A NOTE ON A PERTURBATION ANALYSIS OF ITERATIVE METHODS, WITH AN APPLICATION TO THE SSOR ITERATIVE METHOD

RICHARD S. VARGA
Institute for Computational Mathematics,
Kent Sate University,
Kent, Ohio 44242,
U. S. A.

WILHELM NIETHAMMER
Institut für Praktische Mathematik,
Universität Karlsruhe,
D-7500 Karlsruhe,
Germany.

Dedicated to Professor Ivo Marek on the occasion of his 60th birthday

ABSTRACT

We investigate here iteration matrices $T(\omega)$, for solving systems of linear equations, which, like the SOR iterative method, are dependent on a single relaxation parameter, ω. We use a local perturbation analysis, for ω small, of the extrapolated Jacobi iteration matrix $J(\omega)$, which results in geometrical necessary and sufficient conditions for the convergence or divergence of $T(\omega)$, for ω small. Then, an application of this analysis is given both for the SSOR (symmetric successive overrelation) iterative method and the ADI (alternating directions implicit) method for solving non-Hermitian systems of linear equations.

Keywords: Extrapolated Jacobi iteration, SOR, SSOR, perturbation analysis, optimal paths of relaxation

0. Introduction

During the last few years, there has been renewed interest in the SOR (successive overrelaxation) and the SSOR (symmetric successive overrelaxation) iterative methods, because of their applications to parallel computation and to the preconditioning of matrices in conjunction with the conjugate gradient method (see, for example, Hanke, Neumann, and Niethammer [5]). In addition, there has also been a similar renewed recent interest in the use of iterative methods for real non-

Hermitian, as well as complex non–Hermitian, systems of linear equations (see, for example, Freund [4]).

For an Hermitian matrix A whose diagonal matrix D is positive definite, a well known result of Ostrowski (cf. Ostrowski [8] or Varga [9, p. 77]) gives that each of the SOR and the SSOR iterative methods converges if and only if A is positive definite and the real relaxation factor ω satisfies $0 < \omega < 2$. For related results when A is non–Hermitian with unit diagonal entries, Broyden [1] and Niethammer [6] showed that if the Hermitian part of A, namely $(A + A^*)/2$, is positive definite, then there exists an ω_g in $(0, 2]$ such that the SOR iterative method converges for all ω with $0 < \omega < \omega_g$; in [6], bounds for ω_g were also given. In light of Ostrowski's result which is both necessary and sufficient in the Hermitian case, it is natural to ask if the *converse* of the above result is valid, i.e., if A has unit diagonal entries and if the SOR iterative method converges for all ω in $(0, \omega_g)$ for some $\omega_g > 0$, then is the Hermitian part of A positive definite? Buoni and Varga [2, Corollary 3.5] showed that this converse is not valid in general, and moreover, they derived necessary and sufficient conditions for the simultaneous convergence and divergence of the Jacobi and SOR iterative methods for complex relaxation factors ω. The results of [2] were obtained from a perturbation analysis coupling the spectra of the Jacobi and SOR iteration matrices for complex relaxation factors ω near zero; the convex hull of the eigenvalues of the matrix $D^{-1}A$ played a central role in this analysis.

The motivation for this paper came from the question, if such SOR extension also give rise to SSOR extensions, in particular. Our extensions here, via matrices ω^2*–compatible* with the extrapolated Jacobi matrix (to be defined below), turns out to have applications not only to such SSOR extensions, but also to ADI extensions. For example, it is shown in Corollary 7 that there exists a real interval $(0, \omega_g)$ with $\omega_g > 0$, such that the SSOR iterative method converges for all $\omega \in (0, \omega_g)$, if the eigenvalues of $D^{-1}A$ lie in the open right–half plane. A partial converse to this is also given in Corollary 7. Further, optimal paths of relaxation in the complex plane, in a more general setting than that of [2], will also be discussed in §2.

1. Relaxation methods for non-Hermitian linear systems

Given a matrix $A \in \mathbb{C}^{n \times n}$, we consider a general splitting of A of the form

$$A = D - L - U \qquad (D, L, U \in \mathbb{C}^{n \times n} \text{ with } D \text{ nonsingular}). \tag{1.1}$$

Usually, D is diagonal or block-diagonal matrix with $D^{-1}L$ and $D^{-1}U$ strictly lower and strictly upper triangular matrices, respectively, but his is not needed here. With this splitting of A, the *extrapolated Jacob* iteration matrix $J(\omega) \in \mathbb{C}^{n \times n}$, associated with the splitting of (1.1), is defined for each complex number ω by

$$J(\omega) := I - \omega D^{-1}A, \tag{1.2}$$

so that $J(\omega)$ is an entire function of $\omega \in \mathbb{C}$.

Next, consider any iteration matrix $T(\omega) \in \mathbb{C}^{n\times n}$ which is dependent on the single complex parameter ω, and we assume that there is a largest number $r(T) > 0$ such that $T(\omega)$ is analytic, as a function of ω, in $0 \le |\omega| < r(T)$. (We remark that $r(T)$ can be $+\infty$, as in the above case of the Jacobi matrix $J(\omega)$). Then, the matrix $T(\omega)$ is said to be ω^2*-compatible* with the Jacobi matrix $J(\omega)$ of (1.2) if $T(\omega)$ can be expressed as

$$(1.3) \qquad T(\omega) = J(\omega) - \omega^2\, R_T(\omega),$$

where the matrix $R_T(\omega)$ in $\mathbb{C}^{n\times n}$ is further assumed to be analytic at $\omega = 0$. (Consequently, $R_T(\omega)$ is then also analytic in $0 \le |\omega| < r(T)$.)

As a concrete example of the above definition, consider the successive overrelaxation matrix $\mathcal{L}(\omega)$, defined as usual from (1.1) by

$$(1.4) \quad \mathcal{L}(\omega) := (D - \omega L)^{-1}[(1-\omega)D + \omega U] = (I - \omega D^{-1}L)^{-1}[(1-\omega)I + \omega D^{-1}U].$$

It is clear that if

$$(1.5) \qquad \mu(D^{-1}L) := \min\{|\omega| : \omega \in \mathbb{C} \text{ and } I - \omega D^{-1}L \text{ is singular}\},$$

then $0 < \mu(D^{-1}L) \le \infty$ (with $\mu(D^{-1}L) = \infty$ holding, for example, if $D^{-1}L$ is strictly lower triangular), and $\mathcal{L}(\omega)$ is then analytic in $0 \le |\omega| < \mu(D^{-1}L)$.

Now, it is readily follows from (1.4) and (1.2) that (cf. Buoni and Varga [2])

$$(1.6) \quad \mathcal{L}(\omega) = J(\omega) - \omega^2 D^{-1}L(I - \omega D^{-1}L)^{-1}D^{-1}A, \quad 0 \le |\omega| < \mu(D^{-1}L).$$

In this case (cf. (1.3)), $\mathcal{L}(\omega)$ satisfies (1.3) with

$$R_{\mathcal{L}}(\omega) := D^{-1}L(I - \omega D^{-1}L)^{-1}D^{-1}A,$$

and $R_{\mathcal{L}}(\omega)$ is analytic at $\omega = 0$. Consequently, $\mathcal{L}(\omega)$ is ω^2*-compatible* with $J(\omega)$.

With the above notations and with the usual notations

$$\sigma(F) := \{\lambda : \det(F - \lambda I) = 0\} \quad \text{and} \quad \rho(F) := \max\{|\lambda| : \lambda \in \sigma(F)\}$$

for the *spectrum* $\sigma(F)$ and the *spectral radius* $\rho(F)$, respectively, of a matrix $F \in \mathbb{C}^{n\times n}$, we establish the following result, Proposition 1, which is a slight generalization to ω^2*-compatible* matrices of the result of Buoni and Varga [2, Theorem 2.2]. (The proof of Proposition 1 is given only to indicate the flavor of the proofs to follow; subsequent similar proofs will be omitted.)

Proposition 1. Let the matrix $T(\omega)$ in $\mathbb{C}^{n\times n}$ be ω^2*-compatible* with the Jacobi matrix $J(\omega)$ of (1.2), associated with the splitting of (1.1). If A is nonsingular and if $\{\mu_i(\omega)\}_{j=1}^n$ denotes the eigenvalues of $J(\omega)$ (with multiplicities counted), then there is an ordering $\{\lambda_j(\omega)\}_{j=1}^n$ of the eigenvalues $T(\omega)$ such that

$$(1.7) \qquad |\lambda_j(\omega) - \mu_j(\omega)| = O(\omega^{1+1/n}) \quad (j = 1, 2, \ldots, n),$$

for all complex numbers ω sufficiently small.

Proof. From (1.3), consider the matrix $Q(\omega)$ in $\mathbb{C}^{n\times n}$ defined by

$$Q(\omega) := D^{-1}A + \omega R_T(\omega), \tag{1.8}$$

where the matrices $Q(\omega)$ and $R_T(\omega)$, from (1.3), are both analytic in $0 \leq |\omega| < r(T)$. A classical result of Ostrowski [8, p. 334] gives us that if $\{\gamma_j\}_{j=1}^n = \sigma(D^{-1}A)$ (with multiplicities counted), there is an ordering $\{\xi_j(\omega)\}_{j=1}^n$ of the eigenvalue of $Q(\omega)$ such that

$$|\xi_j(\omega) - \gamma_j| = O(\omega^{1/n}) \quad (j = 1, 2, \ldots, n), \tag{1.9}$$

for all ω sufficiently small in modulus, where n is the order of the matrices $T(\omega)$ and $J(\omega)$. From (1.2), we can write, from (1.3) and (1.8) for any ω with $0 < |\omega| < r(T)$, that

$$\begin{aligned} Q(\omega) &= \frac{1}{\omega}\{(I - J(\omega)) + \omega^2 R_T(\omega)\} \\ &= \frac{1}{\omega}\{(I - J(\omega)) + (J(\omega) - T(\omega)) = \frac{1}{\omega}(I - T(\omega)). \end{aligned} \tag{1.10}$$

For $\omega \neq 0$, $\xi_j(\omega) = \frac{1}{\omega}(1 - \lambda_j(\omega))$ gives a $1-1$ relationship between the eigenvalues $\{\xi_j(\omega)\}_{j=1}^n$ of $Q(\omega)$ and the eigenvalues $\{\lambda_j(\omega)\}_{j=1}^n$ of $T(\omega)$, as does $\gamma_j = \frac{1}{\omega}(1 - \mu_j(\omega))$ similarly relate eigenvalues $\{\gamma_j\}_{j=1}^n$ of $D^{-1}A$ with eigenvalues $\{\mu_j(\omega)\}_{j=1}^n$ of $J(\omega)$. Thus, for each pair of eigenvalues $\xi_j(\omega)$ and $\mu_j(\omega)$ in (1.9), we have

$$|\xi_j(\omega) - \gamma_j| = \frac{1}{|\omega|}|(1 - \lambda_j(\omega)) - (1 - \mu_j(\omega))| = \frac{|\lambda_j(\omega) - \mu_j(\omega)|}{|\omega|} = O(\omega^{1/n}),$$

and the final equality gives the desired result of (1.7). □

We remark that the exponent, namely $1 + (1/n)$, of ω in (1.7) is in general *best possible* as simple examples show. This exponent of ω in (1.7) can be increased to 2 under additional assumptions on the associated matrices (cf. [2, Theorem 2.3] or [5, Theorem 1]).

Consider as before any matrix $T(\omega) \in \mathbb{C}^{n\times n}$ which is analytic in $0 \leq |\omega| < r(T)$. Then, set

$$\begin{cases} \Omega_T := & \{\omega \text{ in } 0 \leq |\omega| < r(T) : \rho(T(\omega)) < 1\}, \text{ and} \\ \mathcal{D}_T := & \{\omega \text{ in } 0 \leq |\omega| < r(T) : \rho(T(\omega)) > 1\}. \end{cases} \tag{1.11}$$

We note that if $T(\omega)$ is ω^2*-compatible* with $J(\omega)$, then $T(0) = I$ from (1.3), so that $\omega = 0$ is not contained in either Ω_T or $\mathcal{D}_T$. In addition, we define

$$K(D^{-1}A) := \text{ closed convex hull of } \sigma(D^{-1}A), \tag{1.12}$$

and

$$\overset{\circ}{K}(D^{-1}A) := \text{ interior of } K(D^{-1}A) \quad \text{(which is possibly empty)}. \tag{1.13}$$

With this notation and with Proposition 1, we state next in Theorem 2 a slight generalization of Theorems 3.4 and 3.6 of [2].

Theorem 2. Let the matrix $T(\omega)$ in $\mathbb{C}^{n\times n}$ be ω^2*–compatible* with $J(\omega)$, where $J(\omega)$ is the extrapolated Jacobi iteration matrix of (1.2), associated with the splitting of (1.1). Then,

$$\Omega_j \cap \Omega_T \neq \phi \text{ if and only if } 0 \notin K(D^{-1}A). \tag{1.14}$$

If $0 \in \overset{\circ}{K}(D^{-1}A)$, then

$$\mathcal{D}_j \cap \mathcal{D}_T \subseteq \{\omega \in \mathbb{C} : 0 < |\omega| < r_0\}, \quad \text{for some } r_0 > 0. \tag{1.15}$$

We remark that (1.14) of Theorem 2 gives that if $0 \notin K(D^{-1}A)$, then there are values of ω in $0 < |\omega| < r(T)$ for which $J(\omega)$ and $T(\omega)$ are *simultaneously convergent*, which is a Stein–Rosenberg–type result. Similarly, if $0 \in \overset{\circ}{K}(D^{-1}A)$, then $J(\omega)$ and $T(\omega)$ are *simultaneously divergent* at every point of the punctured disk $\{\omega \in \mathbb{C} : 0 < |\omega| < r_0\}$, for some $r_0 > 0$, which is also a Stein–Rosenberg–type result.

2. Optimal paths of relaxation

Since we are most interested in the cases where ω is such that $J(\omega)$ and $T(\omega)$ are both convergent matrices, we examine more carefully the case when $0 \notin K(D^{-1}A)$ of (1.4).

To begin, for each fixed $r > 0$, it is evident from compactness considerations that there always exists at least one real $\tilde{\theta}(r)$ such that $\omega = re^{i\tilde{\theta}(r)}$ minimizes the spectral radius of $\rho(J(\omega))$ on the circle $|\omega| = r$. But, under the assumption that $0 \notin K(D^{-1}A)$, it was shown in Buoni and Varga [3, Theorem 2.1], for $r > 0$ sufficiently small, that this $\tilde{\theta}(r)$ is *unique*, and, moreover, that the collection of these points $re^{i\tilde{\theta}(r)}$ determines a smooth *optimal path of relaxation* for $J(re^{i\theta})$ in the complex plane. This unique $\tilde{\theta}(r)$ is determined solely from the geometry of the vertex or edge of the convex set $K(D^{-1}A)$ which is closest to the origin. Specifically, since $0 \notin K(D^{-1}A)$, then

$$\tau := \min\{|\xi| : \xi \in K(D^{-1}A)\} \text{ satisfies } \tau > 0, \tag{2.1}$$

and there exist a unique point $\hat{z} = re^{i\psi}$ (ψ real) of the boundary of $K(D^{-1}A)$, which is closest to the origin. With $\partial K(D^{-1}A)$ denoting the boundary of $K(D^{-1}A)$, two cases arise: Case 1, where $|z| = \tau$ intersects $\partial K(D^{-1}A)$ in a vertex of $K(D^{-1}A)$, and Case 2, where $|z| = \tau$ intersects $\partial K(D^{-1}A)$ in a point of an edge of $K(D^{-1}A)$ which is perpendicular to the ray $\{z = re^{i\psi} : r \geq 0\}$. This is shown in Figure 1 below. (In Case 2, the vertices of the edge of $K(D^{-1}A)$ which is perpendicular to the ray $\{z = re^{i\psi} : r \geq 0\}$, determines two real constants $\mu_1 \geq 0$ and $\mu_2 \leq 0$ with

$|\mu_1| + |\mu_2| > 0$. These two constants μ_1 and μ_2 are used below.)

Figure 1

With the notation that $\mu_1 := \mu_2 := 0$ for Case 1, we state the results of [3, Theorem 2.1].

Theorem 3. For the splitting of (1.1), assume that $0 \notin K(D^{-1}A)$ and let $re^{i\psi}$ be the point of $K(D^{-1}A)$ which is closest to the origin. Then, there exists a positive constant m such that, on the circle $|\omega| = r$ with $0 < r < m$, there is a unique $\tilde{\theta}(r)$, given by

$$\tilde{\theta}(r) := -\psi - r(\mu_1 + \mu_2)/2 - r^3(\mu_1 + \mu_2)^3/48 + O(r^5), \quad \text{as} \quad r \to 0, \tag{2.2}$$

for which

$$\rho(J(re^{i\theta(r)})) = \min_{0 \le \theta \le 2\pi} \rho(J(re^{i\theta})) < 1 \quad (0 < r < m). \tag{2.3}$$

Thus,

$$\lim_{r \to 0} \tilde{\theta}(r) = -\psi, \tag{2.4}$$

and

$$\rho(J(re^{i\tilde{\theta}(r)})) = 1 - r\tau - \frac{r^2\mu_1\mu_2}{2} + \frac{r^3\tau}{4}(\mu_1^2 + \mu_2^2) + O(r^4), \quad \text{as} \quad r \to 0. \tag{2.5}$$

We remark that (2.4) gives us that the uniquely defined optimal path of relaxation for $J(\omega)$, for $|\omega|$ small, is *tangential* to the ray $\{re^{i\psi} : r > 0\}$, as $|\omega| \to 0$.

It was shown in Buoni and Varga [3, Theorem 3.1], using the relationship in (1.6) between the SOR iteration matrix $\mathcal{L}(\omega)$ and the extrapolated Jacobi iteration matrix $J(\omega)$, that if $0 \notin K(D^{-1}A)$, then $\mathcal{L}(\omega)$ inherits a behavior similar to that of $J(\omega)$ given in Theorem 3, for $|\omega|$ small. But the proof of this extends easily to all ω^2*-compatible* matrices $T(\omega)$. Thus, we have

Theorem 4. Let the matrix $T(\omega)$ in $\mathbb{C}^{n\times n}$ be ω^2*–compatible* with the Jacobi matrix $J(\omega)$ of (1.2), associated with the splitting of (1.1). If $0 \notin K(D^{-1}A)$, let $\tau e^{i\psi}$ be the point of $K(D^{-1}A)$ which is closest to the origin. Then, there exists a positive constant m' such that, on the circle $|\omega| = r$ with $0 < r < m'$, there is a $\hat{\theta}(r)$ for which

$$\rho(T(re^{i\hat{\theta}(r)})) = \min_{0\leq\theta\leq 2\pi} \rho(T(re^{i\theta})) < 1. \tag{2.6}$$

Moreover,

$$\lim_{r\to 0} \hat{\theta}(r) = -\psi, \tag{2.7}$$

and

$$\rho(T(re^{i\hat{\theta}(r)}) = 1 - r\tau + O(r^{1+1/n}), \quad \text{as} \quad r \to 0. \tag{2.8}$$

To complete this section, it may be useful in applications to have a geometrical result which gives the convergence of an iteration matrix $T(\omega)$ on *intervals* of specific rays $\{z = re^{i\theta} : r > 0\}$. This new result is given in

Theorem 5. For the splitting of (1.1), assume that $0 \notin K(D^{-1}A)$, and let θ any real number such that the rotated set $e^{i\theta}K(D^{-1}A)$ lies in the open right-half plane, i.e.,

$$Re(e^{-i\theta}\zeta_j) > 0 \quad \text{for all} \quad \zeta \in \sigma(D^{-1}A). \tag{2.9}$$

If the matrix $T(\omega)$ in $\mathbb{C}^{n\times n}$ is ω^2*–compatible* with the Jacobi matrix $J(\omega)$ of (1.2), associated with the splitting of (1.2), there exists a positive constant $M(\theta)$ such that $T(\omega)$ is convergent for all $\omega = re^{i\theta}$ with $0 < r < M(\theta)$. Conversely, if

$$Re(e^{-i\theta}\zeta_j) < 0 \quad \text{for some} \quad \zeta_j \in \sigma(D^{-1}A), \tag{2.10}$$

there is no $M'(\theta)$ such that $T(\omega)$ is convergent for all $\omega = re^{i\theta}$ with $0 < r < M'(\theta)$.

Proof. Because of rotations, there is no loss of generality in assuming that $\theta = 0$ satisfies (2.9), i.e., $K(D^{-1}A)$ lies in the open right–half plane, so that

$$Re\ \zeta_j > 0 \quad \text{for all} \quad \zeta_j \in \sigma(D^{-1}A). \tag{2.11}$$

Then from (1.2), each eigenvalue $\mu_j(\omega)$ of $J(\omega)$ can be expressed as $\mu_j(\omega) = 1 - \omega R_j e^{i\psi_j}$ where $\{R_j e^{i\psi_j}\}_{ja01}^{n}$ are the eigenvalues of $D^{-1}A$. For ω real, we have

$$|\mu_j(\omega)|^2 = 1 - 2\omega R_j \cos\psi_j + \omega^2 R_j^2 \quad (1 \leq j \leq n). \tag{2.12}$$

Assuming (2.11), it follows that

$$\beta := \min_{1\leq j\leq n} \frac{2\cos\psi_j}{R_j} > 0. \tag{2.13}$$

Hence, from (2.12), $\rho(J(\omega)) < 1$ for all $0 < \omega < \beta$. Then, since $T(\omega)$ is ω^2–*compatible* with $J(\omega)$ from Proposition 1, the eigenvalues $\{\lambda_j(\omega)\}_{j=1}^{n}$ of $T(\omega)$ necessarily all satisfy

$$\lambda_j(\omega) = \mu_j(\omega) + O(\omega^{1+\frac{1}{n}}), \quad \text{as} \quad \omega \to 0,$$

so that with (2.12),

$$|\lambda_j(\omega)|^2 = 1 - 2\omega R_j \cos\psi_j + O(\omega^{1+\frac{1}{n}}), \quad \text{as} \quad \omega \to 0.$$

Thus with (2.13), there is a $\omega_g > 0$ such that $T(\omega)$ is convergent for all $0 < \omega < \omega_g$.

Conversely, if (2.10) holds, a similar analysis shows that $|\lambda_j(\omega)|^2 > 1$ for all small $\omega > 0$, which gives the concluding of Theorem 5. □

3. Applications to the SSOR iterative method

With the splitting of A in (1.2), we consider the associated symmetric *successive overrelaxation* (SSOR) iteration matrix $S(\omega)$, defined for our purpose as

$$(3.1)\quad S(\omega) = (D - \frac{\omega}{2}U)^{-1}[(1-\frac{\omega}{2})D + \frac{\omega}{2}L](D-\frac{\omega}{2}L)^{-1}[(1-\frac{\omega}{2}U]$$

$$= (I - \frac{\omega}{2}D^{-1}U)^{-1}[(1-\frac{\omega}{2})I + \frac{\omega}{2}D^{-1}L](I - \frac{\omega}{2}D^{-1}L)^{-1}[(1-\frac{\omega}{2})I + \frac{\omega}{2}D^{-1}U].$$

If, in analogy with (1.5), we set

$$(3.2)\qquad \mu(D^{-1}U) := \min\{|\omega| : I - \omega D^{-1}U \ \text{ is singular}\},$$

so that $0 < \mu(D^{-1}U) \leq \infty$, then $S(\omega)$ is analytic in $0 \leq |\omega| < \min\{\mu(L); \mu(U)\}$.

If we set

$$(3.3)\qquad \tilde{S}(\omega) := (I - \frac{\omega}{2}D^{-1}U)S(\omega)(I - \frac{\omega}{2}D^{-1}U)^{-1},$$

a short calculation using (3.3) and (3.1) shows that, with (1.2),

$$(3.4)\qquad \begin{aligned} \tilde{S}(\omega) \;\; = \;\; & J_\omega + \tfrac{\omega^2}{4}\{I - 3[D^{-1}L + D^{-1}U] + 2[(D^{-1}L)^2 + (D^{-1}U)^2] \\ & +4D^{-1}LD^{-1}U\} + O(\omega^3), \end{aligned}$$

as $\omega \to 0$. But since (3.3) implies that

$$(3.5)\qquad S(\omega) = (I - \frac{\omega}{2}D^{-1}U)^{-1}\,\tilde{S}(\omega)(I - \frac{\omega}{2}D^{-1}U),$$

a further short calculation with (3.4) and (3.5) shows that

$$(3.6)\qquad S(\omega) = J_\omega + O(\omega^2) \quad \text{as} \quad \omega \to 0,$$

which gives us the result of

Proposition 6. The symmetric successive overrelaxation iteration matrix $S(\omega)$ of (3.1) is ω^2–*compatible* with the Jacobi matrix $J(\omega)$ of (1.2).

As a consequence of Proposition 6, Theorems 4 and 5 are directly applicable to the SSOR iteration matrix $S(\omega)$ of (3.1). For example, as a special case of Theorem 5, we have

Corollary 7. For the splitting of (1.1) assume that all the eigenvalues of $D^{-1}A$ lie in the open right–half plane. Then, there is an $\omega_g > 0$ such that the SSOR matrix $S(\omega)$ is convergent for all $0 < \omega < \omega_g$.

We remark that Corollary 7 is then an extension of the results of Broyden [1] and Niethammer [6], which were mentioned in §1.

As a further application, we mention that if the matrix A of (1.1), with D nonsingular, can be expressed as

$$A = H + V \qquad (H, V \in \mathbb{C}^{n \times n}), \tag{3.7}$$

then an associated ADI (alternating direct implicit) iterative matrix can be defined by

$$C(\mu) := (V + \mu D)^{-1}(\mu D - H)(H + \mu D)^{-1}(\mu D - V), \tag{3.8}$$

which is defined for all μ sufficiently large in modulus. Further, if

$$\mu =: \frac{2}{\omega} \quad \text{where} \quad \omega \quad \text{is small and nonzero,} \tag{3.9}$$

then $C(\frac{2}{\omega})$ can be verified to be ω^2–*compatible* with $J(\omega) = I - \omega D^{-1}A$. As such, Theorem 5 applies and gives regions of convergence of $C(\frac{2}{\omega})$, for $\omega \neq 0$ and small.

References

[1] G.G. Broyden, *On Convergence Criteria for the Method of Successive Over–Relaxation.* Math. of Comp. 18, (1964) 136–141.

[2] J.J. Buoni, R.S. Varga, *Theorems of Stein–Rosenberg Type.* Numerical Mathematics, (R. Ansorge, K. Glashoff, B. Werner, eds.) ISNM 49, Birkhäuser Verlag, Basel, (1979) 65–75.

[3] J.J. Buoni, R.S. Varga, *Theorems of Stein–Rosenberg Type.* II. *Optimal paths of relaxation in the complex plane. Elliptic Problem Solvers* (Martin H. Schultz, ed.) Academic Press, Inc., New York, (1981) 231–240.

[4] R. Freund, *Krylov Subspace Methods for Complex Non–Hermitian Linear Systems,* Habilitationsschrift Universität Würzburg (1991).

[5] M. Hanke, M. Neumann, and W. Niethammer, *On the spectrum of the SOR operator for symmetric positive definite matrices,* Linear Algebra Appl. **154–156** (1991) 457–472.

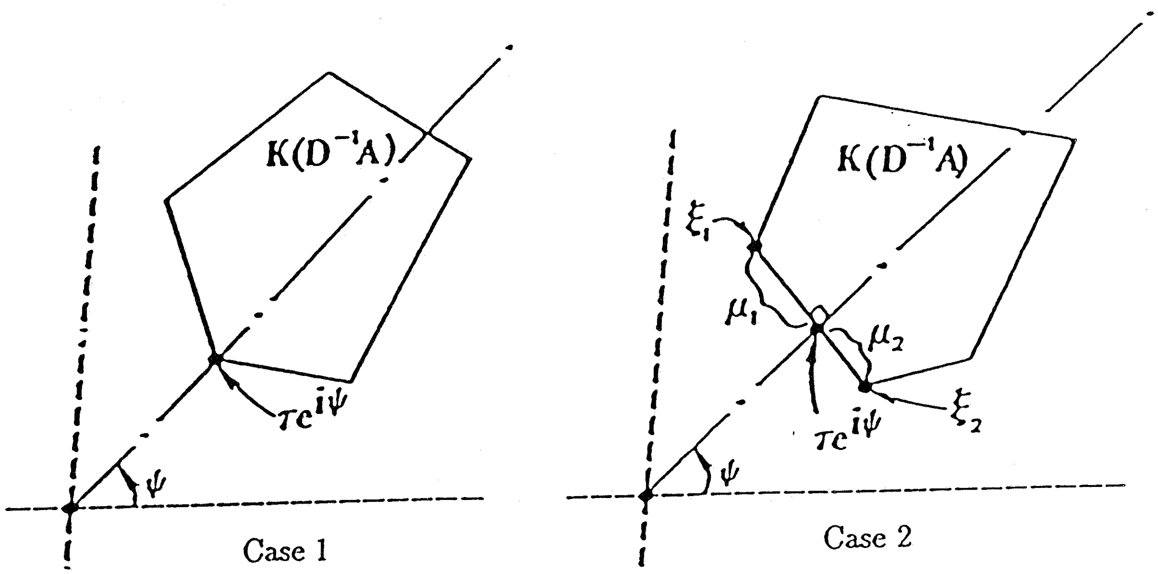
[6] W. Niethammer, *Relaxation bei nichtsymmetrischen Matrizen.* Math. Zeitschr. **86** (1964) 34–40.

[7] W. Niethammer, *Relaxation bei komplexen Matrizen.* Math. Zeitschr. **85** (1964) 319–327.

[8] A.M. Ostrowski, *Solution of Equations in Euclidean and Banach Spaces.* Academic Press Inc., New York (1973).

[9] R.S. Varga, *Matrix Iterative Analysis.* Prentice–Hall Inc., Englewood Cliffs, N.J. (1962).

[10] R.S. Varga, *A Survey of Recent Results on Iterative Methods for Solving Large Sparse Linear Systems.* Elliptic Problem Solvers II (G. Birkhoff and A. Schoenstadt, eds) Academic Press, Inc., New York (1984) 197–217.



Case 1

Case 2

Figure 1