Studia Scientiarum Mathematicarum Hungarica 29 (1994), 415–432

SOME 2-PERIODIC TRIGONOMETRIC INTERPOLATION PROBLEMS ON EQUIDISTANT NODES II: CONVERGENCE

A. SHARMA, J. SZABADOS[1] and R. S. VARGA

1. Introduction

Let $n, p, q \geqq 1$ be integers with

$$N := n(p+q), \quad M = \left[\frac{n}{2}\right]$$

and let

$$(1) \qquad x_n = x_k(n) := \frac{k\pi}{n} \quad (k = 0, 1, \dots, 2n-1).$$

Let $\mathbf{m}_1 = (m_1, \dots, m_p), \mathbf{m}_2 = (m_{p+1}, \dots, m_{p+q})$ be two sequences of integers such that

$$(2) \qquad 0 = m_1 < m_2 < \dots < m_p, \quad 0 \leqq m_{p+1} < \dots < m_{p+q}.$$

The problem of 2-periodic trigonometric interpolation is to reconstruct the unique trigonometric polynomial

$$(3) \quad t_M(x) = a_0 + \sum_{k=1}^{M} (a_k \cos kx + b_k \sin kx) \quad (a_M b_M = 0 \quad \text{if} \quad N \quad \text{is even}),$$

from the data

$$t_M^{(m_\mu)}(x_{2k}), \quad t_M^{(m_\nu)}(x_{2k+1}) \quad (\mu = 1, \dots, p;\ \nu = p+1, \dots, p+q),$$

for given sequences of integers $\mathbf{m}_1, \mathbf{m}_2$ satisfying (2).

1980 *Mathematics Subject Classification* (1985 Revision). Primary 42A15; Secondary 41A10.

Key words and phrases. 2-periodic interpolation, fundamental polynomials, convergence.

[1]Research was supported by Hungarian National Research Fund Grant No. 1801, and by NSERC Grant No. A3094.

Recently in [2], we gave necessary and (separately) sufficient conditions for the regularity (or unique solvability) of the above problem. For studying the convergence problem, we shall suppose that the sufficient conditions in [2] are satisfied. We shall state these conditions for the sake of completeness, but in a different form.

We shall say that a finite sequence of non-negative integers $\mathbf{m} = (m_1, m_2, \ldots, m_p)$ such that

$$(3) \qquad 0 \leqq m_1 < m_2 < \cdots < m_p \quad \text{and} \quad m_i + m_{i+1} \text{ is odd } \quad (i = 1, \ldots, p-1)$$

is EE, EO, OE or OO according as m_1, m_p are both even; m_1 is even, m_p is odd; m_1 is odd, m_p is even or m_1, m_p are both odd, respectively. A sequence with only one element will be either EE or OO depending on the parity of the element. It is clear from this definition that EE and OO sequences have odd cardinality while EO and OE sequences have even cardinality. With this definition we can now state

THEOREM A [2]. *The 2-periodic trigonometric interpolation problem on the nodes* (1) *corresponding to the sequences* $\mathbf{m}_1, \mathbf{m}_2$ *is regular in the following cases:*

		$\mathbf{m}_1$	$\mathbf{m}_2$	n	*Type of* T_M
I	(a)	EO	EE	*odd*	—
II	(a)	EE	OE	*odd*	—
III	(b)	EO	EO	*arb.*	$a_M = 0$
IV	(b)	EE	OO	*odd*	$a_M = 0$
V	(b)	EE	EE	*arb.*	$b_M = 0$
VI	(b)	EO	OE	*arb.*	$a_M = 0$
VII	(c)	EE	EO	*even*	$b_M = 0$

REMARK. Since by supposition $\mathbf{m}_1$ begins with an even number, there are only 8 possible combinations of $\mathbf{m}_1, \mathbf{m}_2$ in the table. The pair $\mathbf{m}_1 = EO$, and $\mathbf{m}_2 = OO$ is not in the above table, because in this case even the necessary conditions of regularity of Theorems 1 and 2 in [2] are not satisfied. The conditions which are necessary for regularity as given in [2] imply that

$$(4) \qquad e - o = 0, 1 \text{ or } 2$$

where e and o denote the cardinality of even and odd numbers in the set $\mathbf{m}_1 \cup \mathbf{m}_2$, respectively. But if $\mathbf{m}_1 = EO$, $\mathbf{m}_2 = OO$, then $o - e = 1$, which contradicts (4).

These conditions are sufficient, but not necessary as can be seen by the examples in [2] and also by the results in [4]. In [3] it was shown that if $\mathbf{m}_1 := (0, m_1, \ldots, m_p)$ and if $\mathbf{m}_2 = (m_1, \ldots, m_p)$, then the problem is regular if and only if p is even and then $b_M = 0$. In this case no condition like $m_i + m_{i+1}$ odd is needed.

The object of this note is twofold: First we want to construct the fundamental polynomials of interpolation in all the above cases. Secondly, we want to examine the convergence of the interpolant.

In Section 2, we find the fundamental polynomials when n is odd and $p+q=2s+1$ (cases covered in (a)). In Section 3, we find the fundamental polynomials when $p+q$ is even and n is arbitrary. This covers the four situations listed in (b) in the Table. Lastly, Section 4 is devoted to the last case (c) in the Table when n is even and $p+q$ is odd. Section 5 deals with the problem of convergence.

2. Fundamental polynomials (n odd, $p+q$ odd)

Here $n = 2r+1$, $p+q = 2s+1$ and $M = ns+r$. This case covers the first two cases in the table. We shall denote the fundamental polynomials by $\varrho_\nu(x)$ which are going to be determined by the followig conditions for $\nu = 1, \ldots, p$:

$$\tag{2.1} \begin{cases} \varrho_\nu^{(m_j)}(x_{2k}) = \delta_{k0}\delta_{\nu j}, & j = 1, \ldots, p; \quad k = 0, 1, \ldots, n-1 \\ \varrho_\nu^{(m_j)}(x_{2k+1}) = 0, & j = p+1, \ldots, p+q; \quad k = 0, 1, \ldots, n-1. \end{cases}$$

For $\nu = p+1, \ldots, p+q$, conditions (2.1) will be replaced by

$$\tag{2.2} \begin{cases} \varrho_\nu^{(m_j)}(x_{2k}) = 0, & j = 1, \ldots, p; \; k = 0, 1, \ldots, n-1 \\ \varrho_\nu^{(m_j)}(x_{2k+1}) = \delta_{k0}\delta_{\nu j}, & j = p+1, \ldots, p+q; \; k = 0, 1, \ldots, n-1. \end{cases}$$

Putting $z = e^{ix}$, we may set

$$\tag{2.3} \varrho_\nu(x) = z^{-M} \sum_{\lambda=0}^{2s} z^{\lambda n} Q_{\nu\lambda}(Z), \quad Q_{\nu\lambda}(z) = \sum_{j=0}^{n-1} a_{\lambda j}(\nu) z^j.$$

Conditions (2.1) (equivalently (2.2)) give the following system of $2s+1$ differential equations to determine $Q_{\nu\lambda}(z)$:

(2.4)

$$\begin{cases} \sum_{\lambda=0}^{2s} (\Theta + \lambda n - M)^{m_\mu} Q_{\nu\lambda}(z) = \dfrac{i^{-m_\nu}}{n} \dfrac{z^n - 1}{z-1} \delta_{\mu\nu}, & \mu = 1, \ldots, p \\ \sum_{\lambda=0}^{2s} (-1)^\lambda (\Theta + \lambda n - M)^{m_\mu} Q_{\nu\lambda}(z) = -\dfrac{i^{-m_\nu}}{n} \dfrac{z^n + 1}{z+1} \delta_{\mu\nu}, & \mu = p+1, \ldots, p+q \end{cases}$$

where $\Theta = z\frac{d}{dz}$. If we denote the determinant of this system by $\Delta(\Theta)$ and the co-factor of the $(\nu, \lambda+1)$ term by $\Delta_{\nu,\lambda+1}(\Theta)$, then we have

$$Q_{\nu\lambda}(z) = \begin{cases} \frac{i^{-m_\nu}}{n} \frac{\Delta_{\nu,\lambda+1}(\Theta)}{\Delta(\Theta)} \frac{z^n - 1}{z-1}, & \nu = 1, \dots, p \\ -\frac{i^{-m_\nu}}{n} \frac{\Delta_{\nu,\lambda+1}(\Theta)}{\Delta(\Theta)} \frac{z^n + 1}{z+1}, & \nu = p+1, \dots, p+q. \end{cases}$$

Since $\Theta z^j = jz^j$, we see easily that

$$\frac{\Delta_{\nu,\lambda+1}(\Theta)}{\Delta(\Theta)} z^j = \frac{1}{n^{m_\nu}} \frac{D_{\nu,\lambda+1}(\alpha_j)}{D(\alpha_j)} z^j, \quad \alpha_j = \frac{j-r}{n}$$

where

$$(2.5) \qquad D(\alpha) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ (\alpha-s)^{m_2} & (\alpha-s+1)^{m_2} & \dots & (\alpha+s)^{m_2} \\ \dots & \dots & \dots & \dots \\ (\alpha-s)^{m_p} & (\alpha-s+1)^{m_p} & \dots & (\alpha+s)^{m_p} \\ (\alpha-s)^{m_{p+1}} & -(\alpha-s+1)^{m_{p+1}} & \dots & (\alpha+s)^{m_{p+1}} \\ \dots & \dots & \dots & \dots \\ (\alpha-s)^{m_{p+q}} & -(\alpha-s+1)^{m_{p+q}} & \dots & (\alpha+s)^{m_{p+q}} \end{vmatrix}$$

and $D_{\nu,\lambda+1}(\alpha)$ is the cofactor of $(\nu, \lambda+1)$ terms in $D(\alpha)$. In the last q rows in $D(\alpha)$, the columns have alternately positive, negative signs.

Setting

$$(2.6) \qquad \alpha_{\lambda j}(\nu) := \frac{D_{\nu,\lambda+1}(\alpha_j)}{D(\alpha_j)}, \qquad \begin{matrix} \nu = 1, \dots, p+q; \quad \lambda = 0, 1, \dots, 2s \\ j = 0, 1, \dots, n-1 \end{matrix}$$

we see that

$$(2.7) \qquad Q_{\nu\lambda}(z) = \begin{cases} \frac{i^{-m_\nu}}{n^{1+m_\nu}} \sum_{j=0}^{n-1} a_{\lambda j}(\nu) z^j, & \nu = 1, \dots, p \\ -\frac{i^{-m_\nu}}{n^{1+m_\nu}} \sum_{j=0}^{n-1} (-1)^j a_{\lambda j}(\nu) z^j, & \nu = p+1, \dots, p+q \end{cases}$$

since n is odd.

It has been shown in [2] that under the hypothesis of Theorem 1 the determinant $D(\alpha) \neq 0$ for $|\alpha| \leqq 1/2$. Also, if e and o denote the number of even and odd integers in the sequence $m_1, \dots, m_{p+q}$, then

$$e - 1 = o$$

is necessary for regularity. Multiplying the rows in $D(-\alpha)$ corresponding to odd m_ν's by (-1) and then doing s elementary column operators, we see that

$$(-1)^o D(-\alpha) = (-1)^s D(\alpha)$$

which proves that

$$D(-\alpha) = D(\alpha). \tag{2.8}$$

Similarly, we can see that

$$D_{\nu,\lambda+1}(-\alpha) = (-1)^{m_\nu} D_{\nu,2s+1-\lambda}(\alpha), \quad \nu = 1, \ldots, p+q. \tag{2.9}$$

From (2.6), (2.8) and (2.9), we have

$$a_{\lambda j}(\nu) = (-1)^{m_\nu} a_{2s-\lambda,2r-j}(\nu), \quad \nu = 1, \ldots, p+q. \tag{2.10}$$

Combining (2.7) and (2.10), we obtain

$$\begin{aligned} \varrho_\nu(x) = \frac{(-1)^{m_\nu/2}}{n^{1+m_\nu}} \Big[a_{sr}(\nu) + 2 \sum_{j=1}^{r} a_{s,r-j}(\nu) \cos jx + \\ + 2 \sum_{\lambda=1}^{s} \sum_{j=0}^{n-1} a_{s-\lambda,j}(\nu) \cos(\lambda n + r - j)x \Big] \end{aligned} \tag{2.11}$$

when m_ν is even and $1 \leqq \nu \leqq p$. When m_ν is odd, because of (2.10) we see that $\varrho_\nu(x)$ is a sine series. More precisely, we have

$$\begin{aligned} \varrho_\nu(x) = \frac{2(-1)^{\frac{m_\nu-1}{2}}}{n^{1+m_\nu}} \Big[\sum_{j=1}^{r} a_{s,r-j} \sin jx + \\ + \sum_{\lambda=1}^{s} \sum_{j=0}^{n-1} a_{s-\lambda,j}(\nu) \sin(\lambda n + r - j)x \Big] \end{aligned} \tag{2.12}$$

when m_ν is odd and $1 \leqq \nu \leqq p$.

For $\nu = p+1, \ldots, p+q$, $\varrho_\nu(x)$ is obtained from (2.11) or (2.12) according as m_ν is even or odd, respectively, by replacing x by $x - \frac{\pi}{n}$.

3. Fundamental polynomials ($p+q$ even)

Here $p+q = 2s+2$ and $M = ns+n$. So we set

$$\varrho_\nu(x) := z^{-M} \Big[\sum_{\lambda=0}^{2s+1} z^{\lambda n} Q_{\nu\lambda}(z) + C_\nu z^{(2s+2)n} \Big], \quad Q_{\nu\lambda}(z) = \sum_{j=0}^{n-1} a_{\lambda j}(\nu) z^j. \tag{3.1}$$

The conditions which determine $\varrho_\nu(x)$ are given by (2.1) or (2.2) and lead to the following systems of equations:

$$(3.2)\quad \begin{cases} \sum_{\lambda=0}^{2s+1} (\Theta+\lambda n - M)^{m_\mu} Q_{\nu\lambda}(z) + C_\nu M^{m_\mu} = \dfrac{i^{-m_\nu}}{n}\dfrac{z^n-1}{z-1}\delta_{\mu\nu} \\ \qquad (\mu=1,2,\dots,p), \\ \sum_{\lambda=0}^{2s+1} (-1)^\lambda(\Theta+\lambda n - M)^{m_\mu} Q_{\nu\lambda}(z) + C_\nu M^{m_\mu} = \dfrac{i^{-m_\nu}}{n}\dfrac{z^n+1}{z-\omega}\omega^{n-1}\delta_{\mu\nu} \\ \qquad (\mu=p+1,\dots,p+q), \\ \delta_0 Q_{\nu 0} + (-1)^{1+\varepsilon}C_\nu = 0, \end{cases}$$

where δ_0 in the last equation is the point evaluation at 0 and $\omega = \exp\frac{i\pi}{n}$. The last condition is a consequence of the fact that the last term in $\varrho_\nu(x)$ is $a_M\cos(Mx+\frac{\varepsilon\pi}{2})$, where $\varepsilon = 0$ or 1.

If we denote the determinant of this system by $\Delta^*(\Theta)$ and the cofactors by $\Delta^*_{kl}(\Theta)$, then we have

$$(3.3)\qquad Q_{\nu,\lambda}(z) = \begin{cases} \dfrac{i^{-m_\nu}}{n}\dfrac{\Delta^*_{\nu,\lambda+1}(\Theta)}{\Delta^*(\Theta)}\dfrac{z^n-1}{z-1}, & \nu=1,\dots,p \\ \dfrac{i^{-m_\nu}}{n}\dfrac{\Delta^*_{\nu,\lambda+1}(\Theta)}{\Delta^*(\Theta)}\dfrac{Z^n-1}{Z-1}, & \nu=p+1,\dots,p+q \end{cases}$$

for $\lambda = 0,1,\dots,2s+1$, where we have put $Z = ze^{-\frac{i\pi}{n}}$. Also C_ν is given by formulae (3.3) when $\lambda = 2s+2$.

Because of the point evaluation operator δ_0 in the determinant $\Delta^*(\Theta)$, it is easy to see that

$$(3.4)\qquad \frac{\Delta^*_{\nu,\lambda+1}(\Theta)}{\Delta^*(\Theta)}z^j = \begin{cases} \dfrac{1}{n^{m_\nu}}\dfrac{\tilde D_{\nu,\lambda+1}(\alpha_j)}{\tilde D(\alpha_j)}z^j, & \lambda=0,1,\dots,2s+1 \\ 0, & \lambda = 2s+2 \end{cases}$$

for $j=1,2,\dots,n-1$ where $\alpha_j = \frac{j}{n}$ $(j=1,\dots,n-1)$ and

$$(3.5)\qquad \tilde D(\alpha) := \begin{vmatrix} 1 & 1 & \dots & 1 \\ (\alpha-s-1)^{m_1} & (\alpha-s)^{m_1} & \dots & (\alpha+s)^{m_1} \\ \dots & \dots & \dots & \dots \\ (\alpha-s-1)^{m_p} & (\alpha-s)^{m_p} & \dots & (\alpha+s)^{m_p} \\ (\alpha-s-1)^{m_{p+1}} & -(\alpha-s)^{m_{p+1}} & \dots & -(\alpha+s)^{m_{p+1}} \\ \dots & \dots & \dots & \dots \\ (\alpha-s-1)^{m_{p+q}} & -(\alpha-s)^{m_{p+q}} & \dots & -(\alpha+s)^{m_{p+q}} \end{vmatrix}$$

and $\tilde D_{\nu,\lambda+1(\alpha)}$ is a cofactor of $\tilde D(\alpha)$. The order of $\tilde D(\alpha)$ is $2s+2$ and the last q rows have alternating sign in the columns.

When $j=0$, we see that

$$\frac{\Delta^*_{\nu,\lambda+1}(\Theta)}{\Delta^*(\Theta)}z^0 = \frac{1}{n^{m_\nu}}\frac{\tilde{D}^*_{\nu,\lambda+1}}{\tilde{D}^*}, \qquad \lambda = 0,1,\dots,2s+2 \tag{3.6}$$

where

$$\tilde{D}^* := \begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ (-s-1)^{m_1} & (-s)^{m_1} & \dots & s^{m_1} & (s+1)^{m_1} \\ \dots & \dots & \dots & \dots & \dots \\ (-s-1)^{m_p} & (-s)^{m_p} & \dots & s^{m_p} & (s+1)^{m_p} \\ (-s-1)^{m_{p+1}} & -(-s)^{m_{p+1}} & \dots & -s^{m_{p+1}} & (s+1)^{m_{p+1}} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 & (-1)^{1+\varepsilon} \end{vmatrix} \tag{3.7}$$

and $\tilde{D}^*_{\nu,\lambda+1}$ is the cofactor of $\tilde{D}^*$. It follows from (3.3) and (3.6) that C_ν is a constant given by

$$C_\nu = \frac{i^{-m_\nu}}{n}\frac{D^*_{\nu,2s+3}}{D^*}, \qquad \nu = 1,\dots,p+q. \tag{3.8}$$

If we now set

$$a_{\lambda j}(\nu) := \begin{cases} \dfrac{D_{\nu,\lambda+1}(\alpha_j)}{D(\alpha_j)}, & 0<\alpha_j \leqq 1 \\ \dfrac{D^*_{\nu,\lambda+1}}{D^*}, & \alpha_j = 0 \end{cases} \qquad (\lambda = 0,1,\dots,2s+2), \tag{3.9}$$

then we have

$$Q_{\nu\lambda}(z) = \frac{i^{-m_\nu}}{n^{1+m_\nu}}\sum_{j=0}^{n=1} \alpha\lambda_j(\nu) z^j \qquad (\lambda = 0,1,\dots,2s+1),$$

$$C_\nu := \frac{i^{-m_\nu}}{n}\frac{D^*_{\nu,2s+3}}{D^*}$$

so that

$$\varrho_\nu(x) = \frac{i^{-m_\nu}}{n} z^{-sn-n}\Big[\sum_{\lambda=0}^{2s+1} z^{\lambda n}\sum_{j=0}^{n-1} a_{\lambda j}(\nu) z^j + a_{2s+2,n}(\nu) z^{(2s+2)n}\Big] \tag{3.10}$$

From (3.5), we see easily that

$$\begin{cases} D(\alpha) = D(1-\alpha) \\ D_{\nu,\lambda+1}(\alpha) = (-1)^{m_\nu} D_{\nu,2s+2-\lambda}(1-\alpha) \quad (\lambda = 0,1,\dots,2s+1). \end{cases} \tag{3.11}$$

Similarly,

$$D^*_{\nu,\lambda+1} = (-1)^{m_\nu} D^*_{\nu,2s+3-\lambda},$$

so that from (3.9) and (3.11) we have

$$a_{\lambda j}(\nu) = (-1)^{m_\nu} a_{2s+2-\lambda,n-j}(\nu), \quad j = 1,\dots,n-1$$
$$a_{\lambda 0}(\nu) = (-1)^{m_\nu} a_{2s+s-\lambda,n}(\nu).$$

Thus we have from (3.10)

(3.12) $$\varrho_\nu(x) =$$

$$= \begin{cases} \dfrac{2(-1)^{\frac{m_\nu-1}{2}}}{n^{1+m_\nu}} \Big[\displaystyle\sum_{\lambda=0}^{s}\sum_{j=0}^{n=1} a_{\lambda j}(\nu)\sin((s+1-\lambda)n-j)x\Big], & m_\nu \text{ odd} \\ \dfrac{(-1)^{\frac{m_\nu}{2}}}{n^{1+m_\nu}} \Big[a_{s+1,0}(\nu) + 2\displaystyle\sum_{\lambda=0}^{s}\sum_{j=0}^{n-1} a_{\lambda j}(\nu)\cos((s+1-\lambda)n-j)x\Big], & m_\nu \text{ even.} \end{cases}$$

4. Fundamental polynomials (n even, $p+q$ odd)

In this case $n = 2r, p+q = 2s+1$ and $M = ns+r$ so that we may set

(4.1) $$\varrho_\nu(x) = z^{-M}\sum_{\lambda=0}^{2s} z^{\lambda n} Q_\lambda(z) + C_\nu z^M, \quad Q_{\nu\lambda}(z) \in \pi_{n-1}.$$

The system of differential equations as in Section 3 is given by

(4.2) $$\begin{cases} \displaystyle\sum_{\lambda=0}^{2s}(\Theta+\lambda n - M)^{m_\mu} Q_{\nu\lambda}(z) + C_\nu M^{m_\mu} = \frac{i^{-m_\nu}}{n}\frac{z^n-1}{z-1}\delta_{\mu\nu} \\ \qquad (\mu = 1,\dots,p) \\ \displaystyle\sum_{\lambda=0}^{2s}(-1)^\lambda(\Theta+\lambda n - M)^{m_\mu} Q_{\nu\lambda}(z) + C_\nu M^{m_\mu} = -\frac{i^{-m_\nu}}{n}\frac{z^n+1}{z+1}\delta_{\mu\nu} \\ \qquad (\mu = p+1,\dots,p+q) \\ \delta_0 Q_{\nu,0}(z) + (-1)^{1+\varepsilon} C_\nu = 0. \end{cases}$$

Then, as in Section 3, we obtain

$$Q_{\nu\lambda}(z) = \begin{cases} \dfrac{i^{-m_\nu}}{n^{1+m_\nu}}\displaystyle\sum_{j=0}^{n-1} a_{\lambda j}(\nu) z^j, & \nu = 1,\dots,p \\ \dfrac{i^{-m_\nu}}{n^{1+m_\nu}}\displaystyle\sum_{j=0}^{n-1} (-1)^j a_{\lambda j}(\nu) z^j, & \nu = p+1,\dots,p+q \end{cases}$$

where

$$a_{\lambda j}(\nu)=\frac{D_{\nu,\lambda+1}(\alpha_j)}{D(\alpha_j)},\quad \alpha_j=\frac{j-r}{n}\quad (\lambda=0,1,\dots,2s)$$
$$(\nu=1,\dots,p+q;\ j=1,\dots,n-1),$$

the determinant $D(\alpha)$ being the same as in (2.5). However, for $j=0$, we have

$$a_{\lambda 0}(\nu)=\frac{D^*_{\nu,\lambda+1}}{D^*}\quad (\lambda=0,1,\dots,2s), \tag{4.4}$$

where

(4.5)

$$D^*=\begin{vmatrix} 1 & 1 & \dots & 1 & 1\\ (\alpha_0-s)^{m_1} & (\alpha_0+1-s)^{m_1} & \dots & (\alpha_0+s)^{m_1} & (\alpha_0+s+1)^{m_1}\\ \dots & \dots & \dots & \dots & \dots\\ (\alpha_0-s)^{m_p} & (\alpha_0+1-s)^{m_p} & \dots & (\alpha_0+s)^{m_p} & (\alpha_0+s+1)^{m_p}\\ (\alpha_0-s)^{m_{p+1}} & -(\alpha_0+1-s)^{m_{p+1}} & \dots & (\alpha_0+s)^{m_{p+1}} & -(\alpha_0+s+1)^{m_{p+1}}\\ \dots & \dots & \dots & \dots & \dots\\ (\alpha_0-s)^{m_{p+q}} & -(\alpha_0+1-s)^{m_{p+q}} & \dots & (\alpha_0+s)^{m_{p+q}} & -(\alpha_0+s+1)^{m_{p+q}}\\ 1 & 0 & \dots & 0 & (-1)^{1+\epsilon} \end{vmatrix}$$

and $D^*_{\nu,\lambda+1}$ is its cofactor. From (4.2), we see, as in Section 3, that

$$C_\nu=\frac{i^{-m_\nu}}{n}\frac{D^*_{\nu,2s+2}}{D^*}\quad (\nu=1,\dots,p+q). \tag{4.6}$$

As in Section 2, (2.10) is valid in this case also for $\nu=1,\dots,p+q$. Thus after some simplification, we obtain

$$\varrho_\nu(x)=\begin{cases} \dfrac{(-1)^{\frac{m_\nu}{2}}}{n^{1+m_\nu}}\Big[a_{sr}(\nu)+2\sum\limits_{j=1}^{r}a_{s,r-j}(\nu)\cos jx+ \\ \qquad +2\sum\limits_{\lambda=1}^{s}\sum\limits_{j=0}^{n-1}a_{s-\lambda,j}(\nu)\cos(\lambda n+r-j)x\Big], & m_\nu \text{ even}\\ \dfrac{2(-1)^{\frac{m_\nu-1}{2}}}{n^{1+m_\nu}}\Big[\sum\limits_{j=1}^{r}a_{s,r-j}(\nu)\sin jx+ \\ \qquad +\sum\limits_{\lambda=1}^{s}\sum\limits_{j=0}^{n=1}a_{s-\lambda,j}(\nu)\sin(\lambda n+r-j)x\Big], & m_\nu \text{ odd.} \end{cases} \tag{4.7}$$

5. Convergence of 2-periodic interpolation

The definition of 2-periodic trigonometric interpolation suggests that in order to prove a general convergence result for continuous 2π-periodic functions, it is enough to consider the linear operator of the form

$$(5.1)\quad L_n(f;x):=\sum_{k=0}^{n-1}[f(x_{2k})\varrho_1(x-x_{2k})+\delta_{0,m_{p+1}}f(x_{2k+1})\varrho_{m_{p+1}}(x-x_{2k})]$$

which has the following properties:

$$\left.\begin{array}{l} L_n^{(m_\nu)}(f;x_{2k})=\delta_{\nu 1}f(x_{2k}) \quad (\nu=1,\dots,p) \\ L_n^{(m_\nu)}(f;x_{2k+1})=\delta_{\nu,p+1}f(x_{2k+1})\delta_{0,m_{p+1}} \quad (\nu=1,\dots,p+q) \end{array}\right\} (k=0,1,\dots,n-1).$$

The convergence properties of this operator depend on the order of magnitude of the fundamental functions $\varrho_\nu(x)$ determined in the previous sections. We first prove a lemma on the fundamental polynomials.

LEMMA 1. *Under the conditions of Theorem A in case (a) and (c) with the additional assumption that $m_{p+1}>0$, we have*

$$(5.2)\qquad \left\|\sum_{k=0}^{2n-1}|\varrho_\nu(x-x_k)|\right\|=O(n^{-m_\nu}\log n),\quad \nu=1\dots,p+q.$$

(Here and in what follows $\|\cdot\|$ means the sup norm.)

PROOF. Assume that m_ν is even. (The proof for the case m_ν odd is similar.) Then $\varrho_\nu(x)$ is given by the form (2.11) or (4.7). Since the coefficients $a_{\lambda j}(\nu)$ of this polynomial are obtained as the ratio of two determinants where the determinant in the denominator is a non-vanishing function of the variable α in the closed interval $[-\frac{1}{2},\frac{1}{2}]$, we have

$$(5.3)\qquad |\alpha_{\lambda j}(\nu)|=O(1).$$

Hence separating terms corresponding to $j=0$, we can write $\varrho_\nu(x)$ in the form

$$(5.4)\qquad \varrho_\nu(x)=\frac{(-1)^{m_\nu/2}}{n^{1+m_\nu}}\Big[2{\sum_{\lambda=0}^{s}}'\sum_{j=1}^{n-1}a_{s-\lambda,j}(\nu)\cos(\lambda n-j)x+O(1)\Big],$$

where $\sum'$ indicates that when $\lambda=0$, the factor 2 should be dropped. For a fixed λ, all the coefficients $a_{s-\lambda,j}(\nu)$ are determined by the same formula (e.g. (2.6) in Section 2 and Section 4). Therefore

$$\Delta a_{s-\lambda,j}(\nu):=a_{s-\lambda,j}(\nu)-a_{s-\lambda,j+1}(\nu)=O(n^{-1}),$$

$$\text{where}\begin{cases} j=1,\dots,r-1, & \text{if } \lambda=0\\ j=1,\dots,n-2, & \text{if } \lambda=1,\dots,s.\end{cases}$$

Using Abel-transform on (5.4), we obtain

$$\sum_{j=1}^{n-1} a_{s-\lambda,j}(\nu)\cos(\lambda n+r-j)x=$$

$$=\sum_{j=1}^{n-2}\Delta a_{s-\lambda,j}(\nu)\sum_{l=1}^{j}\cos(\lambda n+r-l)x+a_{s-\lambda,n-1}(\nu)\sum_{l=1}^{n-1}\cos(\lambda n+r-l)x=$$

$$=O(n^{-1})\sum_{j=1}^{n-2}\left|\frac{\sin\frac{j}{2}x\sin(\lambda n+r-\frac{j+1}{2})x}{\sin\frac{x}{2}}\right|+O(1)\left|\frac{\sin\frac{n-1}{2}x\sin(\lambda n+r-\frac{n}{2})x}{\sin\frac{x}{2}}\right|,$$

whence again from (5.4), we have

$$\sum_{k=0}^{2n-1}|\varrho_\nu(x-x_k)|=$$

$$=O(n^{-1-m_\nu})\sum_{\lambda=0}^{s}\left\{O(n^{-1})\sum_{j=1}^{n-2}\sum_{k=0}^{2n-1}\left|\frac{\sin\frac{j}{2}(x-x_k)}{\sin\frac{x-x_k}{2}}\right|+\right.$$

$$\left.+O(1)\sum_{k=1}^{2n-1}\left|\frac{\sin\frac{n-1}{2}(x-x_k)}{\sin\frac{x-x_k}{2}}\right|\right\}+O(n^{-m_\nu})=$$

$$=O(n^{-1-m_\nu})\{O(n^{-1})O(n)O(n\log n)+O(1)O(n\log n)\}+O(n^{-m_\nu})=$$

$$=O(n^{-m_\nu}\log n).$$

We can now state our convergence theorem.

THEOREM 1. *Under the conditions of Theorem A in cases (a) and (c), we have*

$$\|f(x)-L_n(f,x)\|=O(E_m(f)\log n)+ +O\Big(\frac{\log n}{n^\mu}\Big)\sum_{k=0}^{n}(k+1)^{\mu-1}E_k(f), \tag{5.5}$$

where

$$\begin{cases}\mu=\begin{cases}\min(m_2,m_{p+2}) & \text{if } m_{p+1}=0,\\ \min(m_2,m_{p+1}) & \text{if } m_{p+1}>0,\end{cases}\\ m=\dfrac{n}{(\log n)1/\mu},\end{cases} \tag{5.6}$$

and $E_k(f)$ is the error of best trigonometric approximation of order k to $f(x)$.

PROOF. Let $p_m(x)$ be the trigonometric polynomial of best approximation of order m to $f(x)$. Since in the cases concerned the problem of 2-periodic interpolation is regular, we evidently have

$$\begin{aligned} p_m(f,x) = L_n(p_m,x) + \sum_{k=0}^{n-1}\Big\{\sum_{\nu=2}^{p} p_m^{(m_\nu)}(x_{2k})\varrho_\nu(x-x_{2k}) + \\ + \sum_{\nu=p+1}^{p+q}{}' p_m^{(m_\nu)}(x_{2k+1}\varrho_\nu(x-x_{2k})\Big\} \end{aligned} \tag{5.7}$$

where and in what follows the prime on the summation indicates that the term corresponding to $\nu = p+1$ should be omitted if $m_{p+1} = 0$.

According to Lemma 2 [5], we have

$$\|p_m^{(j)}(x)\| = O\Big(\sum_{k=0}^{m}(k+1)^{j-1}E_k(f)\Big).$$

Thus from Lemma 1 and the definitions of m and μ in (5.6) we see from (5.7) that

$$\begin{aligned} \|p_m - L_n(p_m,x)\| &= O\Big(\sum_{\nu=2}^{p+q}{}'\|p_m^{(m_\nu)}\|\Big\|\sum_{k=0}^{2n-1}\varrho_\nu(x-x_k)\Big\|\Big) = \\ &= O\Big(\sum_{\nu=2}^{p+q}{}'\frac{\log n}{n^{m_\nu}}\sum_{k=0}^{m}(k+1)^{m_\nu-1}E_k(f)\Big) = \\ &= O\Big(\sum_{\nu=2}^{p+q}{}'\frac{\log n}{n^{m_\nu}}m^{m_\nu-\mu}\sum_{k=0}^{m}(k+1)^{\mu-1}E_k(f)\Big) = \\ &= O\Big(\sum_{\nu=2}^{p+q}{}'\frac{(\log n)^{2-\frac{m_\nu}{\mu}}}{n^{\mu}}\sum_{k=0}^{m}(k+1)^{\mu-1}E_k(f)\Big) = \\ &= O\Big(\frac{\log n}{n^{\mu}}\sum_{k=0}^{m}(k+1)^{\mu-1}E_k(f)\Big). \end{aligned}$$

Hence it follows that

$$\begin{aligned} \|f - L_n(f)\| &\leqq \|f - p_m\| + \|p_m - L_n(p_m)\| + \|L_n(p_m - f)\| \leqq \\ &\leqq E_m(f) + O\Big(\frac{\log n}{n^{\mu}}\sum_{k=0}^{m}(k+1)^{\mu+1}E_k(f)\Big) + O(E_m(f)\log n). \qquad \square \end{aligned}$$

6. Convergence (continued)

Theorem 1 shows that in order to have convergence in cases (a) and (c) of Theorem A, we have to assume $E_n(f) = o(\frac{1}{\log n})$. However, in some cases, this condition can be dropped. We shall now turn to these cases.

LEMMA 2. *Under the conditions of Theorem A (in cases III, V, and VI), with n even and $m_{p+1} > 0$, we have*

$$\text{(6.1)} \qquad \Big\| \sum_{k=0}^{2n-1} |\varrho_1(x - x_k)| \Big\| = O(1), \quad \Big\| \sum_{k=0}^{2n-1} |\varrho_\nu(x - x_k)| \Big\| = O\Big(\frac{\log n}{n^{m_\nu}}\Big)$$

$$(\nu = 2, \ldots, n-1).$$

PROOF. Set $\alpha_j = \frac{j}{n}$ $(j = 0, 1, \ldots, n-1)$ and

$$\text{(6.2)} \qquad a_{\lambda j}(\nu) := \frac{D_{\nu,\lambda+1}(\alpha_j)}{D(\alpha_j)}, \qquad \begin{matrix} 0 \leqq \alpha_j \leqq 1; \ \nu = 1, \ldots, p+q \\ \lambda = 0, 1, \ldots, 2s+1. \end{matrix}$$

Then from (3.11) we get for $\nu = 1$,

$$\text{(6.3)} \quad \varrho_1(x) = \frac{1}{n}\Big[a_{s,n-1}(1) + 2\sum_{\lambda=0}^{s}\sum_{j=0}^{n-1} a_{\lambda j}(1)\cos((s+1-\lambda)n - j)x\Big] + O\Big(\frac{1}{n}\Big).$$

Since $a_{\lambda j}(\nu)$ $(\lambda = 0, 1, \ldots, 2s+1)$ is an analytic function of α_j in a domain containing the interval $[0,1]$ and since $\alpha_{j+1} - \alpha_{j-1} = O(\frac{1}{n})$, it follows easily by the mean-value theorem that

$$\text{(6.4)} \qquad \begin{matrix} \Delta^2 a_{\lambda j}(\nu) := a_{\lambda,j-1}(\nu) - 2a_{\lambda,j}(\nu) + a_{\lambda,j+1}(\nu) + O\Big(\frac{1}{n^2}\Big) \\ \nu = 1, \ldots, p+q; \quad \lambda = 0, 1, \ldots, 2s+1; \\ j = 1, \ldots, n-2. \end{matrix}$$

From (3.4) we see that

$$\text{(6.5)} \qquad \tilde{D}_{1,\lambda+1}(1) = 0, \quad \lambda \neq s$$

since the cofactor of $(1, \lambda+1)$ term in $\tilde{D}(1)$ will contain a zero column (here we use the fact that $m_{p+1} > 0$). This together with (3.11) yields

$$\text{(6.6)} \qquad \tilde{D}_{1,\lambda+2}(0) = \tilde{D}_{1,2s+1-\lambda}(1) = 0, \quad \lambda \neq s.$$

Therefore we have

$$
\begin{aligned}
&|a_{\lambda,n-2}(1)-2a_{\lambda,n-1}(1)+a_{\lambda+1,0}(1)|\leqq\\
&\leqq|a_{\lambda,n-2}(1)-a_{\lambda,n-1}(1)|+|a_{\lambda,n-1}(1)-a_{\lambda+1,0}(1)|=\\
&=O\Big(\frac{1}{n}\Big)+\Big|\frac{\tilde D_{1,\lambda+1}(\alpha_{n-1})}{\tilde D(\alpha_{n-1})}-\frac{\tilde D_{1,\lambda+2}(0)}{\tilde D(0)}\Big|=\\
&=O\Big(\frac{1}{n}\Big)+\Big|\frac{\tilde D_{1,\lambda+1}(\alpha_{n-1})}{\tilde D(\alpha_{n-1})}\Big|\quad\text{on using (6.6)}\\
&=O\Big(\frac{1}{n}\Big),\quad \lambda\neq s.
\end{aligned}\tag{6.7}
$$

Similarly, we have

$$|a_{\lambda,n-1}(1)-2a_{\lambda+1,0}(1)+a_{\lambda+1,n}(1)|=O\Big(\frac{1}{n}\Big),\quad \lambda\neq s.\tag{6.8}$$

Thus if we write

$$\varrho_1(x)=\frac{1}{n}\Big[\beta_0+2\sum_{j=1}^{(s+1)n}\beta_j\cos jx\Big]+O\Big(\frac{1}{n}\Big),\tag{6.9}$$

(notice that $\beta_{(s+1)n}=a_{0,0}(1)=\frac{\tilde D_{1,1}(0)}{\tilde D(0)}=0$ from (3.5)), then we see from (6.7) and (6.8) that

$$\Delta^2\beta_j=\begin{cases} O\Big(\dfrac{1}{n^2}\Big) & \text{if } j \text{ or } j+1 \text{ are multiples of } n\\ O\Big(\dfrac{1}{n}\Big) & \text{otherwise.}\end{cases}\tag{6.10}$$

We now apply a double Abel summation to (6.9) to obtain

$$
\begin{aligned}
\varrho_1(x)=&\frac{1}{n}\Big[\sum_{j=1}^{(s+1)n-2}\Delta^2\beta_j\Big(\frac{\sin\frac{jx}{2}}{\sin\frac{x}{2}}\Big)^2+\\
&+\Delta\beta_{(s+1)n-1}\Big(\frac{\sin\frac{(s+1)n-2}{2}x}{\sin\frac{x}{2}}\Big)^2\Big]+O\big(\frac{1}{n}\big).
\end{aligned}\tag{6.11}
$$

Since

$$\sum_{k=0}^{2n-1}\Big(\frac{\sin\frac{j(x-x_k)}{2}}{\sin\frac{x-x_k}{2}}\Big)^2=2nj,\quad j=1,2,\dots$$

we see from (6.10) and (6.11) that

$$\sum_{k=0}^{2n-1}|\varrho_1(x-x_k)|=\frac{1}{n}\Big[2n\sum_{j=1}^{(s+1)n-2} j\Delta^2\beta_j+\Delta\beta_{(s+1)n-1}O(n^2)\Big]+O(1)= \tag{6.12}$$
$$=O(1).$$

Now assume that $2\leqq\nu\leqq p+q$ and that m_ν is odd (the case where m_ν is even is similar). Again, from (3.12) we have

$$|\varrho_\nu(x)|\leqq n^{-1-m_\nu}\Big|\sum_{\lambda=0}^{s}\sum_{j=0}^{n-1}a_{\lambda j}(\nu)\sin((s+1-\lambda)n-j)x\Big|+O(n^{-1-m_\nu})=$$
$$=n^{-1-m_\nu}\Big|\sum_{j=1}^{(s+1)n-1}\gamma_j(\nu)\sin jx\Big|+O(n^{-1-m_\nu})$$

where, as in (6.7), we have

$$\Delta_{\gamma j}:=\gamma_{j+1}-\gamma_j=\begin{cases}O\Big(\dfrac{1}{n}\Big) & \text{if } j \text{ or } j+1 \text{ are not multiples of } n,\\ O(1) & \text{otherwise.}\end{cases}$$

Using Abel transform once, we have

$$|\varrho_\nu(x)|\leqq n^{-1-m_\nu}\sum_{j=1}^{(s+1)n-1}|\Delta\gamma_j|\Big|\frac{\sin jx\sin(j+1)x}{\sin\frac{x}{2}}\Big|+$$
$$+|\gamma_{s+1)n}|\Big|\frac{\sin(s+1)nx\sin(s+2)nx}{\sin\frac{x}{2}}\Big|.$$

Since

$$\sum_{k=1}^{2n-1}\Big|\frac{\sin j(x-x_k)\sin(j+1)(x-x_k)}{\sin\frac{x-x_k}{2}}\Big|=O(n\log n),$$

we obtain

$$\sum_{k=0}^{2n-1}|\varrho(x-x_k)|=O(n^{-1-m_\nu})O(n\log n)\sum_{j=1}^{(s+1)n-1}|\Delta(\gamma_j)|= \tag{6.13}$$
$$=O\Big(\frac{\log n}{n^{m_\nu}}\Big).$$

The result follows from (6.12) and (6.13).

Now we are able to prove our main result on the convergence of some 2-periodic interpolation operators.

THEOREM 2. *Under the conditions of Lemma 2 above, if we set*

$$(6.14)\qquad L_n(f,x) := \sum_{k=0}^{n-1} f(x_{2k})\varrho_1(x-x_{2k}),$$

then we have

$$(6.15)\qquad \|f(x)-L_n(f,x)\| = O\Big(\frac{1}{m^\mu}\Big)\sum_{k=0}^{m}(k+1)^{\mu-1}E_k(f),$$

where

$$\mu=\min(m_2,m_{p+1}),\qquad m=\Big[\frac{n}{(\log n)^{1/\mu}}\Big]$$

for all continuous functions $f(x)$, where $E_k(f)$ is the best trigonometric approximation of order k to $f(x)$.

REMARK. (6.15) implies that $\lim_{n\to\infty}\|f(x)-L_n(f,x)\|=0$ and, in particular, if $f(x)\in \mathrm{Lip}\,\alpha$, then

$$\|f(x)-L_n(f;x)\| = O\Big(\frac{(\log n)^{\alpha/\mu}}{n^\mu}\Big).$$

The proof is the same as that of Theorem 1, but with a reference to Lemma 2 instead of Lemma 1.

In connection with Theorem A case (a), we do not have a general convergence theorem similar to that of case (c), because condition (6.6) is not always guaranteed. Nevertheless, in some special cases, it is still possible as in the following example.

EXAMPLE. Let $p=2$, $q=1$ in the case of $(0,m_2;m_3)$ interpolation where m_2 is odd and $m_3\geqq 2$ is even. From the general formula (2.12), we obtain

$$\varrho_1(x)=\frac{1}{n}\Big[a_{1,r}(1)+2\sum_{j=1}^{r}a_{1,r-j}(1)\cos jx+2\sum_{j=0}^{n-1}a_{0,j}(1)\cos(n+r-j)x\Big].$$

From (2.5), we see that

$$D_{1,1}\Big(\frac{1}{2}\Big)=\frac{3^{m_3}+3^{m_2}}{2^{m_1+m_2}}=D_{1,2}\Big(-\frac{1}{2}\Big)$$

so that $a_{1,0}(1)=a_{0,n-1}(0)$ and the method of proof of Lemma 2 works, yielding the following estimate for the operator (5.1):

$$\|f(x)-L_n(f,x)\| = O\Big(\frac{\log n}{n^\mu}\Big)\sum_{k=0}^{m}(k+1)^{\mu-1}E_k(f),$$

where $\mu = \min(m_2, m_3)$ and $m = [\frac{n}{(\log n)^\mu}]$.

The condition $m_3 > 0$ in this example cannot be dropped, as the example of (0,1;0) shows. Here the fundamental polynomials are easy to calculate. Indeed we have

$$\varrho_1(x) = \frac{\sin nx \sin \frac{nx}{2} \cos \frac{x}{2}}{2n^2 \sin^2 \frac{x}{2}}, \quad \varrho_2(x) = \frac{\sin nx \sin \frac{nx}{2}}{n \sin \frac{x}{2}},$$

$$\varrho_3(x) = -\frac{\sin^2 \frac{nx}{2} \cos \frac{nx}{2}}{n \sin \frac{x-x_1}{2}}.$$

We see that

$$\left\| \sum_{k=0}^{n-1} |\varrho_1(x - x_{2k})| \right\| = O(1), \quad \left\| \sum_{k=0}^{n-1} |\varrho_2(x - x_{2k})| \right\| = O\left(\frac{\log n}{n}\right)$$

but

$$\begin{aligned} \left\| \sum_{k=0}^{n-1} |\varrho_3(x - x_{2k})| \right\| &\geqq \sum_{k=0}^{n-1} \left| \varrho_3\left(\frac{\pi}{2n} - x_{2k}\right) \right| \\ &\geqq \frac{1}{n} \sin^2 \frac{\pi}{4} \cos \frac{\pi}{4} \sum_{k=0}^{n-1} \frac{1}{\sin \frac{4k+1}{2n}\pi} \\ &\geqq c \log n. \end{aligned}$$

Thus in the case of (0,1;0), the interpolant

$$L_n(f, x) := \sum_{k=0}^{n-1} f(x_{2k}) \varrho_1(x - x_{2k}) + \sum_{k=0}^{n-1} f(x_{2k+1}) \varrho_3(x - x_{2k})$$

cannot converge uniformly for all continuous 2π-periodic functions.

7. Conclusion

Unfortunately, we do not have estimates for the fundamental polynomials in cases III to VI in Theorem A when n is odd.

Finally, we mention that there is an error in the formulation of Theorem 3 in [4]. In the notation of the present paper there we considered the problem of $(0; m_1)$-interpolation with $p = 1$, $q = 1$. It was proved there that if m_1 is odd, then the problem of $(0; m_1)$-interpolation on the nodes $x_k = \frac{k\pi}{n}$, $k = 0, 1, \ldots, 2n-1$ is regular if and only if n is odd and $\varepsilon = 1$. Furthermore if m_1 is even, then the necessary and sufficient condition for regularity is $\varepsilon = 0$ and where n could be even or odd. The correct statement of Theorem 3 in [4] is then as follows:

Let m_1, n and ε satisfy either of the conditions listed above. Then for any $f(x) \in C_{2\pi}$, we have

$$\left\| f(x) - \sum_{j=0}^{n-1} f(x_{2j}) \varrho_1(x - x_{2j}) \right\| =$$

$$= O\left(n^{\frac{1-(-1)^{m_1}}{2}} E_{[m/4]}(f) + n^{-m_1} \sum_{k=0}^{m} (k+1)^{m_1 - 1} E_k(f) \right).$$

We mention that this statement when m_1 is odd is a special case of Theorem A case IV, i.e., in this case the relation

$$\left\| \sum_{k=0}^{2n-1} |\varrho_1(x - x_k)| \right\| = O(n)$$

is proved. We suspect that this latter relation holds in all cases IV to VI (n odd).

REFERENCES

[1] LORENTZ, G. G., JETTER, K. and RIEMENSCHNEIDER, S. D., *Birkhoff interpolation*, Encyclopedia of Mathematics and its Applications, **19** (Chapter 2, Sec. 2.4 and Chapter 11), Addison-Wesley Publ. Co., Reading, Mass., 1983. *MR* **84g**:41002

[2] SHARMA, A., SZABADOS, J. and VARGA, R. S., Some 2-periodic trigonometric interpolation problems on equidistant nodes, *Analysis* **11** (1991), 165–190.

[3] SHARMA, A. and VARGA, R. S., On a particular 2-periodic lacunary trigonometric interpolation problem on equidistant nodes, *Resultate Math.* **16** (1989), 383–404. *MR* **91c**:42005

[4] SHARMA, A., SZABADOS, J. and VARGA, R. S., 2-periodic lacunary trigonometric interpolation: the $(0; M)$ case, *Constructive Theory of Functions '87* (Varna, 1987), Publishing House of the Bulgarian Academy of Sciences, Sofia, 1988, 420–427. *MR* **90e**:42010

[5] SZABADOS, J., On the rate of convergence of a lacunary trigonometric interpolation process, *Acta Math. Hungar.* **47** (1986), 361–370. *MR* **87k**:42006

(Received June 14, 1991)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALBERTA
EDMONTON, ALBERTA
T6G 2G1
CANADA

MTA MATEMATIKAI KUTATÓINTÉZETE
POSTAFIÓK 127
H–1364 BUDAPEST
HUNGARY

DEPARTMENT OF MATHEMATICS
KENT STATE UNIVERSITY
KENT, OH 44242
U.S.A.