k}/b_n^*(n)\right]},
$$

for $x$ in $[0, 1]$. Next, set

$$
(4.3)\qquad P_{2n}(x) := \left(x^{2n} + \sum_{k=0}^{n-1} b_k^*(n)x^{2k}/b_n^*(n)\right) + \sum_{k=1}^{n} a_k^*(n)x^{2k-1}/b_n^*(n),
$$

which is a monic polynomial in $\pi_{2n}$ with all its Taylor coefficients positive (cf. Proposition 1). Clearly,

$$
(4.4)\qquad
\begin{aligned}
P_{2n}(x) + P_{2n}(-x) &= 2\left[x^{2n} + \sum_{k=0}^{n-1} b_k^*(n)x^{2k}/b_n^*(n)\right], \\
P_{2n}(x) - P_{2n}(-x) &= 2\left[\sum_{k=1}^{n} a_k^*(n)x^{2k-1}/b_n^*(n)\right],
\end{aligned}
$$

so that on $[0, 1]$, (4.2) becomes

$$
(4.5)\qquad -|x| + s_{2n,2n}^*(x) = \frac{-2xP_{2n}(-x) + 2a_0^*(n)/b_n^*(n)}{P_{2n}(x) + P_{2n}(-x)}.
$$

Next, if $\{t_l^{(n)}\}_{l=0}^{2n+1}$ again denotes the alternation set in $[0, 1]$ of length $2n+2$ for the extreme points of $-\sqrt{t} + r_{n,n}^*(t)$, where $r_{n,n}^*(t) = p_n^*(t)/q_n^*(t)$, then (2.26) holds with

$$
(4.6)\qquad 0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{2n+1}^{(n)} = 1.
$$

Hence, because of the transformation $t = x^2$ in Proposition 3, then $\{x_l(n) := \sqrt{t_l^{(n)}}\}_{l=0}^{2n+1}$ is necessarily the alternation set for the extreme points of $-|x|+s_{2n,2n}^*(x)$ on $[0, 1]$, which gives that

$$-|x_l(n)| + s^*_{2n,2n}(x_l(n)) = (-1)^l E_{2n,2n}(|x|;[-1,+1]) \qquad (l = 0, 1, \ldots, 2n+1). \tag{4.7}$$

Since $x_0(n) = 0$, then evaluating the right side of (4.5) at the point $x_0(n) = 0$ can be seen to yield

$$a^*_0(n) = E_{2n,2n}(|x|;[-1,+1]). \tag{4.8}$$

With this, we now prove that all the zeros of $P_{2n}(x)$ of (4.3) are real and distinct in the interval $(-1, 0)$, for any $n \geq 1$. From (4.2), (4.7), and (4.8), we have, with the notation of (2.3), that

$$-x_l(n)P_{2n}(-x_l(n)) = (-1)^l[q^*_n(x^2_l(n)) + (-1)^{l+1}]\cdot E_{2n,2n}(|x|;[-1,+1])/b^*_n(n), \tag{4.9}$$

for all $l = 0, 1, \ldots, 2n+1$. But since the coefficients $b^*_k(n)$ of $q^*_n(x)$ are all positive with $b^*_0(n) = 1$ from Proposition 1, it is evident that the quantity in brackets in (4.9) is positive for all $l = 1, 2, \ldots, 2n+1$, and this shows that $P_{2n}(-x_l(n))$ *oscillates* in sign at the consecutive $2n+1$ points $\{x_l(n)\}_{l=1}^{2n+1}$ of $(0, 1]$. Thus, $P_{2n}(x)$ has $2n$ distinct real zeros in $(-1, 0)$, and we can write

$$P_{2n}(x) = \prod_{j=1}^{2n}(x + \xi_j(2n)) \qquad (n = 1, 2, \ldots), \tag{4.10}$$

where $0 < \xi_1(2n) < \xi_2(2n) < \cdots < \xi_n(2n) < 1$. Then, (4.5) can be written as

$$-|x| + s^*_{2n,2n}(x) = \frac{-2x\prod_{j=1}^{2n}\left(\frac{\xi_j(2n)-x}{\xi_j(2n)+x}\right)}{1+\prod_{j=1}^{2n}\left(\frac{\xi_j(2n)-x}{\xi_j(2n)+x}\right)} + \frac{2a^*_0(n)/b^*_n(n)}{P_{2n}(x)+P_{2n}(-x)}, \tag{4.11}$$

for $x$ in $[0, 1]$. Now, the ratios $a^*_0(n)/b^*_n(n)$ were exceedingly small in our numerical results (for example, $a^*_0(32)/b^*_{32}(32) = 7.305 \cdot 10^{-221}$) and this suggests dropping the last term in (4.11), which thereby defines a modified rational approximation, $\tilde{s}_{2n,2n}(x)$, to $|x|$ on $[-1, +1]$:

$$-|x| + \tilde{s}_{2n,2n}(x) = \frac{-2x\prod_{j=1}^{2n}\left(\frac{\xi_j(2n)-x}{\xi_j(2n)+x}\right)}{1+\prod_{j=1}^{2n}\left(\frac{\xi_j(2n)-x}{\xi_j(2n)+x}\right)}. \tag{4.12}$$

The above form, of course, reminds one of Newman's original proof in [15], and this form is also explicitly used in Vyacheslavov's proof [23]. We remark, however, that there is *one small difference* between the error curves of $-|x| + s^*_{2n,2n}(x)$ and $-|x| + \tilde{s}_{2n,2n}(x)$, namely, on $[0, 1]$, $-|x| + s^*_{2n,2n}(x)$ has $2n+2$ equioscillations with one extreme point at $x = 0$, while from (4.12) it is evident that $-|x| + \tilde{s}_{2n,2n}(x)$ vanishes at $x = 0$. In essence, $-|x| + \tilde{s}_{2n,2n}(x)$ loses the extreme point at $x = 0$ because of having dropped the last term in (4.11).

We mention that the high-precision coefficients (cf. (2.3)) of

$$\{r^*_{n,n}(t) := p^*_n(t)/q^*_n(t)\}_{n=1}^{40},$$

along with their zeros, poles, and extreme points, as well as the polynomials (cf. (4.3)) $\{P_{2n}(x)\}_{n=1}^{40}$, along with their zeros, are much too lengthy to reproduce here. These are however available upon request from the Institute for Computational Mathematics, Kent State University, Kent, OH 44242.

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