International Series of Numerical Mathematics, Vol. 119, ©1994 Birkhäuser Verlag Basel

An Extension of a Result of Rivlin on Walsh Equiconvergence (Faber Nodes)

R. Brück* A. Sharma† R.S. Varga‡

Dedicated to Walter Gautschi on the occasion of his 65th birthday

Abstract. We continue our investigations of generalizations of Walsh's equiconvergence theorem. The setting is a compact set E of the complex plane, whose complement is simply connected in the extended complex plane, and the Faber polynomials associated with E. Here, we study equiconvergence phenomena for differences of interpolating polynomials, defined by Lagrange (and Hermite) interpolants in zeros of associated Faber polynomials.

1 INTRODUCTION

We begin with the well-known Walsh equiconvergence theorem [10, p. 153]. With $D_\tau := \{z \in \mathbb{C} : |z| < \tau\}$ and with any R satisfying $1 < R < \infty$, let A_R denote the set of all functions f which are analytic in D_R, but not in $\bar{D}_R$. Then, the Walsh equiconvergence theorem, simply stated, asserts that if $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is in A_R, then

$$\lim_{n\to\infty} [L_n(z;f) - S_n(z;f)] = 0, \qquad z \in D_{R^2}, \tag{1.1}$$

the convergence being uniform and geometric on every disk $\bar{D}_\mu$ with $\mu < R^2$, where $L_n(z;f)$ denotes the Lagrange interpolant to f in the $(n+1)^{st}$ roots of unity and where $S_n(z;f) := \sum_{k=0}^{n} a_k z^k$. (Since the convergence to zero in (1.1) takes place in the domain D_{R^2} which is *larger* than the domain D_R of analyticity of f, the result of (1.1) is said to exhibit *overconvergence*.) Rivlin [9] extended (1.1) by replacing $L_n(z;f)$ by polynomials $P_{m,n}(z;f)$ which best approximate f, in the ℓ_2-sense over all polynomials of degree n, in the $(m+1)^{st}$ roots of unity, where, for a fixed positive integer q (i.e., $q \in \mathbb{N}$), $m := q(n+1) - 1$ for all $n \in \mathbb{N}$. He showed that

$$\lim_{n\to\infty} [P_{m,n}(z;f) - S_n(z;f)] = 0, \qquad z \in D_{R^{1+q}}, \tag{1.2}$$

the convergence being uniform and geometric on every disk $\bar{D}_\mu$ with $\mu < R^{1+q}$.

*Mathematisches Institut, Justus-Liebig-Universität Giessen, Arndtstrasse 2, D-35392 Giessen, Germany

†Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

‡Institute for Computational Mathematics, Kent State University, Kent, Ohio 44242-0001, USA

In a recent paper [2], we studied the problem of Walsh equiconvergence for domains more general than disks. More precisely, if $\mathbb{C}_\infty$ denotes the extended complex plane, let E be a compact subset (not a point) of the complex plane whose complement, $\mathbb{C}_\infty \backslash E$, is simply connected. By the Riemann mapping theorem, there exists a conformal map ψ of $\{w \in \mathbb{C} : |w| > 1\}$ onto $\mathbb{C}_\infty \backslash E$, where the mapping is normalized at infinity by $\psi(\infty) = \infty$ and $c := \psi'(\infty) > 0$. (The quantity c is called the *capacity* of E.) For $1 < R < \infty$, let $C_R := \{z = \psi(w) : |w| = R\}$ be an outer level curve of E, and let A_R now denote the class of functions which are analytic in $G_R := \text{Int} C_R$, but not in $\bar{G}_R$. If F_k denotes the k^{th} Faber polynomial associated with E and if

$$f(z) = \sum_{k=0}^{\infty} a_k F_k(z) \tag{1.3}$$

is the Faber expansion of f with respect to E, we set

$$S_n(z; f) := \sum_{k=0}^{n} a_k F_k(z). \tag{1.4}$$

As in [2], if we assume that the boundary ∂E of E is a Jordan curve, then the conformal map ψ can be extended to a homeomorphism of $\{w \in \mathbb{C} : |w| \geq 1\}$ onto $\mathbb{C} \backslash \text{Int} E$, so that we may define, for each $m \in \mathbb{N}$, the $m+1$ points $\{z_{k,m}\}_{k=0}^{m}$, where

$$z_{k,m} := \psi(w_{k,m}), \qquad w_{k,m} := \exp\left(\frac{2\pi i k}{m+1}\right), \qquad k = 0, \ldots, m. \tag{1.5}$$

The points $z_{k,m}$ are called the $(m+1)^{st}$ *Fejér nodes* with respect to E. Following Pommerenke [8], we call ∂E an r_0-*analytic curve* $(0 \leq r_0 < 1)$ if the conformal map ψ admits a univalent continuation to $\{w \in \mathbb{C} : |w| > r_0\}$. For $f \in A_R$, let (1.3) be the Faber expansion of f with respect to E, and let $L_m(z; f)$ denote the Lagrange interpolant of f in the Fejér nodes (1.5). If $L_m(z; f) = \sum_{k=0}^{m} b_k F_k(z)$ is the Faber expansion of $L_m(z; f)$, set

$$S_n\left(z; L_m(\cdot; f)\right) := \sum_{k=0}^{n} b_k F_k(z). \tag{1.6}$$

Our extension in [2] of Rivlin's theorem [9] studied the region of equiconvergence of the difference $S_n\left(z; L_m(\cdot; f)\right) - \sum_{k=0}^{n} a_k F_k(z)$.

If we set

$$S_{m,n,j}(z; f) := \sum_{k=0}^{n} a_{k+j(m+1)} F_k(z), \qquad j \in \mathbb{N}_0, \tag{1.7}$$

then for any integer $\ell \in \mathbb{N}$, we considered the difference

$$\Delta_{m,n,\ell}(z; f) := S_n\left(z; L_m(\cdot; f)\right) - \sum_{j=0}^{\ell-1} S_{m,n,j}(z; f). \tag{1.8}$$

The following theorem was proved in [2].

Theorem A [2] *Let ∂E be an r_0-analytic curve for some $r_0 \in [0,1)$, let $f \in A_R$, let $m = q(n+1)-1$ for a fixed $q \in \mathbb{N}$, and let $\Delta_{m,n,\ell}(z;f)$ be given by (1.8). Then,*

$$\lim_{n\to\infty} \Delta_{m,n,\ell}(z;f) = 0, \qquad z \in G_\lambda, \tag{1.9}$$

the convergence being uniform and geometric on every subset $\bar{G}_\mu$ for $1 \le \mu < \lambda$, where

$$\lambda := \min\{R^{1+\ell q}; R/r_0^q; R^q/r_0^{q-1}\}, \tag{1.10}$$

with $0^k := 0$ for any nonnegative integer k and $1/0 := \infty$.

Remark. From (1.10), we see that if $q = 1$ and if $0 < r_0 < 1$, then $\lambda = R$, and also that $\lambda \to R$ for arbitrary q as $r_0 \to 1$. Thus, Theorem A gives no overconvergence in these cases. Indeed, the first author has shown in [1] that there is no overconvergence in the case $q = \ell = 1$ and $r_0 > 0$.

In the special case of $E = D_1$ where $r_0 = 0$, (1.10) reduces to $\lambda = R^{1+\ell q}$. For $\ell = 1$, this gives the result of Rivlin [9, Theorem 1] and for $q = 1$, this gives a result of Cavaretta, Sharma and Varga [3, Theorem 1]. If $q \ge \ell + 1$ and $r_0 \le \frac{1}{R^\ell}$, then (1.10) gives $\lambda = R^{1+\ell q}$, which means that we have the same λ as in the case when E is chosen to be the closed unit disk $\bar{D}_1$ (where $r_0 = 0$).

The object of this paper is to investigate the situation where the nodes of the Lagrange interpolant $L_m(z;f)$ are the zeros of the Faber polynomial $F_{m+1}(z)$, rather than the Fejér nodes of (1.5). In §2, we list some properties of Faber polynomials and state Theorem 1 and outline its proof. Section 3 deals with operators analogous to those of Theorem A, but based on Hermite interpolation using Fejér nodes and Faber nodes, respectively. It consists of a statement of Theorem B (which was given without proof in [2]) and the statement and proof of its analogue using Faber nodes. In §4, we give Theorem C (given without proof in [2]) and prove an analogous result (Theorem 3) using Faber nodes. Sections 5 and 6 are devoted to the proofs of Theorems B and C, respectively.

2 FABER NODES

In this section, we establish an analogue of Theorem A, where we replace the $m+1$ Fejér nodes of (1.5) by the $m+1$ zeros of the $(m+1)^{st}$ Faber polynomial with respect to E. We call these zeros *Faber nodes.* It is well known [5, p. 584], for an arbitrary compact set E (not a point) for which $\mathbb{C}_\infty \backslash E$ is simply connected, that the associated Faber polynomials $\{F_n(z)\}_{n\ge 0}$ satisfy

$$\lim_{n\to\infty} |F_n(\psi(w))|^{1/n} = |w|, \qquad |w| > 1, \tag{2.1}$$

uniformly on every closed subset of $\{w \in \mathbb{C} : |w| > 1\}$. For any fixed $R > 1$, (2.1) implies that the zeros of $F_n(z)$ all lie in G_R for any n sufficiently large, which further implies that all accumulation points of the zeros of $\{F_n(z)\}_{n\geq 1}$ must lie in E. Thus, $L_n^*(z;f)$, defined to be the Lagrange interpolant of f in the zeros of the Faber polynomial $F_{n+1}(z)$, is then well defined for all large n. If ∂E is r_0-analytic, then (2.1) holds for $|w| > r_0$ and thus, the Faber nodes all lie in the interior of E for every n sufficiently large. It is known [7] that if E is convex, but not a line segment, then all Faber nodes lie in the interior of E. In the case when E is the line segment $[-1,1]$, it is well known that the Faber polynomials for E coincide with the classical Chebyshev polynomials of the first kind.

Let $L_m^*(z;f)$ denote the Lagrange interpolant to $f \in A_R$ in the $m+1$ Faber nodes, i.e., the zeros of the Faber polynomial $F_{m+1}(z)$ of degree $m+1$. Set

$$\Delta^*_{m,n,\ell}(z;f) := S_n\left(z; L_m^*(\cdot;f)\right) - \sum_{j=0}^{\ell-1} S_{m,n,j}(z;f), \qquad \ell \in \mathbb{N}, \tag{2.2}$$

where $S_{m,n,j}(z;f)$ is given by (1.7) and where $S_n\left(z; L_m^*(\cdot;f)\right)$ is the Faber expansion of $L_m^*(z;f)$ up to degree n. We now establish:

Theorem 1 *Let $f \in A_R$, let $m = q(n+1)-1$ for a fixed $q \in \mathbb{N}$, and let $\Delta^*_{m,n,\ell}(z;f)$ be given by (2.2), where $L_m^*(z;f)$ is the Lagrange interpolant of f in the Faber nodes (i.e., the zeros of $F_{m+1}(z)$ with respect to a compact set E). Then,*

$$\lim_{n\to\infty} \Delta^*_{m,n,\ell}(z;f) = 0, \qquad z \in G_{R^q}, \tag{2.3}$$

the convergence being uniform and geometric on every subset $\bar{G}_\mu$ for $1 \leq \mu < R^q$.

Proof: We proceed along the lines of the proof of Theorem A of [2]. From the well-known Hermite interpolation formula, we have, for any $r' \in (1,R)$ and any $z \in \mathbb{C}$, that

$$L_m^*(z;f) = \frac{1}{2\pi i} \int\limits_{|\zeta|=r'} f\left(\psi(\zeta)\right) \frac{\psi'(\zeta)}{\psi(\zeta)-z} \cdot \frac{w_m\left(\psi(\zeta)\right) - w_m(z)}{w_m\left(\psi(\zeta)\right)}\, d\zeta, \tag{2.4}$$

where $w_m(z) := c^{m+1}F_{m+1}(z)$. From [5, eq. (5.2)], the Faber coefficients a_k of f have the integral representation of

$$a_k = \frac{1}{2\pi i} \int\limits_{|\zeta|=r'} \frac{f\left(\psi(\zeta)\right)}{\zeta^{k+1}}\, d\zeta, \qquad k = 0,1,2,\ldots. \tag{2.5}$$

Then, from (1.4) and (2.5), we have

$$S_n(z;f) = \frac{1}{2\pi i} \int\limits_{|\zeta|=r'} f\left(\psi(\zeta)\right) \sum_{k=0}^{n} \frac{F_k(z)}{\zeta^{k+1}}\, d\zeta, \qquad z \in \mathbb{C}. \tag{2.6}$$

From (2.4) and (2.6), we obtain

$$S_n\left(z; L_m^*(\cdot; f)\right)$$
$$=\frac{1}{2\pi i}\int\limits_{|\zeta|=r'} f\left(\psi(\zeta)\right)\left(\frac{1}{2\pi i}\int\limits_{|t|=r}\frac{\psi'(\zeta)}{\psi(\zeta)-\psi(t)}\cdot\frac{w_m(\psi(\zeta))-w_m(\psi(t))}{w_m(\psi(\zeta))}\sum_{k=0}^{n}\frac{F_k(z)}{t^{k+1}}\,dt\right)d\zeta, \tag{2.7}$$

where we choose r and r' such that $1 < r < r' < R$. Furthermore, from (1.7) and (2.5) we easily derive that

$$\sum_{j=0}^{\ell-1} S_{m,n,j}(z; f) = \frac{1}{2\pi i}\int\limits_{|\zeta|=r'} f\left(\psi(\zeta)\right)\frac{\zeta^{\ell(m+1)}-1}{\zeta^{(\ell-1)(m+1)}(\zeta^{m+1}-1)}\sum_{k=0}^{n}\frac{F_k(z)}{\zeta^{k+1}}\,d\zeta, \quad z\in\mathbb{C}. \tag{2.8}$$

But, as a consequence of the residue theorem (applied in $|t| > r$), it follows that

$$\frac{1}{2\pi i}\int\limits_{|t|=r}\frac{\psi'(\zeta)}{\psi(\zeta)-\psi(t)}\frac{dt}{t} = \frac{1}{\zeta}, \qquad |\zeta| > r,$$

which allows us to express (2.8) as

$$\sum_{j=0}^{\ell-1} S_{m,n,j}(z; f) = \frac{1}{2\pi i}\int\limits_{|\zeta|=r'} f\left(\psi(\zeta)\right)\times$$
$$\left(\frac{1}{2\pi i}\int\limits_{|t|=r}\frac{\psi'(\zeta)}{\psi(\zeta)-\psi(t)}\cdot\frac{\zeta}{t}\cdot\frac{\zeta^{\ell(m+1)}-1}{\zeta^{(\ell-1)(m+1)}(\zeta^{m+1}-1)}\sum_{k=0}^{n}\frac{F_k(z)}{\zeta^{k+1}}\,dt\right)d\zeta. \tag{2.9}$$

From (2.7) and (2.9), we thus have an integral representation for $\Delta^*_{m,n,\ell}(z; f)$ of (2.2) as the difference of two double integrals.

For the Faber polynomials F_{n+1}, it is known from [5, eq. (2.7)] that $F_0(z) = 1$ and

$$F_n\left(\psi(w)\right) = w^n + \sum_{\nu=1}^{\infty}\alpha_{n,\nu}w^{-\nu}, \qquad |w| > 1,\ n\in\mathbb{N}, \tag{2.10}$$

uniformly on closed subsets of $\{w\in\mathbb{C} : |w| > 1\}$, where, from [5, eq. (4.9)],

$$|\alpha_{n,k}| \le \sqrt{\frac{n}{k}}, \qquad n, k\in\mathbb{N}. \tag{2.11}$$

From (2.11), it readily follows that

$$\left|\sum_{\nu=1}^{\infty}\alpha_{n,\nu}w^{-\nu}\right| \le \frac{\sqrt{n}}{|w|-1}, \qquad |w| > 1,\ n\in\mathbb{N}. \tag{2.12}$$

Next, as previously, we set

$$w_n(z) := c^{n+1} F_{n+1}(z).$$

From (2.10) and (2.12), it is easy to deduce that, for any t and ζ with $1 < |t| < |\zeta|$,

$$\frac{w_m(\psi(\zeta)) - w_m(\psi(t))}{w_m(\psi(\zeta))} = \frac{\zeta^{m+1} - t^{m+1}}{\zeta^{m+1}} \left(1 + O(1)\frac{\sqrt{m}}{|\zeta|^m}\right), \qquad m \to \infty, \quad (2.13)$$

uniformly on closed subsets of $\{\zeta \in \mathbb{C} : |\zeta| > 1\} \times \{t \in \mathbb{C} : |t| > 1\}$. Choosing ρ such that $1 < \rho < r < r' < R$ and using (2.7), (2.9), (2.10) and (2.13), we obtain (with $z = \psi(w)$)

$$\begin{aligned} \Delta^*_{m,n,\ell}(z;f) &=: \sum_{j=1}^{4} \Delta^{*(j)}_{m,n,\ell}(z;f) \\ &= \frac{1}{2\pi i} \int\limits_{|\zeta|=r'} f(\psi(\zeta)) \left(\frac{1}{2\pi i} \int\limits_{|t|=r} \frac{\psi'(\zeta)}{\psi(\zeta)-\psi(t)} \sum_{j=1}^{4} K^{(j)}_{m,n,\ell}(w,\zeta,t)\, dt \right) d\zeta, \end{aligned}$$

where

$$\begin{aligned} K^{(1)}_{m,n,\ell}(w,\zeta,t) := & \frac{\zeta^{m+1}-t^{m+1}}{\zeta^{m+1}} \left(1 + O\left(\frac{1}{\rho^m}\right)\right) \sum_{k=1}^{n}\sum_{\nu=1}^{\infty} \alpha_{k,\nu} w^{-\nu} t^{-k-1} \\ & - \frac{\zeta}{t} \cdot \frac{\zeta^{\ell(m+1)}-1}{\zeta^{(\ell-1)(m+1)}(\zeta^{m+1}-1)} \sum_{k=1}^{n}\sum_{\nu=1}^{\infty} \alpha_{k,\nu} w^{-\nu} \zeta^{-k-1}, \end{aligned} \quad (2.14)$$

$$K^{(2)}_{m,n,\ell}(w,\zeta,t) := \frac{t^{n+1}-w^{n+1}}{(t-w)t^{n+1}} - \frac{\zeta}{t} \cdot \frac{\zeta^{n+1}-w^{n+1}}{(\zeta-w)\zeta^{n+1}} \cdot \frac{\zeta^{\ell(m+1)}-1}{\zeta^{(\ell-1)(m+1)}(\zeta^{m+1}-1)}, \quad (2.15)$$

$$K^{(3)}_{m,n,\ell}(w,\zeta,t) := O\left(\frac{1}{\rho^m}\right) \frac{\zeta^{m+1}-t^{m+1}}{\zeta^{m+1}} \cdot \frac{t^{n+1}-w^{n+1}}{(t-w)t^{n+1}}, \quad (2.16)$$

and

$$K^{(4)}_{m,n,\ell}(w,\zeta,t) := -\frac{t^{m+1}}{\zeta^{m+1}} \cdot \frac{t^{n+1}-w^{n+1}}{(t-w)t^{n+1}}. \quad (2.17)$$

(i) Because $1 < |t| < |\zeta|$, then letting n tend to infinity gives

$$\begin{aligned} K_1(w,\zeta,t) : &= \lim_{n\to\infty} K^{(1)}_{m,n,\ell}(w,\zeta,t) \\ &= \sum_{k=1}^{\infty}\sum_{\nu=1}^{\infty} \alpha_{k,\nu} w^{-\nu} t^{-k-1} - \frac{\zeta}{t} \sum_{k=1}^{\infty}\sum_{\nu=1}^{\infty} \alpha_{k,\nu} w^{-\nu} \zeta^{-k-1}, \end{aligned}$$

where the two double series on the right side are convergent, uniformly on closed subsets of $\{w \in \mathbb{C} : |w| > 1\} \times \{t \in \mathbb{C} : |t| > 1\}$. The residue theorem now implies that

$$\frac{1}{2\pi i} \int\limits_{|t|=r} \frac{\psi'(\zeta)}{\psi(\zeta) - \psi(t)} K_1(w, \zeta, t)\, dt = 0,$$

for all $\zeta \in \mathbb{C}$ with $|\zeta| > r > 1$. Thus, $\lim\limits_{n\to\infty} \Delta^{*(1)}_{m,n,\ell}(z; f) = 0$, locally uniformly on $\mathbb{C}$.

(ii) Again using the residue theorem, we see that for $|w| > r'$,

$$\frac{1}{2\pi i} \int\limits_{|t|=r} \frac{\psi'(\zeta)}{\psi(\zeta) - \psi(t)} K^{(2)}_{m,n,\ell}(w, \zeta, t)\, dt = K^{(2)}_{m,n,\ell}(w, \zeta, \zeta)$$

$$= \frac{\zeta^{n+1} - w^{n+1}}{(\zeta - w)\zeta^{n+1}} \cdot \frac{1 - \zeta^{(\ell-1)(m+1)}}{\zeta^{(\ell-1)(m+1)}(\zeta^{m+1} - 1)} = \begin{cases} 0, & \ell = 1, \\ O(1)\left(\frac{|w|}{(r')^{1+q}}\right)^n, & \ell \geq 2. \end{cases}$$

But, as r' is any number satisfying $1 < r' < R$, it follows that $\lim\limits_{n\to\infty} \Delta^{*(2)}_{m,n,\ell}(z; f) = 0$, uniformly on $\bar{G}_\mu$ for every $1 \leq \mu < R^{1+q}$ when $\ell \geq 2$.

(iii) Also, it is obvious from the expression for $K^{(3)}_{m,n,\ell}(w, \zeta, t)$ that for $|w| > r'$,

$$K^{(3)}_{m,n,\ell}(w, \zeta, t) = O(1)\left(\frac{|w|}{r\rho^q}\right)^n,$$

so that $\lim\limits_{n\to\infty} \Delta^{*(3)}_{m,n,\ell}(z; f) = 0$, uniformly on $\bar{G}_\mu$ for $1 \leq \mu < R^{1+q}$.

(iv) Finally, in order to estimate $\Delta^{*(4)}_{m,n,\ell}$, consider

$$F_{m,n}(w, \zeta) := \frac{1}{2\pi i} \int\limits_{|t|=r} \frac{\psi'(\zeta)}{\psi(\zeta) - \psi(t)} K^{(4)}_{m,n,\ell}(w, \zeta, t)\, dt.$$

Since, from [5, eq. (2.9)],

$$\frac{\psi'(\zeta)}{\psi(\zeta) - \psi(t)} = \frac{1}{\zeta - t} + \sum_{\nu=1}^{\infty} \sum_{k=1}^{\infty} \alpha_{\nu,k} t^{-k} \zeta^{-\nu-1} \tag{2.18}$$

holds uniformly on closed subsets of $\{t \in \mathbb{C} : |t| > 1\} \times \{\zeta \in \mathbb{C} : |\zeta| > 1\}$, where the coefficients $\alpha_{\nu,k}$ are defined by (2.10), it follows from (2.17) that

$$F_{m,n}(w, \zeta)$$
$$= -\frac{1}{2\pi i} \int\limits_{|t|=r} \frac{1}{\zeta^{m+2}} \left\{ \sum_{k=0}^{\infty} \sum_{j=0}^{n} \zeta^{-k} w^j t^{k+m-j} + \sum_{\nu=1}^{\infty} \sum_{k=1}^{\infty} \sum_{j=0}^{n} \alpha_{\nu,k} \zeta^{-\nu} w^j t^{m-k-j} \right\} dt.$$

Since $k + m - j = k + q(n + 1) - 1 - j \geq 0$, the integral of the double sum above vanishes and the triple sum gives a contribution to the integral only when $k = m + 1 - j$, so that

$$F_{m,n}(w, \zeta) = -\frac{1}{\zeta^{m+2}} \sum_{\nu=1}^{\infty} \sum_{j=0}^{n} \alpha_{\nu,m+1-j} w^j \zeta^{-\nu}. \tag{2.19}$$

Again using (2.11), we obtain

$$F_{m,n}(w, \zeta) = O(1) \frac{1}{(r')^m} \sum_{\nu=1}^{\infty} \sum_{j=0}^{n} \sqrt{\frac{\nu}{m+1-j}}\, |w|^j (r')^{-\nu}$$

$$= O(1) \frac{1}{(r')^m} \sum_{\nu=1}^{\infty} \frac{\sqrt{\nu}}{(r')^\nu} \sum_{j=0}^{n} |w|^j = O(1) \left(\frac{|w|}{(r')^q} \right)^n,$$

which implies that $\lim_{n \to \infty} \Delta^{*(4)}_{m,n,\ell}(z; f) = 0$, uniformly on $\bar{G}_\mu$ for every $1 \leq \mu < R^q$. On combining the above results of (i)-(iv), we have the desired result of (2.3) of Theorem 1. ■

Remarks. (1) We do not know if $\lambda = R^q$ is best possible in (2.3). However, we can improve our result if ∂E is r_0-analytic. In this case, we have from [5, eq. (4.2)] that $\alpha_{n,k} = O(\beta^{n+k})$, for every β such that $r_0 < \beta < 1$. Then (2.12) can be improved by the bound

$$\left| \sum_{\nu=1}^{\infty} \alpha_{n,\nu} w^{-\nu} \right| = O(\beta^n), \qquad |w| > \beta,\ n \in \mathbb{N}, \tag{2.20}$$

which leads, for $\beta < |t| < |\zeta|$, to

$$\frac{w_m(\psi(\zeta)) - w_m(\psi(t))}{w_m(\psi(\zeta))} = \frac{\zeta^{m+1} - t^{m+1}}{\zeta^{m+1}} \left(1 + O\left(\left(\frac{\beta}{|\zeta|} \right)^m \right) \right), \qquad m \to \infty. \tag{2.21}$$

An examination of the proof of Theorem 1 then shows that the estimates of $\Delta^{*(1)}_{m,n,\ell}(z; f)$ and $\Delta^{*(2)}_{m,n,\ell}(z; f)$ remain unchanged. But from (2.16) we now have, because of (2.21), that

$$K^{(3)}_{m,n,\ell}(w, \zeta, t) = O(1) \left(\frac{|w|\beta^q}{r(r')^q} \right)^n,$$

so that $\lim_{n \to \infty} \Delta^{*(3)}_{m,n,\ell}(z; f) = 0$, uniformly on $\bar{G}_\mu$ for every $1 \leq \mu < R(R/r_0)^q$.

Finally, from (2.19) and $|\zeta| = r'$, we have

$$F_{m,n}(w,\zeta) = O(1)\frac{1}{(r')^m}\sum_{\nu=1}^{\infty}\sum_{j=0}^{n}\beta^{\nu+m-j}|w|^j(r')^{-\nu}$$

$$= O(1)\left(\frac{\beta}{r'}\right)^m\sum_{\nu=1}^{\infty}\left(\frac{\beta}{r'}\right)^\nu\sum_{j=0}^{n}\left(\frac{|w|}{\beta}\right)^j = O(1)\left(\frac{|w|\beta^{q-1}}{(r')^q}\right)^n.$$

Thus, $\lim_{n\to\infty}\Delta^{*(4)}_{m,n,\ell}(z;f) = 0$, uniformly on $\bar{G}_\mu$ for every $1 \le \mu < r_0(R/r_0)^q$. Combining all the above, we have

$$\lim_{n\to\infty}\Delta^{*}_{m,n,\ell}(z;f) = 0, \qquad z \in G_\lambda,$$

where

$$\lambda := \begin{cases} r_0(R/r_0)^q, & \ell = 1, \\ \min\left\{r_0(R/r_0)^q; R^{q+1}\right\}, & \ell \ge 2. \end{cases} \tag{2.22}$$

(2) A further improvement may be achieved in the case of an ellipse E_δ (where $\delta > 1$ and where ∂E_δ is the image of the circle $\{w \in \mathbb{C} : |w| = \delta\}$ under the map $w \to \frac{1}{2}(w + \frac{1}{w})$, and is therefore r_0-analytic with $r_0 = 1/\delta$). Then, we have $F_n(\psi(w)) = w^n + \frac{1}{\delta^{2n}w^n}$, and an examination of the proof of Theorem 1 shows that in this case

$$\lambda = \begin{cases} r_0(R/r_0)^{2q-1}, & \ell = 1, \\ \min\{R(R/r_0)^{2q-1}; R^{q+1}\}, & \ell \ge 2. \end{cases} \tag{2.23}$$

(3) The previous remark also applies to the case of the segment $E = [-1, 1]$ (where $\delta = 1$) and gives

$$\lambda = \begin{cases} R^{2q-1}, & \ell = 1, \\ R, & q = 1, \\ R^{q+1}, & \ell \ge 2. \end{cases} \tag{2.24}$$

For $\ell = 1$, this is Theorem 2 of Rivlin [9].

(4) The main reason for the different results in Theorem A and Theorem 1 (in the case when ∂E is r_0-analytic) may be explained as follows. If E is the closed unit disk D_1, then the Fejér nodes coincide with the roots of unity while the Faber nodes are all zero, so that $L^*_n(z;f) \equiv S_n(z;f)$.

3 HERMITE INTERPOLATION

In [2], we stated without proof a result analogous to Theorem A, after replacing Lagrange interpolation with Hermite interpolation in Fejér nodes. For $s \in \mathbb{N}$ and $f \in A_R$, we denote by $H_{s(m+1)-1}(z;f)$ the Hermite interpolation polynomial to

$f, f', \ldots, f^{(s-1)}$ in the $(m+1)^{st}$ Fejér nodes on a compact set E. Then for $p, q \in \mathbb{N}$ with $sq \geq p$, and $m = q(n+1) - 1$, we considered, as in [2], the operator

$$\Delta_{m,n}^{p,s}(z; f) := S_{p(n+1)-1}\left(z; H_{s(m+1)-1}(\cdot, f)\right) - S_{p(n+1)-1}(z; f). \tag{3.1}$$

With the above notation and the notation that $\lfloor t \rfloor$ denotes the integral part of the real number t, we state the following corrected form of:

Theorem B [2] *Let ∂E be an r_0-analytic curve for some $r_0 \in [0, 1)$, let $f \in A_R$, let $m = q(n+1) - 1$ for a fixed $q \in \mathbb{N}$, let $s, p \in \mathbb{N}$ be such that $sq \geq p$, and let $\Delta_{m,n}^{p,s}(z; f)$ be given by (3.1). Then,*

$$\lim_{n \to \infty} \Delta_{m,n}^{p,s}(z; f) = 0, \qquad z \in G_\lambda, \tag{3.2}$$

the convergence being uniform and geometric on every subset $\bar{G}_\mu$ for $1 \leq \mu < \lambda$, where

$$\lambda := \begin{cases} \min\left\{R/r_0^{q/p}; R^{sq/p}/r_0^{(q/p)-1}; R^\sigma\right\}, & \text{if } q \geq p, \\ \min\left\{R/r_0^{q/p}; R^{sq/p}; R^\sigma\right\}, & \text{if } q < p, \end{cases} \tag{3.3}$$

where, if $p = tq + \tau$ with $t \in \mathbb{N}_0$ and $0 \leq \tau < q$, then $\sigma := (qs + \tau)/p$ if $0 < \tau < p$ and $\sigma := q(s+1)/p$ if $\tau = 0$.

Here, we shall consider the Hermite interpolant to $f, f', \ldots, f^{(s-1)}$ in the $(m+1)^{st}$ Faber nodes. We denote it by $H^*_{s(m+1)-1}(z; f)$. We set

$$\Delta_{m,n}^{*p,s}(z; f) := S_{p(n+1)-1}\left(z; H^*_{s(m+1)-1}(\cdot; f)\right) - S_{p(n+1)-1}(z; f), \tag{3.4}$$

where $S_{p(n+1)-1}(z; f)$ denotes the $(p(n+1)-1)^{st}$ section of the Faber expansion of f.

We next establish:

Theorem 2 *Let $f \in A_R$, let $m = q(n+1) - 1$ for a fixed $q \in \mathbb{N}$, let $s, p \in \mathbb{N}$ be such that $sq \geq p$, and let $\Delta_{m,n}^{*p,s}(z; f)$ be given by (3.4) for the Faber nodes with respect to a compact set E. Then,*

$$\lim_{n \to \infty} \Delta_{m,n}^{*p,s}(z; f) = 0, \quad z \in G_\lambda, \tag{3.5}$$

the convergence being uniform and geometric on every subset $\bar{G}_\mu$ for $1 \leq \mu < \lambda$, where

$$\lambda := \min\{R^{1+q/p}; R^{sq/p}\}. \tag{3.6}$$

Proof: In analogy to (2.7) and (2.6), it is easy to see that

$$S_{p(n+1)-1}(z; H^*_{s(m+1)-1}(\cdot, f)) = \frac{1}{2\pi i} \int_{|\zeta|=r'} f(\psi(\zeta)) \times \left(\frac{1}{2\pi i} \int_{|t|=r} \frac{\psi'(\zeta)}{\psi(\zeta) - \psi(t)} \cdot \frac{w_m^s(\psi(\zeta)) - w_m^s(\psi(t))}{w_m^s(\psi(\zeta))} \sum_{k=0}^{p(n+1)-1} \frac{F_k(z)}{t^{k+1}} \, dt \right) d\zeta, \tag{3.7}$$

and

$$S_{p(n+1)-1}(z;f)=\frac{1}{2\pi i}\int\limits_{|\zeta|=r'} f(\psi(\zeta))\left(\frac{1}{2\pi i}\int\limits_{|t|=r}\frac{\psi'(\zeta)}{\psi(\zeta)-\psi(t)}\cdot\frac{\zeta}{t}\sum_{k=0}^{p(n+1)-1}\frac{F_k(z)}{\zeta^{k+1}}\,dt\right)d\zeta, \tag{3.8}$$

where $1<r<r'<R$.

Writing $w_m(z)=c^{m+1}F_{m+1}(z)$, we use (2.10) and (2.12) to show that

$$\frac{w_m^s(\psi(\zeta))-w_m^s(\psi(t))}{w_m^s(\psi(\zeta))}=\frac{\zeta^{s(m+1)}-t^{s(m+1)}}{\zeta^{s(m+1)}}\left(1+O(1)\frac{\sqrt{m}}{|\zeta|^m}\right), \tag{3.9}$$

uniformly on closed subsets of $\{\zeta\in\mathbb{C}:|\zeta|>1\}\times\{t\in\mathbb{C}:|t|>1\}$.

Choosing ρ such that $1<\rho<r<r'<R$ and putting (3.7), (3.8), (3.9) and (2.10) together, we obtain (with $z=\psi(w)$)

$$\Delta_{m,n}^{*p,s}(z;f)=\sum_{j=1}^{4}\Delta_{m,n,j}^{*p,s}(z;f), \tag{3.10}$$

where

$$\Delta_{m,n,j}^{*p,s}(z;f):=\frac{1}{2\pi i}\int\limits_{|\zeta|=r'} f(\psi(\zeta))\left(\frac{1}{2\pi i}\int\limits_{|t|=r}\frac{\psi'(\zeta)}{\psi(\zeta)-\psi(t)}K_{m,n,j}(w,\zeta,t)\,dt\right)d\zeta,$$

$$j=1,2,3,4.$$

The kernels $K_{m,n,j}(w,\zeta,t)\quad(j=1,2,3,4)$ are given explicitly as follows:

$$K_{m,n,1}(w,\zeta,t):=\frac{\zeta^{s(m+1)}-t^{s(m+1)}}{\zeta^{s(m+1)}}\left(1+O\left(\frac{1}{\rho^m}\right)\right)\sum_{k=1}^{p(n+1)-1}\sum_{\nu=1}^{\infty}\frac{\alpha_{k,\nu}}{w^\nu t^{k+1}}$$
$$-\frac{\zeta}{t}\sum_{k=1}^{p(n+1)-1}\sum_{\nu=1}^{\infty}\frac{\alpha_{k,\nu}}{w^\nu\zeta^{k+1}}, \tag{3.11}$$

$$K_{m,n,2}(w,\zeta,t):=\frac{t^{p(n+1)}-w^{p(n+1)}}{(t-w)t^{p(n+1)}}-\frac{\zeta}{t}\cdot\frac{\zeta^{p(n+1)}-w^{p(n+1)}}{(\zeta-w)\zeta^{p(n+1)}}, \tag{3.12}$$

$$K_{m,n,3}(w,\zeta,t):=O\left(\frac{1}{\rho^m}\right)\frac{\zeta^{s(m+1)}-t^{s(m+1)}}{\zeta^{s(m+1)}}\cdot\frac{t^{p(n+1)}-w^{p(n+1)}}{(t-w)t^{p(n+1)}}, \tag{3.13}$$

and

$$K_{m,n,4}(w,\zeta,t) := -\frac{t^{s(m+1)}}{\zeta^{s(m+1)}} \cdot \frac{t^{p(n+1)} - w^{p(n+1)}}{(t-w)t^{p(n+1)}}. \tag{3.14}$$

As in the proof of Theorem 1, it can be shown that $\lim_{n\to\infty} \Delta^{*p,s}_{m,n,1}(z;f) = 0$, locally uniformly on $\mathbb{C}$. Using the residue theorem, we can see, exactly as in the proof of Theorem 1, that $\Delta^{*p,s}_{m,n,2}(z;f) \equiv 0$. Moreover it is obvious that for $|w| > r'$,

$$K_{m,n,3}(w,\zeta,t) = O(1)\left(\frac{|w|^p}{\rho^q r^p}\right)^n, \tag{3.15}$$

so that $\lim_{n\to\infty} \Delta^{*p,s}_{m,n,3}(z;f) = 0$, uniformly on $\bar{G}_\mu$ for $1 \le \mu < R^{1+(q/p)}$.

Finally, in order to estimate $\Delta^{*p,s}_{m,n,4}(z;f)$, we consider

$$F_{m,n}(w,\zeta) := \frac{1}{2\pi i}\int_{|t|=r} \frac{\psi'(\zeta)}{\psi(\zeta)-\psi(t)} K_{m,n,4}(w,\zeta,t)\,dt, \tag{3.16}$$

and recall from (2.18) that

$$\frac{\psi'(\zeta)}{\psi(\zeta)-\psi(t)} = \frac{1}{\zeta - t} + \sum_{\nu=1}^{\infty}\sum_{k=1}^{\infty} \alpha_{\nu,k} t^{-k}\zeta^{-\nu-1}. \tag{3.17}$$

The integrand in (3.16), when expanded in powers of t and on using (3.17), becomes

$$-\frac{1}{\zeta^{s(m+1)+1}}\left[\sum_{k=0}^{\infty}\sum_{j=0}^{p(n+1)-1} \zeta^{-k} w^j t^{k+s(m+1)-j-1} + \sum_{\nu=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j=0}^{p(n+1)-1} \alpha_{\nu,k}\zeta^{-\nu} w^j t^{-k+s(m+1)-j-1}\right].$$

Since $k + s(m+1) - j - 1 = k + sq(n+1) - j - 1 \ge 0$, the integral of the double sum vanishes and only the triple sum gives a contribution to the integral $F_{m,n}(w,\zeta)$ when $k = s(m+1) - j$. Thus, we have

$$F_{m,n}(w,\zeta) = -\frac{1}{\zeta^{s(m+1)+1}} \sum_{\nu=1}^{\infty}\sum_{j=0}^{p(n+1)-1} \alpha_{\nu,s(m+1)-j} w^j \zeta^{-\nu}. \tag{3.18}$$

Again using the inequality of (2.11), we obtain

$$F_{m,n}(w,\zeta) = O(1)\frac{1}{(r')^{sm}} \sum_{\nu=1}^{\infty}\sum_{j=0}^{p(n+1)-1} \sqrt{\frac{\nu}{s(m+1)-j}}\ |w|^j (r')^{-\nu}$$

$$= O(1)\frac{1}{(r')^{sm}} \sum_{\nu=1}^{\infty}\frac{\sqrt{\nu}}{(r')^{\nu}} \sum_{j=0}^{p(n+1)-1} |w|^j = O(1)\left(\frac{|w|^p}{(r')^{sq}}\right)^n,$$

which implies that $\lim_{n\to\infty} \Delta^{*p,s}_{m,n,4}(z;f) = 0$, uniformly on $\bar{G}_\mu$ for every $1 \le \mu < R^{sq/p}$. Combining the above results then gives the desired result of (3.6) of Theorem 2. ∎

Remarks. **(1)** We do not know whether λ in (3.6) is best possible. However, if ∂E is r_0-analytic, we can improve our result. In this case we have for every β such that $r_0 < \beta < 1$,

$$\frac{w^s_m(\psi(\zeta)) - w^s_m(\psi(t))}{w^s_m(\psi(\zeta))} = \frac{\zeta^{s(m+1)} - t^{s(m+1)}}{\zeta^{s(m+1)}}\left(1 + O\left(\left(\frac{\beta}{|\zeta|}\right)^m\right)\right).$$

Then, an examination of the above proof shows that in this case the estimates of $\Delta^{*p,s}_{m,n,1}(z;f)$ and $\Delta^{*p,s}_{m,n,2}(z;f)$ remain unchanged. But

$$K_{m,n,3}(w,\zeta,t) = O(1)\left(\frac{\beta^q |w|^p}{r^{q+p}}\right),$$

so that $\lim_{n\to\infty} \Delta^{*p,s}_{m,n,3}(z;f) = 0$, uniformly on $\bar{G}_\mu$ for every $1 \le \mu < R(R/r_0)^{q/p}$.

Finally, from (3.18) (since now $\alpha_{n,k} = O(\beta^{n+k})$), we obtain

$$F_{m,n}(w,\zeta) = O(1)\frac{1}{(r')^{sm}}\sum_{\nu=1}^{\infty}\sum_{j=0}^{p(n+1)-1} \beta^{\nu+s(m+1)-j}|w|^j (r')^{-\nu}$$

$$= O(1)\left(\frac{\beta}{r'}\right)^{sm}\sum_{\nu=1}^{\infty}\left(\frac{\beta}{r'}\right)^{\nu}\sum_{j=0}^{p(n+1)-1}\left(\frac{|w|}{\beta}\right)^j = O(1)\left(\frac{|w|^p\beta^{sq-p}}{(r')^{sq}}\right)^n,$$

and this implies that $\lim_{n\to\infty} \Delta^{*p,s}_{m,n,4}(z;f) = 0$, uniformly on $\bar{G}_\mu$ for every $1 \le \mu < r_0(R/r_0)^{sq/p}$. Therefore, when the boundary curve ∂E is an r_0-analytic curve,

$$\lim_{n\to\infty} \Delta^{*p,s}_{m,n}(z;f) = 0, \qquad z \in G_\lambda,$$

where

$$\lambda := \min\left\{r_0\left(\frac{R}{r_0}\right)^{sq/p}; R\left(\frac{R}{r_0}\right)^{q/p}\right\}.$$

(2) A further improvement may be achieved in the case of an ellipse E_δ (where $\delta > 1$ and where ∂E_δ is r_0-analytic with $r_0 = 1/\delta$). An examination of the proof of Theorem 2 shows that in this case,

$$\lambda := \min\left\{r_0\left(\frac{R}{r_0}\right)^{(2sq/p)-1}; R\left(\frac{R}{r_0}\right)^{2q/p}\right\}.$$

This also applies to the case of a segment $E = [-1,1]$ (where $\delta = 1$) and gives

$$\lambda := \min\left\{R^{(2sq/p)-1}; R^{1+(2q/p)}\right\}.$$

4 MIXED HERMITE AND LAGRANGE INTERPOLATION

We next consider the case where Faber sections of Hermite interpolants are compared with Faber sections of Lagrange interpolants. This is analogous to Theorem C of [2] which was stated without proof. More precisely, for $s, p \in \mathbb{N}$ with $s \geq \max\{p; 2\}$, we set

$$D_{p,s,n}(z; f) := S_{p(n+1)-1}\left(z; \{H_{s(n+1)-1}(\cdot; f) - L_{s(n+1)-1}(\cdot; f)\}\right), \tag{4.1}$$

where $H_{s(n+1)-1}(z; f)$ and $L_{s(n+1)-1}(z; f)$ are Hermite and Lagrange interpolants in Fejér nodes with respect to a compact set E. The following theorem was announced in [2].

Theorem C [2] *Let ∂E be an r_0-analytic curve for some $r_0 \in [0, 1)$, let $f \in A_R$, let $s, p \in \mathbb{N}$ be such that $s \geq \max\{p, 2\}$, and let $D_{p,s,n}(z; f)$ be given by (4.1). Then,*

$$\lim_{n\to\infty} D_{p,s,n}(z; f) = 0, \qquad z \in G_\lambda, \tag{4.2}$$

the convergence being uniform and geometric on every subset $\bar{G}_\mu$ for $1 \leq \mu < \lambda$, where

$$\lambda := \begin{cases} \left.\begin{matrix} R^{s+2} & \text{for } p = 1 \text{ and } s \text{ even} \\ R^{(s+1)/p}; & \text{otherwise} \end{matrix}\right\} & \text{if } r_0 = 0, \\ \min\{R/r_0^{1/p}; R^{s/p}\}, & \text{if } r_0 > 0. \end{cases} \tag{4.3}$$

We shall next prove an analogue of Theorem C, using Hermite interpolation and Lagrange interpolation in Faber nodes. Let $H^*_{s(n+1)-1}(z; f)$ be the Hermite interpolant to f in the zeros of $(F_{n+1}(z))^s$, and let $L^*_{s(n+1)-1}(z; f)$ denote the Lagrange interpolant to f in the zeros of $F_{s(n+1)}(z)$. As in earlier sections, $S_{p(n+1)-1}(z; f)$ denotes the Faber section of degree $p(n+1) - 1$ of the Faber expansion of f. Set

$$D^*_{p,s,n}(z; f) := S_{p(n+1)-1}\left(z; \{H^*_{s(n+1)-1}(\cdot; f) - L^*_{s(n+1)-1}(\cdot; f)\}\right). \tag{4.4}$$

We next establish:

Theorem 3 *Let $f \in A_R$, let $s, p \in \mathbb{N}$ be such that $s \geq \max\{p, 2\}$, and let $D^*_{p,s,n}(z; f)$ be given by (4.4). Then,*

$$\lim_{n\to\infty} D^*_{p,s,n}(z; f) = 0, \qquad z \in G_\lambda, \tag{4.5}$$

the convergence being uniform and geometric on every subset $\bar{G}_\mu$ for $1 \leq \mu < \lambda$, where

$$\lambda := R^{1+(1/p)}. \tag{4.6}$$

Proof: It is easy to verify that the following integral representation holds:

$$D^*_{p,s,n}(z;f) = \frac{1}{2\pi i}\int_{|\zeta|=r'} f(\psi(\zeta))\left(\frac{1}{2\pi i}\int_{|t|=r} \frac{\psi'(\zeta)}{\psi(\zeta)-\psi(t)} K_{s,n}(\zeta,t)\sum_{k=0}^{p(n+1)-1}\frac{F_k(z)}{t^{k+1}}\,dt\right)d\zeta, \tag{4.7}$$

where $K_{s,n}(\zeta,t)$ is the difference between the kernels of Hermite and Lagrange interpolants and where $1 < r < r' < R$. Then,

$$\begin{aligned} K_{s,n}(\zeta,t) :&= \frac{w_n^s(\psi(\zeta)) - w_n^s(\psi(t))}{w_n^s(\psi(\zeta))} - \frac{w_{s(n+1)-1}(\psi(\zeta)) - w_{s(n+1)-1}(\psi(t))}{w_{s(n+1)-1}(\psi(\zeta))} \\ &= \frac{w_{s(n+1)-1}(\psi(t))}{w_{s(n+1)-1}(\psi(\zeta))} - \frac{w_n^s(\psi(t))}{w_n^s(\psi(\zeta))}. \end{aligned} \tag{4.8}$$

Using (2.10) and (2.12), we then have

$$K_{s,n}(\zeta,t) = \frac{t^{s(n+1)}}{\zeta^{s(n+1)}} O\left(\frac{1}{\rho^{n+1}}\right),$$

uniformly on closed subsets of $\{\zeta \in \mathbb{C} : |\zeta| > 1\} \times \{t \in \mathbb{C} : |t| > 1\}$, where ρ is chosen such that $1 < \rho < r < r' < R$. Moreover (with $z = \psi(w)$),

$$\begin{aligned} K_{s,n}(\zeta,t)\sum_{k=0}^{p(n+1)-1}\frac{F_k(z)}{t^{k+1}} &= O\left(\frac{1}{\rho^{n+1}}\right)\frac{t^{s(n+1)}}{\zeta^{s(n+1)}}\cdot\frac{t^{p(n+1)} - w^{p(n+1)}}{(t-w)t^{p(n+1)}} \\ &\quad + O\left(\frac{1}{\rho^{n+1}}\right)\frac{t^{s(n+1)}}{\zeta^{s(n+1)}}\sum_{k=1}^{p(n+1)-1}\sum_{\nu=1}^{\infty}\alpha_{k,\nu}w^{-\nu}t^{-k-1}. \end{aligned} \tag{4.9}$$

Since $|t| < |\zeta|$ and because of (2.11), the second term on the right side of (4.9) is bounded above by

$$O\left(\frac{1}{\rho^{n+1}}\right)\left(\frac{|t|}{|\zeta|}\right)^{s(n+1)} \cdot \sum_{k=1}^{p(n+1)-1}\sqrt{k}|t|^{-k-1} \cdot \sum_{\nu=1}^{\infty}\frac{1}{\sqrt{\nu}}|w|^{-\nu},$$

and this tends to zero as $n \to \infty$. The first term on the right of (4.9) can be estimated by

$$O(1)\left(\frac{|w|^p}{\rho r^p}\right)^n,$$

which gives the desired result of (4.6) of Theorem 3. ∎

Remarks. **(1)** We do not know whether λ of (4.6) is best possible. However, we are able to improve our result if ∂E is r_0-analytic. In this case, we have

$$\lim_{n\to\infty} D^*_{p,s,n}(z;f) = 0, \qquad z \in G_\lambda,$$

with $\lambda := R(R/r_0)^{1/p}$.

(2) A further improvement may be achieved in the case of an ellipse E_δ (where $\delta > 1$ and where ∂E_δ is r_0-analytic with $r_0 = 1/\delta$). An examination of the above proof shows that in this case λ is given by

$$\lambda := R(R/r_0)^{2/p}.$$

This also applies to the case of the segment $E = [-1,1]$ (where $\delta = 1$) and gives $\lambda := R^{1+2/p}$.

5 PROOF OF THEOREM B

Since the proof of Theorem B was not given in [2], we outline it briefly here. Observe that the formula for $\Delta^{p,s}_{m,n}(z;f)$ remains the same as the difference of (3.7) and (3.8). We have to keep in mind that $w_n(\psi(t))$, based on Fejér nodes, satisfies (from Lemma 3.1 in [4] where ∂E is a r_0-analytic curve, with $r_0 \in [0,1)$)

$$w_m(\psi(w)) = c^{m+1}(w^{m+1}-1)(1+O(\beta^m)), \tag{5.1}$$

for any β such that $r_0 < \beta < 1$, uniformly on closed subsets of $\{w \in \mathbb{C} : |w| > r_0\}$, which gives

$$\frac{w_m(\psi(\zeta)) - w_m(\psi(t))}{w_m(\psi(\zeta))} = \frac{\zeta^{m+1}-t^{m+1}}{\zeta^{m+1}-1}(1+O(\beta^m)), \tag{5.2}$$

and

$$\frac{w^s_m(\psi(\zeta)) - w^s_m(\psi(t))}{w^s_m(\psi(\zeta))} = \frac{(\zeta^{m+1}-1)^s - (t^{m+1}-1)^s}{(\zeta^{m+1}-1)^s}(1+O(\beta^m)), \tag{5.3}$$

uniformly on closed subsets of $\{\zeta \in \mathbb{C} : |\zeta| > 1\} \times \{t \in \mathbb{C} : |t| > 1\}$. Here again, we write

$$\Delta^{p,s}_{m,n}(z;f) = \sum_{j=1}^{4} \Delta_{m,n,j}(z;f),$$

where for $j = 1,2,3,4$ we have

$$\Delta_{m,n,j}(z;f) = \frac{1}{2\pi i}\int\limits_{|\zeta|=r'} f(\psi(\zeta))\left(\frac{1}{2\pi i}\int\limits_{|t|=r} \frac{\psi'(\zeta)}{\psi(\zeta)-\psi(t)} K_{m,n,j}(w,\zeta,t)\,dt\right)d\zeta. \tag{5.4}$$

The kernels $K_{m,n,j}(w,\zeta,t)$ are defined as follows:

$$K_{m,n,1}(w,\zeta,t) := \frac{(\zeta^{m+1}-1)^s-(t^{m+1}-1)^s}{(\zeta^{m+1}-1)^s}\,(1+O(\beta^m))\sum_{k=1}^{p(n+1)-1}\sum_{\nu=1}^{\infty}\frac{\alpha_{k,\nu}}{w^\nu t^{k+1}}$$
$$-\frac{\zeta}{t}\sum_{k=1}^{p(n+1)-1}\sum_{\nu=1}^{\infty}\frac{\alpha_{k,\nu}}{w^\nu \zeta^{\nu+1}}, \tag{5.5}$$

$$K_{m,n,2}(w,\zeta,t) := \frac{t^{p(n+1)}-w^{p(n+1)}}{(t-w)t^{p(n+1)}}-\frac{\zeta}{t}\cdot\frac{\zeta^{p(n+1)}-w^{p(n+1)}}{(\zeta-w)\zeta^{p(n+1)}}, \tag{5.6}$$

$$K_{m,n,3}(w,\zeta,t) := O(\beta^m)\frac{(\zeta^{m+1}-1)^s-(t^{m+1}-1)^s}{(\zeta^{m+1}-1)^s}\cdot\frac{t^{p(n+1)}-w^{p(n+1)}}{(t-w)t^{p(n+1)}}, \tag{5.7}$$

and

$$K_{m,n,4}(w,\zeta,t) := -\frac{(t^{m+1}-1)^s}{(\zeta^{m+1}-1)^s}\cdot\frac{t^{p(n+1)}-w^{p(n+1)}}{(t-w)t^{p(n+1)}}. \tag{5.8}$$

The estimate of $\Delta_{m,n,1}(z;f)$ is obtained as in the proof of Theorem 2, and we obtain $\lim_{n\to\infty}\Delta_{m,n,1}(z;f)=0$, locally uniformly on $\mathbb{C}$. Further, using the residue theorem, we obtain

$$\frac{1}{2\pi i}\int\limits_{|t|=r}\frac{\psi'(\zeta)}{\psi(\zeta)-\psi(t)}K_{m,n,2}(w,\zeta,t)dt = K_{m,n,2}(w,\zeta,\zeta)=0$$

for all w,ζ, which yields $\lim_{n\to\infty}\Delta_{m,n,2}(z;f)=0$ for all $z\in\mathbb{C}$. For the estimate of $\Delta_{m,n,3}(z;f)$, we examine the kernel $K_{m,n,3}(w,\zeta,t)$ and we see that for $|w|>r$,

$$K_{m,n,3}(w,\zeta,t)=O(1)\left(\frac{|w|^p\beta^q}{r^p}\right)^n,$$

and thus $\lim_{n\to\infty}\Delta_{m,n,3}(z;f)=0$, uniformly on $\bar{G}_\mu$ for $1\le\mu<R/r_0^{q/p}$. To find the estimate of $\Delta_{m,n,4}(z;f)$, we use (2.18) to obtain

$$\frac{1}{2\pi i}\int\limits_{|t|=r}\frac{\psi'(\zeta)}{\psi(\zeta)-\psi(t)}K_{m,n,4}(w,\zeta,t)\,dt = I_1(w,\zeta)+I_2(w,\zeta), \tag{5.9}$$

where

$$I_1(w,\zeta) := \frac{1}{2\pi i}\int\limits_{|t|=r}\frac{1}{\zeta-t}K_{m,n,4}(w,\zeta,t)dt,$$

and

$$I_2(w,\zeta) := \frac{1}{2\pi i} \int\limits_{|t|=r} A(\zeta,t) K_{m,n,4}(w,\zeta,t)dt,$$

with $A(\zeta,t) := \sum_{\nu=1}^{\infty}\sum_{k=1}^{\infty} \alpha_{\nu,k} t^{-k}\zeta^{-\nu-1}$. Expanding the integrand in $I_1(w,\zeta)$ in powers of t, we obtain

$$\frac{1}{\zeta-t}K_{m,n,4}(w,\zeta,t) = \frac{-1}{(\zeta^{m+1}-1)^s}\sum_{k=0}^{\infty}\sum_{j=0}^{s}\sum_{\nu=0}^{p(n+1)-1}\binom{s}{j}\frac{(-1)^{s-j}w^\nu}{\zeta^{k+1}}t^{k+j(m+1)-\nu-1},$$

and its integral, with respect to t, will vanish except when $k = \nu - j(m+1) \geq 0$.

On setting $k = \nu - j(m+1) \geq 0$ in the above display, it follows, as $s \geq p/q$ and $m+1 = q(n+1)$ from the hypotheses of Theorem B, that the above summation index j satisfies $0 \leq j \leq \min\left\{\frac{p(n+1)-1}{q(n+1)}\ ;\ s\right\} = \frac{p}{q} - \frac{1}{q(n+1)}$. Thus,

$$I_1(w,\zeta) = \frac{-1}{\zeta(\zeta^{m+1}-1)^s}\sum_{j=0}^{\lfloor \frac{p}{q}-\frac{1}{q(n+1)}\rfloor}\binom{s}{j}(-1)^{s-j}\zeta^{jq(n+1)}\sum_{\nu=jq(n+1)}^{p(n+1)-1}\left(\frac{w}{\zeta}\right)^\nu.$$

On writing $p = qt + \tau$, where t is a nonnegative integer and where $0 \leq \tau < q$, it follows that

$$I_1(w,\zeta) = \begin{cases} O(1)\left(\dfrac{|w|^p}{(r')^{sq+\tau}}\right)^n & \text{if } 0 < \tau < q, \\[2ex] O(1)\left(\dfrac{|w|^p}{(r')^{q(s+1)}}\right)^n & \text{if } 0 = \tau, \end{cases} \tag{5.10}$$

which implies that $\lim_{n\to\infty} \Delta^{(1)}_{m,n,4}(z;f) = 0$ uniformly on $\bar{G}_\mu$ for every $1 \leq \mu < R^\lambda$, where

$$\lambda := \begin{cases} \dfrac{qs+\tau}{p} & \text{if } 0 < \tau < q, \\[2ex] \dfrac{q(s+1)}{p} & \text{if } 0 = \tau. \end{cases}$$

In the special case when $p < q$, then $p = tq + \tau$ implies $t = 0$ and $\tau = p > 0$, so that the first display in (5.10) applies with $\tau = p$.

In order to estimate $I_2(w,\zeta)$, we again expand the integrand in powers of t and obtain

$$A(\zeta,t)K_{m,n,4}(w,\zeta,t)$$
$$= \frac{-1}{(\zeta^{m+1}-1)^s}\sum_{\mu=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j=0}^{s}\sum_{\nu=0}^{p(n+1)-1}\binom{s}{j}(-1)^{s-j}\alpha_{\mu,k}\zeta^{-\mu-1}w^\nu t^{-k+j(m+1)-\nu-1}.$$

Then, all the integrals vanish except when $k = j(m+1) - \nu \geq 1$. Then, as $s \geq p/q$,

$$I_2(w,\zeta) = \frac{-1}{(\zeta^{m+1}-1)^s} \sum_{\mu=1}^{\infty} \sum_{\nu=0}^{p(n+1)-1} \sum_{\substack{j=1 \\ jq(n+1)\geq \nu+1}}^{s} \binom{s}{j} \frac{(-1)^{s-j} \alpha_{\mu,j(m+1)-\nu} w^{\nu}}{\zeta^{\mu+1}},$$

and using the known fact [5, eq. (4.2)] that $\alpha_{\mu,k} = O(1)\beta^{\mu+k}$, we obtain

$$I_2(w,\zeta) = O(1)\frac{1}{(r')^{sqn}} \sum_{\nu=0}^{p(n+1)-1} \sum_{\substack{j=1 \\ jq(n+1)\geq \nu+1}}^{s} \beta^{jq(n+1)} \left(\frac{|w|}{\beta}\right)^{\nu}$$

$$= O(1)\frac{1}{(r')^{sqn}} \sum_{\ell=0}^{p-1} \sum_{\nu=0}^{n} \sum_{\substack{j=1 \\ jq(n+1)\geq \nu+1+\ell(n+1)}}^{s} \beta^{jq(n+1)} \left(\frac{|w|}{\beta}\right)^{\nu+\ell(n+1)}$$

$$= O(1)\frac{1}{(r')^{sqn}}\beta^n \sum_{\ell=0}^{p-1} |w|^{\ell(n+1)} \sum_{\nu=0}^{n} \left(\frac{|w|}{\beta}\right)^{\nu} = O(1)\left(\frac{|w|^p}{(r')^{sq}}\right)^n, \tag{5.11}$$

since $jq - \ell \geq \frac{\nu+1}{n+1}$ which implies $jq - \ell \geq 1$. Therefore, we obtain

$$\Delta^{(2)}_{m,n,4}(z;f) := \frac{1}{2\pi i} \int_{|\zeta|=r'} f(\psi(\zeta))\, I_2(w,\zeta) d\zeta \to 0 \qquad \text{as } n \to \infty,$$

uniformly on $\bar{G}_\mu$ for every $1 \leq \mu < R^{sq/p}$.

In the special case $q \geq p$, we have $jq(n+1) \geq jp(n+1) \geq \nu + 1 + \ell(n+1)$ for all $j = 1, \ldots, p$, $\nu = 0, \ldots, n$, and $\ell = 0, \ldots, p-1$ so that

$$I_2(w,\zeta) = O(1)\left(\frac{|w|^p \beta^{q-p}}{(r')^{sq}}\right)^n,$$

which shows that $\lim_{n\to\infty} \Delta^{(2)}_{m,n,4}(z;f) = 0$, uniformly on $\bar{G}_\mu$ for every $1 \leq \mu < R^{sq/p}/r_0^{-1+q/p}$. Combining the above results then gives the desired result (3.3) of Theorem B. ■

Remarks. **(1)** For arbitrary p, q, s, as $r_0 \to 1$, we obtain $\lambda \to R$. In the case $s = p$ and $r_0 > 0$, we obtain

$$\lambda = \begin{cases} \min\{R^{1+q}; R/r_0^{q/p}; R^q/r_0^{-1+q/p}\}, & \text{if } q \geq p, \\ \min\{R^q; R/r_0^{q/p}\}, & \text{if } q < p. \end{cases} \tag{5.12}$$

If $r_0 = 0$, i.e., $E = \bar{D}_1$, we have $\lambda = R^{1+sq/p}$ for $q \geq p$ and $\lambda = R^{(1+sq)/p}$ for $q < p$. If in addition $s = p$ and $q = 1$, Theorem B gives a special case of Theorem 3 of [3].

(2) We do not know whether λ of (3.3) is best possible. However, we are able to improve our result if E is the ellipse E_δ. An examination of the proof then yields that Theorem B holds with

$$\lambda = \begin{cases} \min\{R^{1+sq/p}; R^{1+q/p}/r_0^{2q/p}; R^{-1+(s+1)q/p}/r_0^{2(-1+q/p)}\}, & q \geq p, \\ \min\{R^{1+q/p}/r_0^{2q/p}; R^{sq/p}\}, & q < p. \end{cases} \tag{5.13}$$

(3) The previous remark also applies when $\delta = 1$, i.e., $E = [-1, 1]$, and we obtain

$$\lambda = \begin{cases} \min\{R^{1+q/p}; R^{(s+1)q/p-1}\}, & q \geq p, \\ \min\{R^{1+q/p}; R^{sq/p}\}, & q < p. \end{cases} \tag{5.14}$$

6 PROOF OF THEOREM C

As in the proof of Theorem 3 (Sec. 4), the form of the integral representation of the operator $D_{p,s,n}(z; f)$ is given by (4.7) where the kernel $K_{s,n}(\zeta, t)$ is given by (4.8). Since $w_n(\psi(t))$ is based on Fejér nodes, we have

$$\begin{aligned} K_{s,n}(\zeta, t) &= \frac{t^{s(n+1)} - 1}{\zeta^{s(n+1)} - 1}(1 + O(\beta^{sn})) - \frac{(t^{n+1} - 1)^s}{(\zeta^{n+1} - 1)^s}(1 + O(\beta^n)) \\ &= K_{s,n}^{(1)}(\zeta, t) + K_{s,n}^{(2)}(\zeta, t), \end{aligned} \tag{6.1}$$

where

$$K_{s,n}^{(1)}(\zeta, t) := \frac{t^{s(n+1)} - 1}{\zeta^{s(n+1)} - 1} - \frac{(t^{n+1} - 1)^s}{(\zeta^{n+1} - 1)^s} \tag{6.2}$$

and

$$K_{s,n}^{(2)}(\zeta, t) := \frac{t^{s(n+1)} - 1}{\zeta^{s(n+1)} - 1}O(\beta^{sn}) - \frac{(t^{n+1} - 1)^s}{(\zeta^{n+1} - 1)^s}O(\beta^n). \tag{6.3}$$

Since ∂E is an r_0-analytic curve, then $1 > \beta > r_0$ and with $z = \psi(w)$ for $|w| > 1$, we have

$$\begin{aligned} \sum_{k=0}^{p(n+1)-1} \frac{F_k(z)}{t^{k+1}} &= \frac{t^{p(n+1)} - w^{p(n+1)}}{(t - w)t^{p(n+1)}} + \sum_{k=1}^{p(n+1)-1} \sum_{\nu=1}^{\infty} \alpha_{k,\nu} w^{-\nu} t^{-k-1} \\ &=: A_{p,n,1}(w, t) + A_{p,n,2}(w, t). \end{aligned}$$

Since

$$\lim_{n\to\infty} A_{p,n,2}(w, t) = \sum_{k=1}^{\infty}\sum_{\nu=1}^{\infty} \alpha_{k,\nu} w^{-\nu} t^{-k-1},$$

uniformly on closed subsets of $\{w \in \mathbb{C} : |w| > 1\} \times \{t \in \mathbb{C} : |t| > 1\}$, we obtain

$$\lim_{n\to\infty} K_{s,n}^{(j)}(\zeta, t) A_{p,n,2}(w, t) = 0 \qquad (j = 1, 2), \tag{6.4}$$

uniformly for $|\zeta| > |t| \geq \delta$, $|w| \geq \delta$ for any $\delta > 1$. It is therefore enough to consider

$$\begin{cases} K^{(1)}_{p,s,n}(w,\zeta,t) := K^{(2)}_{s,n}(\zeta,t)A_{p,n,1}(w,t), \\ K^{(2)}_{p,s,n}(w,\zeta,t) := K^{(1)}_{s,n}(\zeta,t)A_{p,n,1}(w,t). \end{cases} \tag{6.5}$$

We now set

$$D^{(j)}_{p,s,n}(z;f) := \frac{1}{2\pi i}\int_{|\zeta|=r'} f(\psi(\zeta))\left(\frac{1}{2\pi i}\int_{|t|=r}\frac{\psi'(\zeta)}{\psi(\zeta)-\psi(t)}K^{(j)}_{p,s,n}(w,\zeta,t)\,dt\right)d\zeta. \tag{6.6}$$

Estimate of $D^{(1)}_{p,s,n}(z;f)$

From (6.3), we see that $K^{(2)}_{s,n}(\zeta,t) = O(\beta^n)$, uniformly for $|\zeta| > |t| \geq \delta > 1$, so that for $|w| > r$, we obtain

$$K^{(1)}_{p,s,n}(w,\zeta,t) = O(1)\left(\frac{\beta|w|^p}{r^p}\right)^n,$$

and thus $\lim_{n\to\infty} D^{(1)}_{p,s,n}(z;f) = 0$, uniformly on $\bar{G}_\mu$ for every $1 \leq \mu < R/r_0^{1/p}$. Note that if $r_0 = 0$, then $D^{(1)}_{p,s,n}(z;f) \equiv 0$.

Estimate of $D^{(2)}_{p,s,n}(z;f)$

Since

$$\frac{\psi'(\zeta)}{\psi(\zeta)-\psi(t)} = \frac{1}{\zeta - t} + \sum_{\mu=1}^{\infty}\sum_{k=1}^{\infty}\alpha_{\mu,k}t^{-k}\zeta^{-\mu-1}, \tag{6.7}$$

uniformly on closed subsets of $\{t \in \mathbb{C} : |t| > 1\} \times \{\zeta \in \mathbb{C} : |\zeta| > 1\}$ where $\alpha_{\mu,k} = O(\beta^{\mu+k})$, we can write

$$\frac{\psi'(\zeta)}{\psi(\zeta)-\psi(t)}K^{(2)}_{p,s,n}(w,\zeta,t) = \sum_{j=1}^{3} B^{(j)}_{p,s,n}(w,\zeta,t),$$

where we have set

$$\begin{cases} B^{(1)}_{p,s,n}(w,\zeta,t) := \dfrac{t^{s(n+1)}-1}{\zeta^{s(n+1}-1}\cdot\dfrac{t^{p(n+1)}-w^{p(n+1)}}{(\zeta-t)(t-w)t^{p(n+1)}}, \\[2ex] B^{(2)}_{p,s,n}(w,\zeta,t) := -\dfrac{(t^{n+1}-1)^s}{(\zeta^{n+1}-1)^s}\cdot\dfrac{t^{p(n+1)}-w^{p(n+1)}}{(\zeta-t)(t-w)t^{p(n+1)}}, \\[2ex] B^{(3)}_{p,s,n}(w,\zeta,t) := K^{(2)}_{p,s,n}(w,\zeta,t)\displaystyle\sum_{\mu=1}^{\infty}\sum_{k=1}^{\infty}\alpha_{\mu,k}t^{-k}\zeta^{-\mu-1}. \end{cases} \tag{6.8}$$

We now evaluate

$$I_j(w,\zeta) := \frac{1}{2\pi i} \int\limits_{|t|=r} B^{(j)}_{p,s,n}(w,\zeta,t)dt \qquad (j=1,2,3).$$

In order to find $I_1(w,\zeta)$, we note that $(\zeta^{s(n+1)}-1)B^{(1)}_{p,s,n}(w,\zeta,t)$ when expanded in powers of t yields

$$\sum_{k=0}^{\infty}\sum_{\nu=0}^{p(n+1)-1} \zeta^{-k-1}w^{\nu}t^{k+s(n+1)-\nu-1} - \sum_{k=0}^{\infty}\sum_{\nu=0}^{p(n+1)-1} \zeta^{-k-1}w^{\nu}t^{k-\nu-1}.$$

In the first sum, $k+s(n+1)-\nu-1 \geq 0$ and so its integral vanishes. The integral over the second sum gives a contribution only when $k=\nu$ and so

$$\begin{aligned} I_1(w,\zeta) &= -\sum_{\nu=0}^{p(n+1)-1} \zeta^{-\nu-1}w^{\nu}/(\zeta^{(s(n+1)}-1) \\ &= -\frac{\zeta^{p(n+1)}-w^{p(n+1)}}{(\zeta-w)(\zeta^{s(n+1)}-1)\zeta^{p(n+1)}}. \end{aligned} \tag{6.9}$$

The integrand in $I_2(w,\zeta)$, after multiplying by $(\zeta^{n+1}-1)^s$, has the following expansion in powers of t :

$$-\sum_{k=0}^{\infty}\sum_{j=0}^{s}\sum_{\nu=0}^{p(n+1)-1} \binom{s}{j}(-1)^{s-j}\zeta^{-k-1}w^{\nu}t^{k+j(n+1)-\nu-1}.$$

On integrating with respect to t on the circle $|t|=r$, all the integrals vanish except when $k=\nu-j(n+1)\geq 0$ (then $\nu \geq j(n+1)$ and thus $j \leq p-1$) and therefore

$$I_2(w,\zeta) = \frac{-1}{(\zeta^{n+1}-1)^s}\sum_{j=0}^{p-1}\sum_{\nu=j(n+1)}^{p(n+1)-1}\binom{s}{j}(-1)^{s-j}w^{\nu}\zeta^{j(n+1)-\nu-1}$$

$$= -\frac{w^{p(n+1)}\sum_{j=0}^{p-1}\binom{s}{j}(-1)^{s-j}\zeta^{j(n+1)} - \zeta^{p(n+1)}\sum_{j=0}^{p-1}\binom{s}{j}(-1)^{s-j}w^{j(n+1)}}{(\zeta-w)(\zeta^{n+1}-1)^s\zeta^{p(n+1)}}. \tag{6.10}$$

From (6.9) and (6.10), we obtain after a slight rearrangement

$$I_1(w,\zeta)+I_2(w,\zeta)=\frac{w^{p(n+1)}-\zeta^{p(n+1)}}{(\zeta-w)\zeta^{p(n+1)}}\cdot\frac{(\zeta^{n+1}-1)^s-(-1)^s(\zeta^{s(n+1)}-1)}{(\zeta^{s(n+1)}-1)(\zeta^{n+1}-1)^s}$$

$$-\frac{w^{p(n+1)}\sum_{j=1}^{p-1}\binom{s}{j}(-1)^{s-j}\zeta^{j(n+1)}-\zeta^{p(n+1)}\sum_{j=1}^{p-1}\binom{s}{j}(-1)^{s-j}w^{j(n+1)}}{(\zeta-w)(\zeta^{n+1}-1)^s\zeta^{p(n+1)}}$$

$$=: J_1(w,\zeta)+J_2(w,\zeta).$$

Since

$$(\zeta^{n+1}-1)^s-(-1)^s(\zeta^{s(n+1)}-1)=\begin{cases}2\zeta^{s(n+1)}\left(1+O\left(\frac{1}{\zeta^{n+1}}\right)\right), & \text{if } s \text{ is odd},\\ -s\zeta^{(s-1)(n+1)}\left(1+O\left(\frac{1}{\zeta^{n+1}}\right)\right), & \text{if } s \text{ is even},\end{cases}$$

it follows that

$$J_1(w,\zeta)=\begin{cases}O(1)\left(\frac{|w|^p}{(r')^{p+s}}\right)^n, & s \text{ is odd},\\ O(1)\left(\frac{|w|^p}{(r')^{p+s+1}}\right)^n, & s \text{ is even}.\end{cases}\tag{6.11}$$

Furthermore, $J_2(w,\zeta)\equiv 0$ for $p=1$ and for $p>1$,

$$J_2(w,\zeta)=O(1)\left(\frac{|w|^p}{(r')^{s+1}}\right)^n.\tag{6.12}$$

Therefore combining (6.11) and (6.12), we obtain

$$\frac{1}{2\pi i}\int_{|\zeta|=r'} f(\psi(\zeta))\,[I_1(w,\zeta)+I_2(w,\zeta)]\,d\zeta\to 0,\qquad \text{as } n\to\infty,$$

uniformly on $\bar{G}_\mu$ for every $1\le\mu<\lambda$, where

$$\lambda=\begin{cases}R^{s+2}, & \text{for } p=1 \text{ and } s \text{ even},\\ R^{(s+1)/p}, & \text{otherwise}.\end{cases}$$

This proves the theorem for $r_0=0$ because then $I_3(w,\zeta)\equiv 0$.

Estimate of $I_3(w,\zeta)$

In order to estimate $I_3(w,\zeta)$, we expand $(\zeta^{s(n+1)}-1)B^{(3)}_{p,s,n}(w,\zeta,t)$ in powers of t. Then we have

$$(\zeta^{s(n+1)}-1)B^{(3)}_{p,s,n}(w,\zeta,t)=S_1+S_2+S_3,$$

where

$$\begin{cases} S_1 := \displaystyle\sum_{\mu=1}^{\infty}\sum_{k=1}^{\infty}\sum_{\nu=0}^{p(n+1)-1} \alpha_{\mu,k}\zeta^{-\mu-1}w^{\nu}t^{-k+s(n+1)-\nu-1}, \\ S_2 := -\displaystyle\sum_{\mu=1}^{\infty}\sum_{k=1}^{\infty}\sum_{\nu=0}^{p(n+1)-1} \alpha_{\mu,k}\zeta^{-\mu-1}w^{\nu}t^{-k-\nu-1}, \\ S_3 := -\left(1+O\left(\dfrac{1}{\zeta^{n+1}}\right)\right)\displaystyle\sum_{\mu=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j=0}^{s}\sum_{\nu=0}^{p(n+1)-1}\binom{s}{j}\frac{(-1)^{s-j}\alpha_{\mu,k}w^{\nu}}{\zeta^{\mu+1}}t^{-k+j(n+1)-\nu-1}. \end{cases} \tag{6.13}$$

Since the power of t in the sum S_2 is $-k-\nu-1\le -2$, we have no contribution from the integral of S_2. Term by term integration of the first term S_1 is non-zero only when $k=s(n+1)-\nu$ and so its contribution is

$$S_{11}=\frac{1}{\zeta^{s(n+1)}-1}\sum_{\mu=1}^{\infty}\sum_{\nu=0}^{p(n+1)-1}\alpha_{\mu,s(n+1)-\nu}\zeta^{-\mu-1}w^{\nu}.$$

Similarly, the integration of S_3 with respect to t vanishes except when $k=j(n+1)-\nu$ (note that then $j(n+1)\ge\nu+1$ and then $j\ne 0$). This yields

$$S_{31}=\left(1+O\left(\frac{1}{\zeta^{n+1}}\right)\right)\frac{1}{\zeta^{s(n+1)}-1}\sum_{\mu=1}^{\infty}\sum_{\nu=0}^{p(n+1)-1}\sum_{\substack{j=1\\ j(n+1)\ge\nu+1}}^{s}\binom{s}{j}(-1)^{s-j}\alpha_{\mu,j(n+1)-\nu}\zeta^{-\mu-1}w^{\nu}.$$

Since $\alpha_{\mu,k}=O(1)\beta^{\mu+k}$, we obtain

$$S_{11}=O(1)\frac{1}{(r')^{sn}}\sum_{\mu=1}^{\infty}\left(\frac{\beta}{r'}\right)^{\mu}\beta^{sn}\sum_{\nu=0}^{p(n+1)-1}\left(\frac{w}{\beta}\right)^{\nu}=O(1)\left(\frac{\beta^{s-p}|w|^{p}}{(r')^{s}}\right)^{n},$$

and

$$S_{31} = O(1)\frac{1}{(r')^{sn}}\sum_{\mu=1}^{\infty}\left(\frac{\beta}{r'}\right)^{\mu}\sum_{\substack{j=1\\ j(n+1)\geq \nu+1}}^{s}\sum_{\nu=0}^{p(n+1)-1}\beta^{j(n+1)}\left(\frac{w}{\beta}\right)^{\nu}$$

$$= O(1)\frac{1}{(r')^{sn}}\sum_{\nu=0}^{p(n+1)-1}\beta^{\nu+1}\left(\frac{w}{\beta}\right)^{\nu} = O(1)\left(\frac{|w|^p}{(r')^s}\right)^n.$$

It therefore follows that

$$\frac{1}{2\pi i}\int_{|\zeta|=r'} f(\psi(\zeta))\, I_3(w,\zeta)\, d\zeta \to 0, \qquad \text{as } n\to\infty,$$

uniformly on $\bar{G}_\mu$ for every $1 \leq \mu < R^{s/p}\min\{1/r_0^{-1+s/p}, 1\} = R^{s/p}$. Combining the above results then gives the desired result of (4.3) of Theorem C. ∎

Remarks. **(1)** For $s = p$ and $r_0 > 0$, we have from (4.3) that $\lambda = R$, so that Theorem C gives no overconvergence. Also $\lambda \to R$ as $r_0 \to 1$ for arbitrary p and s. If $0 < r_0 < \frac{1}{R^{s-p}}$, then $\lambda = R^{s/p}$.

(2) If $r_0 = 0$, then λ from (4.3) is best possible as can be seen by the example $f(z) = 1/(R-z)$, but we do not know if this is the case when $r_0 > 0$. However, we are able to improve our result if E is the ellipse E_δ. An examination of the proof shows that Theorem C holds with

$$\lambda = \min\{R^{1+1/p}/r_0^{2/p}; R^{s/p}\}.$$

(3) The above remark also applies when $\delta = 1$, i.e., $E = [-1, 1]$ and we obtain

$$\lambda = \begin{cases} R & \text{for } s = p, \\ R^{1+1/p} & \text{for } s > p. \end{cases}$$

REFERENCES

[1] Brück R. On the failure of Walsh's equiconvergence theorem for Jordan domains. *Analysis*, **13**:229–234, 1993.

[2] Brück R., Sharma A., Varga R.S. An extension of a result of Rivlin on Walsh equiconvergence. In *Advances in Computational Mathematics*: New Delhi, India, 1993, pages 225–234. World Scientific Publishing Co. Ptl. Ltd., Singapore, 1994. H.P. Dikshit and C.A. Micchelli, eds.

[3] Cavaretta A.S. Jr., Sharma A., Varga R.S. Interpolation in the roots of unity: An extension of a theorem of J. L. Walsh. *Resultate Math.*, **3**:155–191, 1980.

[4] Curtiss J.H. Convergence of complex Lagrange interpolation polynomials on the locus of the interpolation points. *Duke Math. J.*, **32**:187–204, 1965.

[5] Curtiss J.H. Faber polynomials and the Faber series. *Amer. Math. Monthly*, **78**:577–596, 1971.

[6] Gaier D. *Vorlesungen über Approximation im Komplexen*. Birkhäuser Verlag, Basel, 1980.

[7] Kövari T., Pommerenke Ch. On Faber polynomials and Faber expansions. *Math. Z.*, **99**:193–206, 1967.

[8] Pommerenke Ch. Über die Verteilung der Fekete-Punkte. *Math. Ann.*, **168**:111–127, 1967.

[9] Rivlin T.J. On Walsh equiconvergence. *J. Approx. Theory*, **36**:334–345, 1982.

[10] Walsh J.L. *Interpolation and Approximation by Rational Functions in the Complex Domain*, 5th Ed. American Math. Soc., Providence, R.I., 1969.