BLOCK DIAGONALLY DOMINANT MATRICES AND GENERALIZATIONS OF THE GERSCHGORIN CIRCLE THEOREM

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1. Introduction. The main purpose of this paper is to give generalizations of the well known theorem of Gerschgorin on inclusion or exclusion regions for the eigenvalues of an arbitrary square matrix A. Basically, such exclusion regions arise naturally from results which establish the nonsingularity of A. For example, if $A = D + C$ where D is a nonsingular diagonal matrix, then Householder [7] shows that $\|D^{-1}C\| < 1$ in some matrix norm is sufficient to conclude that A is nonsingular. Hence, the set of all complex numbers z for which

$$\|(zI - D)^{-1}C\| < 1$$

evidently contains no eigenvalues of A. In a like manner, Fiedler [4] obtains exclusion regions for the eigenvalues of A by establishing the nonsingularity of A through comparisons with M-matrices.[1] Our approach, though not fundamentally different, establishes the nonsingularity of the matrix A by the generalization of the simple concept of a diagonally dominant matrix. But one of our major results (§ 3) is that these new exclusion regions can give significant improvements over the usual Gerschgorin circles in providing bounds for the eigenvalues of A.

2. Block diagonally dominant matrices. Let A be any $n \times n$ matrix with complex entries, which is partitioned in the following manner:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,N} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,N} \\ \vdots & & & \vdots \\ A_{N,1} & A_{N,2} & \cdots & A_{N,N} \end{bmatrix}, \tag{2.1}$$

where the diagonal submatrices $A_{i,i}$ are square of order n_i, $1 \leqq i \leqq N$. For reasons to appear in § 3, the particular choice $N = 1$ of

$$A = [A_{1,1}] \tag{2.1'}$$

will be useful. Viewing the square matrix $A_{i,i}$ as a linear transformation of the n_i-dimensional vector subspace Ω_i into itself, we associate with this subspace the vector norm $\|\boldsymbol{x}\|_{\Omega_i}$, i.e., if $\boldsymbol{x}$ and $\boldsymbol{y}$ are elements of

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[1] For the definition of an M-matrix, see § 4 or [8].

Ω_i, then

$$(2.2)\qquad \begin{cases} \|\boldsymbol{x}\|_{\Omega_i} > 0 \text{ unless } \boldsymbol{x} = \boldsymbol{O}\,; \\ \|\alpha\boldsymbol{x}\|_{\Omega_i} = |\alpha|\,\|\boldsymbol{x}\|_{\Omega_i} \text{ for any scalar } \alpha\,; \\ \|\boldsymbol{x}+\boldsymbol{y}\|_{\Omega_i} \leqq \|\boldsymbol{x}\|_{\Omega_i} + \|\boldsymbol{y}\|_{\Omega_i}\,, \quad 1 \leqq i \leqq N\,. \end{cases}$$

The point here is that we can associate *different* vector norms with different subspaces Ω_i. Now, similarly considering the rectangular matrix $A_{i,j}$ for any $1 \leqq i, j \leqq N$ as a linear transformation from Ω_j to Ω_i, the norm $\|A_{i,j}\|$ is defined as usual by

$$(2.3)\qquad \|A_{i,j}\| \equiv \sup_{\boldsymbol{x}\in\Omega_j, \boldsymbol{x}\neq \boldsymbol{O}} \frac{\|A_{i,j}\boldsymbol{x}\|_{\Omega_i}}{\|\boldsymbol{x}\|_{\Omega_j}}\,.$$

Note that if the partitioning in (2.1) is such that all the matrices $A_{i,j}$ are 1×1 matrices and $\|\boldsymbol{x}\|_{\Omega_i} \equiv |x|$, then the norms $\|A_{i,j}\|$ are just the moduli of the single entries of these matrices. As no confusion arises, we shall drop the subscripts on the different vector norms.[2]

DEFINITION 1. Let the $n \times n$ matrix A be partitioned as in (2.1). If the diagonal submatrices $A_{j,j}$ are nonsingular, and if

$$(2.4)\qquad (\|A_{j,j}^{-1}\|)^{-1} \geqq \sum_{\substack{k=1\\k\neq j}}^{N} \|A_{j,k}\| \quad \text{for all } 1 \leqq j \leqq N\,,$$

then A is *block diagonally dominant*, relative to the partitioning (2.1). If strict inequality in (2.4) is valid for all $1 \leqq j \leqq N$, then A is *block strictly diagonally dominant*, relative to the partitioning of (2.1).

It is useful to point out that the quantity appearing on the lefthand side of (2.4) can also be characterized form (2.3) by

$$(2.5)\qquad (\|A_{j,j}^{-1}\|)^{-1} = \inf_{\boldsymbol{x}\in\Omega_j, \boldsymbol{x}\neq \boldsymbol{O}} \left(\frac{\|A_{j,j}\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\right),$$

whenever $A_{j,j}$ is nonsingular. With (2.5), we can then define $(\|A_{j,j}^{-1}\|)^{-1}$ by continuity to be zero whenever $A_{j,j}$ is singular.

In the special case that all the matrices $A_{i,j}$ are 1×1 matrices and $\|x\| = |x|$, then (2.4) can be written as

$$(2.4')\qquad |A_{j,j}| \geqq \sum_{\substack{k=1\\k\neq j}}^{N} |A_{j,k}| \quad \text{for all } 1 \leqq j \leqq N\,,$$

which is the usual definition of diagonal dominance.

As an example of a matrix which is block strictly diagonally dominant, consider the case $n = 4$, $N = 2$ of

[2] Later, we shall use the notation $\|\boldsymbol{x}\|_p$ to denote the l_p-norm $\|\boldsymbol{x}\|_p \equiv (\sum_i |x_i|^p)^{1/p}$.

$$A = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 \\ \hline 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1 & 0 \end{array}\right], \tag{2.6}$$

where we choose the vector norms $\|x\|_\infty \equiv \max_j |x_j|$. In this case,

$$(\|A_{1,1}^{-1}\|)^{-1} = (\|A_{2,2}^{-1}\|)^{-1} = \tfrac{2}{3}\,, \quad \text{and } \|A_{1,2}\| = \|A_{2,1}\| = \tfrac{1}{3}\,.$$

Obviously, A is not diagonally dominant in the sense of (2.4′).

DEFINITION 2. The $n \times n$ partitioned matrix A of (2.1) is *block irreducible* if the $N \times N$ matrix $B = (b_{i,j} \equiv \|A_{i,j}\|)$, $1 \leqq i, j \leqq N$, is *irreducible*, i.e., the directed graph of B is strongly connected.[3]

THEOREM 1. *If the partitioned matrix A of* (2.1) *is block strictly diagonally dominant, or if A is block irreducible and block diagonally dominant with inequality holding in* (2.4) *for at least one j, then A is nonsingular.*

Proof. The extension to the case where A is block irreducible and block diagonally dominant with strict inequality for at least one j is easy, so we consider for simplicity only the case when A is block strictly diagonally dominant. Suppose that A is singular, i.e., there exists a nonzero vector $\boldsymbol{W}$ with

$$A\begin{bmatrix} W_1 \\ \vdots \\ W_N \end{bmatrix} = \boldsymbol{O}\,; \tag{2.7}$$

here, we have partitioned $\boldsymbol{W}$ conformally with respect to the partitioning of (2.1). But this is equivalent to

$$\sum_{\substack{j=1\\ j\neq i}}^{N} A_{i,j} W_j = -A_{i,i} W_i\,, \qquad 1 \leqq i \leqq N\,. \tag{2.8}$$

Since $\boldsymbol{W}$ is a nonzero vector, normalize $\boldsymbol{W}$ so that $\|W_j\| \leqq 1$ for all $1 \leqq j \leqq N$, and assume that equality is valid for some r, i.e., $\|W_r\| = 1$ where $1 \leqq r \leqq N$. Thus, from (2.3)

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[3] Equivalently, there exists *no* $N \times N$ permutation matrix P such that $PBP^T = \left[\begin{smallmatrix} C & D \\ O & E \end{smallmatrix}\right]$, where C and E are square nonvoid submatrices. For strongly connected directed graphs, see for example [**6**].

$$(2.8') \qquad \| A_{r,r} W_r \| = \Big\| \sum_{\substack{j=1 \\ j \neq r}}^{N} A_{r,j} W_j \Big\| \leqq \sum_{\substack{j=1 \\ j \neq r}}^{N} \| A_{r,j} \| \cdot \| W_j \| \leqq \sum_{\substack{j=1 \\ j \neq r}}^{N} \| A_{r,j} \| .$$

But as $A_{r,r}$ is nonsingular by hypothesis, then putting $A_{r,r} W_r = Z_r$,

$$\| A_{r,r} W_r \| = \frac{\| A_{r,r} W_r \|}{\| W_r \|} = \frac{\| Z_r \|}{\| A_{r,r}^{-1} Z_r \|} \geqq (\| A_{r,r}^{-1} \|)^{-1} ,$$

using (2.3). This combined with (2.8′) gives a contradiction to the assumption (2.4) that A is block strictly diagonally dominant, which completes the proof for the block strictly diagonally dominant case.

Actually, we can regard Theorem 1 as the block analogue of the well known Hadamard theorem on determinants, since Theorem 1 reduces to this result in the case that all the matrices $A_{i,j}$ of (2.1) are 1×1 matrices and $\| x \| \equiv | x |$. It should be pointed out that the result of Theorem 1 itself is a special case of a more general result by Ostrowski [10, Theorem 3, p. 185], and Fiedler [4].

As stated in the introduction, the above theorem leads naturally to a block analogue of the *Gerschgorin Circle Theorem*. If I is the $n \times n$ identity matrix which is partitioned as in (2.1), and I_j is the $n_j \times n_j$ identity matrix, suppose that

$$(2.9) \qquad (\| (A_{j,j} - \lambda I_j)^{-1} \|)^{-1} > \sum_{\substack{k=1 \\ k \neq j}}^{N} \| A_{j,k} \| \quad \text{for all } 1 \leqq j \leqq N .$$

Thus, we have from Theorem 1 that $A - \lambda I$ is nonsingular. Hence, if λ is an eigenvalue of A, then $A - \lambda I$ cannot be block strictly diagonally dominant, which gives us

THEOREM 2. *For the partitioned matrix A of* (2.1), *each eigenvalue λ of A satisfies*

$$(2.10) \qquad (\| (A_{j,j} - \lambda I_j)^{-1} \|)^{-1} \leqq \sum_{\substack{k=1 \\ k \neq j}}^{N} \| A_{j,k} \|$$

for at least one j, $1 \leqq j \leqq N$.

We again remark that if the partitioning of (2.1) is such that all the diagonal submatrices are 1×1 matrices and $\| x \| \equiv | x |$, then Theorem 2 reduces to the well known Gerschgorin Circle Theorem.

3. Inclusion regions for eigenvalues.

In Theorem 2, we saw that each eigenvalue λ of an arbitrary $n \times n$ complex matrix A necessarily satisfied (2.10) for *at least* one j, $1 \leqq j \leqq N$.

DEFINITION 3. For the partitioned $n \times n$ matrix A of (2.1), let the

Gerschgorin set G_j be the set of all complex numbers z such that

$$(3.1) \qquad (\|(A_{j,j} - zI)^{-1}\|)^{-1} \leqq \sum_{\substack{k=1 \\ k \neq j}}^{N} \|A_{j,k}\| , \qquad 1 \leqq j \leqq N .$$

Thus, from (2.5), we conclude that the Gerschgorin set G_j always contains the eigenvalues of $A_{j,j}$, independent to the magnitude of the right side of (3.1) and independent of the vector norms used. Next, it is clear that each Gerschgorin set G_j is closed and bounded. Hence, so is their union

$$(3.2) \qquad G = \bigcup_{j=1}^{N} G_j .$$

Thus, we can speak of the *boundary of* G, as well as the boundary of each G_j. By Theorem 2, all the eigenvalues of A lie in G. Can any eigenvalue λ of A lie on the boundary of G? This can be answered trivially for the particular partitioning of (2.1′). In this case, the right-hand side of (3.1) is vacuously zero, and from (3.1), we see that the set G is a *finite point set* consisting *only* of the eigenvalues of A. In this case, Theorem 2 gives *exact* information about the eigenvalues of A.

It is interesting that Theorem 2 can be strengthened by the assumption that A is block irreducible, which is the analogue of a well known result of Taussky [11].

THEOREM 3. *Let the partitioned matrix A of* (2.1) *be block irreducible, and let λ be an eigenvalue of A. If λ is a boundary point of G, then it is a boundary point of each set G_j, $1 \leqq j \leqq N$.*

Proof. Since λ is an eigenvalue of A, then $\sum_{j=1}^{N} A_{i,j} W_j = \lambda W_i$, and if $\| W_j \| \leqq \| W_r \| = 1$, then as before

$$(3.3) \qquad (\|(A_{r,r} - \lambda I_r)^{-1}\|)^{-1} \leqq \sum_{\substack{j=1 \\ j \neq r}}^{N} \|A_{r,j}\| \, \|W_j\| \leqq \sum_{\substack{j=1 \\ j \neq r}}^{N} \|A_{r,j}\| .$$

But as λ is a boundary point of G, equality must hold throughout (3.3), showing that λ is a boundary point of G_r. Moreover, if $\|A_{r,j}\| \neq 0$, then $\|W_j\| = 1$, and we can repeat the argument with r replaced by j. In this way, we conclude that λ is a boundary point of G_j. From the irreducibility of A, the argument can be extended to every index j, $1 \leqq j \leqq N$, which completes the proof. A similar argument can be applied to complete the proof of Theorem 1.

Another familiar result of Gerschgorin can also be generalized. The proof, depending on a continuity argument, follows that given in [13, p. 287].

THEOREM 4. *If the union $H = \bigcup_{j=1}^{m} G_{p_j}$, $1 \leqq p_j \leqq N$, of m Gerschgorin* *sets is disjoint from the remaining $N - m$ Gerschgorin sets for the partitioned matrix A of* (2.1), *then H contains precisely $\sum_{j=1}^{m} n_{p_j}$ eigenvalues of A.*

The previous example of the matrix of (2.1′) indicated that sharper inclusion regions for the eigenvalues of a matrix A may be obtained from the generalized form of Gerschgorin's Theorem 2. To give another illustration, consider the partitioned matrix

$$A = \left[\begin{array}{cc|cc} 4 & -2 & -1 & 0 \\ -2 & 4 & 0 & -1 \\ \hline -1 & 0 & 4 & -2 \\ 0 & -1 & -2 & 4 \end{array}\right] = \left[\begin{array}{c|c} A_{1,1} & A_{1,2} \\ \hline A_{2,1} & A_{2,2} \end{array}\right] . \tag{3.4}$$

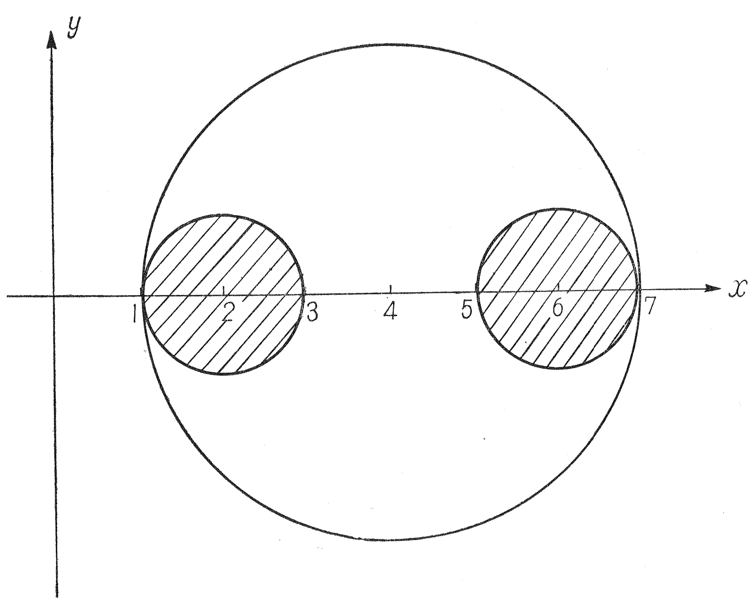
Employing now the vector norm $||\boldsymbol{x}||_2 \equiv (\sum_i |x_i|^2)^{1/2}$, it is apparent that $||A_{1,2}|| = ||A_{2,1}|| = 1$. On the other hand, direct computation shows that

$$(||(A_{i,i} - zI_i)^{-1}||^{-1} = \min\{|6 - z|, |2 - z|\} , \qquad i = 1, 2 .$$

By definition, the set G_1 then consists of the points z for which

$$|6 - z| \leqq 1 , \qquad |2 - z| \leqq 1 ,$$

so that G_1 is itself the union of two *disjoint* circles. The same is true for G_2, since $G_2 = G_1$, as shown in the figure below. The usual Gerschgorin circles



for the matrix A of (3.4) are all given by the single circle $|4 - \lambda| \leqq 3$, which is a circle of radius 3, with center at $z = 4$, as shown above. From this figure, we conclude that the block Gerschgorin result *can* give significant improvements over the usual Gerschgorin circles in providing bounds for eigenvalues. For the matrix A of (3.4), its eigenvalues are

$$\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 5, \lambda_4 = 7 .$$

Note, again from the figure above, that Theorem 3 applies in this case.

At this point, we remark that the previous example was such that each Gerschgorin set G_j consisted of the union of circles. This is a special case of

THEOREM 5. *Let the partitioned matrix A of* (2.1) *be such that its diagonal submatrices $A_{j,j}$ are all normal. If the Euclidean vector norms $\|\boldsymbol{x}\|_2$ are used for each subspace Ω_j, $1 \leqq j \leqq N$, then each Gerschgorin set G_j is the union of n_j circles.*

Proof. Let the eigenvalues of $A_{j,j}$ be σ_l, $1 \leqq l \leqq n_j$. Since $A_{j,j}$ is normal, we can write $(\|(A_{j,j} - zI_j)^{-1}\|)^{-1} = \min_l |\sigma_l - z|$, which, combined with Definition 3, completes the proof.

It is quite simple to obtain the block analogues of well known results on inclusion regions for eigenvalues of $n \times n$ complex matrices. As a first example, the result of A. Brauer [2] on ovals of Cassini easily carries over.

THEOREM 6. *Let the $n \times n$ complex matrix A be partitioned as in* (2.1). *Then, all the eigenvalues of A lie in the union of the* $[N(N-1)]/2$ *point sets $C_{i,j}$ defined by*

$$(3.5)\quad (\|(A_{i,i} - zI_i)^{-1}\| \cdot \|(A_{j,j} - zI_j)^{-1}\|)^{-1} \leqq \left(\sum_{\substack{l=1 \\ l\neq i}}^{N} \|A_{i,l}\|\right)\left(\sum_{\substack{l=1 \\ l\neq j}}^{N} \|A_{j,l}\|\right)$$

where $1 \leqq i, j \leqq N$ and $i \neq j$. Moreover, if A is block irreducible, and λ is an eigenvalue of A not in the interior of $\bigcup_{i\neq j} C_{i,j}$, then λ is a boundary point of each of the point sets $C_{i,j}$.

Other obvious remarks can be made. Clearly, replacing A by A^T leaves the eigenvalues of A invariant. Thus, rows sums can be replaced by column sums in the definition (2.4) of diagonal dominance, and many results using both row and column sums admit easy generalizations. As an illustration, we include the following known [4] generalization of a result by Ostrowski [9].

THEOREM. 7. *Let the $n \times n$ complex matrix A be partitioned as in* (2.1), *and define*

$$R_j \equiv \sum_{\substack{k=1 \\ k \neq j}}^{N} \| A_{j,k} \| ; \quad C_j \equiv \sum_{\substack{k=1 \\ k \neq j}}^{N} \| A_{k,j} \| , \qquad 1 \leqq j \leqq N . \tag{3.6}$$

Then, for any α *with* $0 \leqq \alpha \leqq 1$, *each eigenvalue* λ *of* A *satisfies*

$$(\| (A_{j,j} - \lambda I_j)^{-1} \|)^{-1} \leqq R_j^{\alpha} C_j^{1-\alpha} \tag{3.7}$$

for at least one j, $1 \leqq j \leqq N$.

Also, the important result of Fan and Hoffman [3] carries over with ease.

THEOREM 8. *Let the* $n \times n$ *complex matrix* A *be partitioned as in* (2.1). *Let* $p > 1$, *and* $1/p + 1/q = 1$. *If* $\alpha > 0$ *satisfies*

$$\sum_{i=1}^{N} \left\{ \frac{\left(\sum_{j \neq i} \| A_{i,j} \| \right)^{q}}{\left(\sum_{j \neq i} \| A_{i,j} \|^{p} \right)^{q/p}} \right\} \leqq \alpha^{q} (1 + \alpha^{q}) , \tag{3.8}$$

(whenever 0/0 occurs on the left-hand side, we agree to put $0/0 = 0$), then every eigenvalue λ of A satisfies at least one of the following relations:

$$(\| (A_{j,j} - \lambda I_j)^{-1} \|)^{-1} \leqq \alpha \left(\sum_{\substack{k=1 \\ k \neq j}}^{N} \| A_{j,k} \|^{p} \right)^{1/p} , \qquad 1 \leqq j \leqq N . \tag{3.9}$$

We wish to emphasize that, unlike the cases previously treated where all the matrices $A_{i,j}$ of (2.1) are 1×1 matrices, these new inclusion regions now *depend* on the vector norms used. It seems reasonable, at least theoretically, to minimize these inclusion regions by considering all possible vector norms to produce optimum results. Similarly, there is a great deal of flexibility in the manner in which the matrix A is partitioned, and this perhaps can be used to advantage.

4. Another generalization. Another result, due again to Taussky [12], states that if an $n \times n$ matrix $A = (a_{i,j})$ is strictly diagonally dominant in the usual sense of (2.4′) with positive real diagonal entries $a_{i,i}$ $1 \leqq i \leqq n$, then the eigenvalues λ_j of A satisfy

$$Re \lambda_j > 0 , \qquad 1 \leqq j \leqq n . \tag{4.1}$$

Based on our previous results, we now give a generalization of this result which depends upon the use of *absolute norms* [1]. By this, we mean the following. First, if $\boldsymbol{x}$ is a column vector with complex components x_i, let $|\boldsymbol{x}|$ denote the vector with components $|x_i|$. If

$$\| \boldsymbol{x} \| = \| \, |\boldsymbol{x}| \, \| \tag{4.2}$$

for all vectors $\boldsymbol{x}$, then the norm is an absolute norm.[4] This is equivalent

[4] Clearly, the l_p-norms of footnote 2 are absolute norms.

[1] to the property that if $|\boldsymbol{y}| \geqq |\boldsymbol{x}|$, i.e., each component of $|\boldsymbol{y}| - |\boldsymbol{x}|$ is a nonnegative real number, then

$$\|\boldsymbol{y}\| \geqq \|\boldsymbol{x}\| . \tag{4.2'}$$

Next, if $B = (b_{i,j})$ is a real $m \times m$ matrix with $b_{i,j} \leqq 0$ for all $i \neq j$, and if B is nonsingular with $B^{-1} \equiv (r_{i,j})$ such that $r_{i,j} \geqq 0$ for all $1 \leqq i, j \leqq m$, then B is said to be an *M-matrix* [8].

THEOREM 9. *Let the $n \times n$ complex matrix A be partitioned as in* (2.1), *and let A be block strictly diagonally dominant* (*or block irreducible and block diagonally dominant with strict inequality in* (2.4) *for at least one j*). *Further, assume that each submatrix $A_{j,j}$ is an M-matrix, $1 \leqq j \leqq N$, and the vector norms for each subspace Ω_j are absolute norms. If λ is any eigenvalue of A, then*

$$Re\lambda > 0 . \tag{4.3}$$

Proof. For simplicity, we shall consider again only the case where A is block strictly diagonally dominant. Let z be any complex number with $Re z \leqq 0$. If $A_{j,j}^{-1} \equiv (r_{k,l})$, and $(A_{j,j} - zI_j)^{-1} \equiv (s_{k,l}(z))$, it follows [8] from the assumption that $A_{j,j}$ is an M-matrix that

$$|s_{k,l}(z)| \leqq r_{k,l} , \qquad 1 \leqq k, l \leqq n_j . \tag{4.4}$$

Next, with (4.4) and the assumption of absolute norms, it follows from (4.2) and (4.2′) that

$$\frac{\|(A_{j,j} - zI_j)^{-1}\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leqq \frac{\|A_{j,j}^{-1}|\boldsymbol{x}|\|}{\||\boldsymbol{x}|\|} ,$$

so that from (2.3),

$$(\|(A_{j,j} - zI_j)^{-1}\|)^{-1} \geqq (\|A_{j,j}^{-1}\|)^{-1} .$$

In other words, for any z with $Re z \leqq 0$, then the matrix $A - zI$ continues to be block strictly diagonally dominant, and hence nonsingular. Thus, if λ is an eigenvalue of A, then $Re\lambda > 0$, which completes the proof.

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