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Generalized Ultrametric Matrices — a Class of Inverse *M*-Matrices

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ABSTRACT

Recently, Martínez, Michon, and San Martín introduced the new class of (symmetric) *strictly ultrametric matrices*. They proved that the inverse of a strictly ultrametric matrix is a strictly row and strictly column diagonally dominant Stieltjes matrix. Here, we generalize their result by introducing a class of nonsymmetric matrices, called *generalized ultrametric matrices*. We give a necessary and sufficient condition for the regularity of these matrices and prove that the inverse of a nonsingular generalized ultrametric matrix is a row and column diagonally dominant M-matrix. We establish that a nonnegative matrix is a generalized ultrametric matrix if and only if the matrix is a certain sum of at most rank-two matrices. Moreover, we give a characterization of generalized ultrametric matrices, based on weighted trees. The entries of generalized ultrametric matrices then arise as certain "distances" between the leaves and the root of the tree.

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1. INTRODUCTION

One of the most beautiful properties of a nonsingular M-matrix is that its inverse is a nonnegative matrix. However, the converse of this result is not in general true, i.e., the inverse of a nonsingular nonnegative matrix is not in general an M-matrix. First, Thomas Markham [7] established in 1972 a sufficient condition for a nonnegative symmetric matrix to be an inverse of a Stieltjes matrix (a nonsingular symmetric M-matrix). He introduced a class of nonnegative symmetric matrices called *matrices of type D*, which are inverses of Stieltjes matrices. Since his paper appeared, several authors have studied this so-called inverse-M-matrix problem [3, 5, 6, 12]. However, only a few sufficient conditions were developed. An overview of the inverse-M-matrix problem is given by C. R. Johnson [3].

Recently, Martínez, Michon, and San Martín [8] introduced a new class of symmetric matrices. They proved in [8] that the inverse of a so-called *strictly ultrametric matrix* is a strictly row and strictly column diagonally dominant Stieltjes matrix. Stimulated by the beauty of this result, Nabben and Varga [10] gave a short proof of it using more familiar tools from linear algebra. Moreover, in [10] a characterization of this class of strictly ultrametric matrices was given. This characterization and the use of a rank-one update, which was also used in [10], are the main tools for certain generalizations of the result in [8]. A first generalization was given in [11], where the authors introduced the class of *symmetric ultrametric matrices*. They showed that the inverse of a nonsingular symmetric ultrametric matrix is a (not necessarily strictly) row and column diagonally dominant Stieltjes matrix. Even more, it turns out that the class of matrices of type D, introduced by Markham [7], is contained in the class of symmetric ultrametric matrices; see Section 2.

In this paper we generalize the result of [8] by introducing a class of nonsymmetric matrices, called *generalized ultrametric matrices*. We give here a necessary and sufficient condition for the regularity of these matrices and prove that the inverse of a nonsingular generalized ultrametric matrix is a row and column diagonally dominant M-matrix. Moreover, we establish that a nonnegative matrix is a generalized ultrametric matrix if and only if the matrix is a certain sum of at most rank-two matrices; see Section 3.

Another aim of this paper is to show how generalized ultrametric matrices arise and how they can be characterized. Therefore, we establish another characterization of this class of matrices based on weighted graphs. The entries of generalized ultrametric matrices then arise as "distances" between the leaves and the root of the tree; see Example 2.2 and Theorem 3.3.

After we obtained some of the result in this paper, we learned that McDonald, Neumann, Schneider and Tsatsomeros [9] have considered similar classes of matrices, and that they have obtained some overlapping results.

2. GENERALIZED ULTRAMETRIC MATRICES AND EXAMPLES

In this section, we derive the class of generalized ultrametric matrices and give some examples of these matrices. However, we start by describing the subclasses mentioned above. At the end of this section, we introduce further needed notation.

Martínez, Michon, and San Martín [8] introduced the following class of strictly ultrametric matrices:

DEFINITION 2.1. A matrix $A = [a_{i,j}]$ in $\boldsymbol{R}^{n,n}$ is a strictly ultrametric matrix if

(i) A is symmetric and has nonnegative entries;
(ii) $a_{i,j} \geqslant \min\{a_{i,k};\ a_{k,j}\}$ for all $i, j, k \in N$ (where $N := \{1, 2, \dots, n\}$);
(iii) $a_{i,i} > \max\{a_{i,k} : k \in N \setminus \{i\}\}$ for all $i \in N$,

where, if $n = 1$, (iii) is interpreted as $a_{1,1} > 0$.

They proved in [8] that the inverse of a strictly ultrametric matrix is a strictly (row and column) diagonally dominant Stieltjes matrix. Moreover, they established that the sets consisting of the zero entries of a strictly ultrametric matrix and that of its inverse are exactly the same.

In [11], the authors defined the class of symmetric preultrametric matrices, where equality in (iii) of Definition 2.1 is allowed. Adding a certain regularity condition, the class of symmetric ultrametric matrices was defined in [11]. The inverse of such a matrix becomes a (row and column) diagonally dominant Stieltjes matrix. This class of symmetric ultrametric matrices contains the class of type-D matrices introduced by Markham [7]:

DEFINITION 2.2. A symmetric matrix $A = [a_{i,j}] \in \boldsymbol{R}^{n,n}$ is of type D if there exist real numbers $\{\alpha_i\}_{i=1}^{n}$ such that

$$a_{i,j} = \begin{cases} \alpha_i & \text{if } i \leqslant j, \\ \alpha_j & \text{if } i > j, \end{cases} \qquad \text{where} \quad \alpha_n > \alpha_{n-1} > \cdots > \alpha_1.$$

Markham proved that with $\alpha_1 > 0$, a matrix of type D is nonsingular and its inverse is a (symmetric) tridiagonal Stieltjes matrix.

On comparing Definitions 2.1 and 2.2, we see that if $A = [a_{i,j}]$ in $\boldsymbol{R}^{n,n}$ is a type D matrix (with $\alpha_1 > 0$), then A certainly satisfies (i) and (ii) of Definition 2.1, but the diagonal entries $a_{i,i}$ $(1 \leqslant i < n)$ of A fail to satisfy (iii) of Definition 2.1. However, while a type-D matrix (with $\alpha_1 > 0$) is not a strictly ultrametric matrix, it can be verified to be a symmetric ultrametric matrix in the sense of [11].

Considering nonsymmetric matrices is another way to generalize the result in [8]. This leads to the more general problem of determining which nonsingular nonnegative matrices in $\boldsymbol{R}^{n,n}$ have inverses which are M-matrices. Before we establish our sufficient condition, we mention a necessary condition, due to Fiedler and Pták [1], which states that the entries of the inverse $A = [a_{i,j}]$ of a strictly row diagonally dominant M-matrix satisfy

$$a_{i,i} > \max\{a_{k,i} : k \in N \setminus \{i\}\} \qquad (\text{all } i \in N).$$

However, Example 2.1 below shows that the inverse of a matrix A, which satisfies

$$A \text{ has nonnegative entries;} \tag{2.1a}$$

$$a_{i,j} \geqslant \min\{a_{i,k}, a_{k,j}\} \quad \text{for all} \quad i, j, k \in N, \tag{2.1b}$$

$$a_{i,i} > \max\{a_{i,k} : k \in N \setminus \{i\}\} \quad \text{for all} \quad i \in N, \tag{2.1c}$$

$$a_{i,i} > \max\{a_{k,i} : k \in N \setminus \{i\}\} \quad \text{for all} \quad i \in N, \tag{2.1d}$$

where, if $n = 1$, (2.1c) and (2.1d) are interpreted as $a_{1,1} > 0$, is *not* in general an M-matrix.

EXAMPLE 2.1. Consider the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \qquad \text{where} \quad A^{-1} = \frac{1}{64}\begin{bmatrix} 24 & -8 & 1 & -11 \\ -8 & 24 & -11 & -7 \\ 0 & 0 & 24 & -8 \\ 0 & 0 & -8 & 24 \end{bmatrix}.$$

It can be verified that A satisfies (2.1), but obviously A^{-1} is not an M-matrix, since its (1,3) element is not nonpositive. Thus, an additional property of $A = [a_{i,j}]$ must be added to the assumptions of (2.1) to ensure that A^{-1} is an M-matrix. This additional property can be motivated from the symmetric case, using the following observation:

OBSERVATION 2.1. *Let $A \in \boldsymbol{R}^{n,n}$ be a symmetric nonnegative matrix. Then the following are equivalent*:

(i) *for each triple $\{q, t, s\}$ in N^3, there exists a reordering $\{i, j, k\}$ of the elements of this triple such that*

$$a_{i,j} \geqslant a_{i,k} = a_{j,k};$$

(ii) $a_{i,j} \geqslant \min\{a_{i,k}, a_{k,j}\}$ *for all i, k, $j \in N$.*

Proof. Since (ii) is formulated for all $i, j, k \in N$, (ii) is equivalent with

$$\begin{aligned} a_{i,j} &\geqslant \min\{a_{i,k}; a_{k,j}\}, \\ a_{i,k} &\geqslant \min\{a_{i,j}; a_{j,k}\}, \\ a_{k,j} &\geqslant \min\{a_{k,i}; a_{i,j}\} \end{aligned} \tag{2.2}$$

for a triple $\{i, j, k\} \in N^3$. Thus, if there is a reordering $\{i, j, k\}$ of the elements of each triple $\{q, t, s\}$ such that $a_{i,j} \geqslant a_{i,k} = a_{k,j}$ holds, then, using the symmetry of A, it can be verified that (2.2) holds. Hence, (i) implies (ii).

On the other hand, if (ii) or (2.2) holds, let $\{q, s, t\}$ be any triple in N^3; from the three nonnegative numbers, $a_{q,s}$, $a_{q,t}$, and $a_{s,t}$ choose the largest, say $a_{s,t}$, and set $i := s$, $j := t$, and $k := q$. By definition, $a_{i,j} \geqslant a_{k,i}$ and $a_{i,j} \geqslant a_{k,j}$. But on using (2.2) and the symmetry of A, it follows that $a_{i,k} \geqslant \min\{a_{i,j}; a_{j,k}\} = a_{j,k} \geqslant \min\{a_{j,i}; a_{i,k}\} = a_{i,k}$, i.e., $a_{i,k} = a_{j,k}$, and (ii) implies (i). ■

Using this observation, we make the following definition:

DEFINITION 2.3. A matrix $A = [a_{i,j}] \in \boldsymbol{R}^{n,n}$ is called a *generalized ultrametric matrix* if

(i) A has nonnegative entries;

(ii) $a_{i,j} \geqslant \min\{a_{i,k}; a_{k,j}\}$ for all $i, k, j \in N$;

(iii) $a_{i,i} \geqslant \max\{a_{i,k} : k \in N \setminus \{i\}\}$ for all $i \in N$;

(iv) $a_{i,i} \geqslant \max\{a_{k,i} : k \in N \setminus \{i\}\}$ for all $i \in N$;

(v) each triple $\{q, s, t\}$ in N^3 can be reordered as a triple $\{i, j, k\}$ such that

(v.i) $a_{j,k} = a_{i,k}$ and $a_{k,j} = a_{k,i}$,
(v.ii) $\max\{a_{i,j}; a_{j,i}\} \geqslant \max\{a_{i,k}; a_{k,i}\}$,

where, if $n = 1$, (iii) and (iv) are interpreted as $a_{1,1} \geqslant 0$. A matrix A is called a strictly generalized ultrametric matrix if the above conditions hold with strict inequality in (iii) and (iv).

If A is symmetric, then using Observation 2.1, conditions (ii) and (v) of Definition 2.3 are equivalent. Thus, the set of generalized ultrametric matrices includes all classes of matrices mentioned above. Moreover, if a (not necessarily symmetric) matrix A satisfies (i), (iii), (iv), and (v) of Definition 2.3, then condition (ii) is equivalent to the statement:

the same ordering used in (v) satisfies

$$\min\{a_{i,j};\, a_{j,i}\} \geqslant \min\{a_{i,k};\, a_{k,i}\}. \tag{2.3}$$

Since the definition above of a generalized ultrametric matrix may seem nonintuitive, we give below a description of generalized ultrametric matrices using weighted graphs, which may be more transparent.

Recall that a rooted tree $G = (V, E)$ is a connected undirected graph without cycles, having a root and consisting of the set of vertices V and the set of edges $E \subset \{(x, y): x, y \in V, x \neq y\}$. Let $w \in V$ denote the *root* of the tree, and let $B \subset V$ denote the set of its *leaves*, with cardinality $|B| = n$. To each edge (x, y) of E, we assign two nonnegative numbers:

$$l(x, y) \geqslant 0 \text{ and } r(x, y) \geqslant 0 \qquad [\text{all } (x, y) \in E],$$

and l and r are called *weighting functions* for the rooted tree. Then, for any $b \in B$, let $p(b)$ denote the consecutive distinct edges $\{(v_i, v_{i+1})\}_{i=0}^{m-1}$, with $b = v_0$ and $v_m = w$, which form a path connecting the leaf b to the root w. For any b and $\tilde{b}$ in B, set

$$\begin{aligned} d_l(b, \tilde{b}) &:= \sum_{p(b)\cap p(\tilde{b})} l(v_i, v_{i+1}), \\ d_r(b, \tilde{b}) &:= \sum_{p(b)\cap p(\tilde{b})} r(v_i, v_{i+1}), \\ d(b, b) &:= \max\{d_l(b, b);\, d_r(b, b)\}, \end{aligned} \tag{2.4}$$

where $p(b) \cap p(\tilde{b})$ denotes the common edges of these paths from the leaves b and $\tilde{b}$ to the root w. If we number the leaves from 1 to n and define the matrix $A = [a_{i,j}] \in \boldsymbol{R}^{n,n}$ by

$$a_{i,j} := \begin{cases} d_r(i, j) & \text{for } i < j, \\ d(i, j) & \text{for } i = j, \\ d_l(i, j) & \text{for } i > j, \end{cases} \tag{2.5}$$

we obtain, for every rooted tree and for all weighting functions l and r defined on this rooted tree, a generalized ultrametric matrix; see Theorem 3.3. Conversely, we show in Theorem 3.3 that, for a given generalized ultrametric matrix A, there exist a rooted tree and weighting functions l and r such that the entries of A are given as indicated in (2.4) and (2.5).

EXAMPLE 2.2. Consider the rooted tree in Figure 1 with leaves (1), ..., (10) and weights $r(i, j)$ and $l(i, j)$, shown respectively on the right and left sides of the corresponding edge (i, j).

From the tree we see that $p(2) \cap p(3) = (\alpha, \beta) \cup (\beta, w)$, so that from the definitions of (2.5), the associated matrix $A = [a_{i,j}] \in \boldsymbol{R}^{10,10}$ below has entries $a_{2,3} = 3$ and $a_{3,2} = 9$:

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 11 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 9 & 12 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 7 & 5 & 5 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 5 & 7 & 5 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 5 & 5 & 8 & 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 & 2 & 2 & 7 & 3 & 3 & 3 \\ 0 & 2 & 2 & 2 & 2 & 2 & 6 & 9 & 3 & 3 \\ 0 & 2 & 2 & 2 & 2 & 2 & 6 & 6 & 9 & 7 \\ 0 & 2 & 2 & 2 & 2 & 2 & 6 & 6 & 8 & 9 \end{bmatrix}.$$

In the next sections, we use the following

NOTATION. By ξ_n, we denote the vector $[1, \dots, 1]^T \in \boldsymbol{R}^n$. For $A = [a_{i,j}] \in \boldsymbol{R}^{n,n}$, we set

$$\begin{aligned} \tau(A) &:= \min\{a_{i,j} : i, j \in N\}, \\ \omega(A) &:= \min\{a_{j,i} : a_{i,j} = \tau(A)\}, \\ \delta(A) &:= \omega(A) - \tau(A), \\ \mu(A) &:= \max\{a_{i,j} : i, j \in N\}. \end{aligned} \tag{2.6}$$

If A is symmetric, we note that $\tau(A) = \omega(A)$.

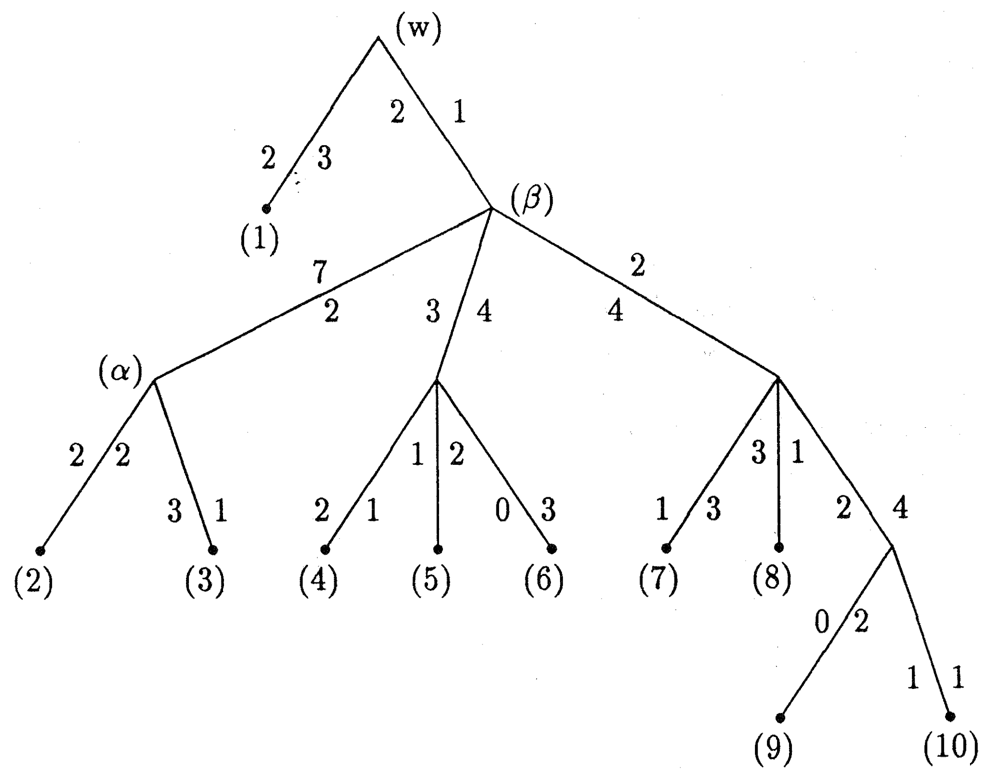

FIG. 1.

3. CHARACTERIZATIONS AND THE INVERSE OF GENERALIZED ULTRAMETRIC MATRICES

We begin this section with the following theorem which describes the nested block structure of a generalized ultrametric matrix. Moreover, Theorem 3.1 indicates a way to construct such a matrix. This theorem, which was already given in [10] for the symmetric case, is one of main tools for considering generalized ultrametric matrices.

THEOREM 3.1. *Let $A = [a_{i,j}] \in \boldsymbol{R}^{n,n}$, $n > 1$, be a generalized ultrametric matrix. Then there exist a positive integer r, with $1 \leqslant r < n$, and a permutation matrix $P \in \boldsymbol{R}^{n,n}$ such that*

$$PAP^T = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} := \begin{bmatrix} C & O \\ \delta(A)\xi_{n-r}\xi_r^T & D \end{bmatrix} + \tau(A)\xi_n\xi_n^T, \quad (3.1)$$

where $C \in \boldsymbol{R}^{r,r}$ and $D \in \boldsymbol{R}^{n-r,n-r}$ are generalized ultrametric matrices and

$$\delta(A) = \omega(A) - \tau(A).$$

Moreover,

$$\omega(A_{1,1}) \geqslant \omega(A) \quad \text{and} \quad \omega(A_{2,2}) \geqslant \omega(A). \tag{3.2}$$

Conversely, if $C \in \boldsymbol{R}^{r,r}$ *and if* $D \in \boldsymbol{R}^{n-r,n-r}$, *with* $1 \leqslant r < n$, *are generalized ultrametric matrices and if* $0 \leqslant \delta \leqslant \omega(C)$ *and if* $0 \leqslant \delta \leqslant \omega(D)$, *then*

$$\begin{bmatrix} C & O \\ \delta \xi_{n-r} \xi_r^T & D \end{bmatrix} + \tau \xi_n \xi_n^T$$

is a generalized ultrametric matrix for each nonnegative real τ.

Proof. First, suppose that $a_{j,j} = \tau(A)$ for some $j \in N$. From (iii) and (iv) of Definition 2.3, all entries in the jth row and jth column of A are $\tau(A)$. By a suitable permutation of indices, we may assume that $j = 1$, and A can then be expressed, in partitioned form, as

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} = \begin{bmatrix} O & O \\ O & D \end{bmatrix} + \tau(A) \xi_n \xi_n^T, \tag{3.3}$$

where $A_{1,1} \in \boldsymbol{R}^{1,1}$ and $A_{2,2} \in \boldsymbol{R}^{n-1,n-1}$. In this case, we see from (2.6) that $\omega(A) = \tau(A)$, so that $\delta(A) = 0$. Then (3.3) has the desired form of (3.1) with $C := O$, and, as $\omega(A_{1,1}) = \tau(A) = \omega(A)$ and $\omega(A_{2,2}) \geqslant \tau(A) = \omega(A)$, the inequalities of (3.2) are then also valid. Moreover, since (from Definition 2.3) any principal submatrix of a generalized ultrametric matrix is again a generalized ultrametric matrix, it follows that the block diagonal submatrices $C = O$ and D in (3.3) are generalized ultrametric matrices. Thus, we may assume, without loss of generality in what follows, that $a_{j,j} > \tau(A)$ for all $j \in N$.

With $a_{j,j} > \tau(A)$ for all $j \in N$, a smallest entry of the nonnegative matrix A must be a nondiagonal entry, and, as in Theorem 1.2 of [10], we may assume, again by a suitable permutation of indices, that $a_{1,n} = \tau(A)$. Then, define

$$S := \{ j \in N : a_{1,j} = \tau(A) \} \quad \text{and} \quad T := \{ j \in N : a_{1,j} > \tau(A) \}.$$

Since $n \in S$ and $1 \in T$, S and T are disjoint nonempty subsets of N with $S \cup T = N$. Again, by a suitable permutation of indices, we may further assume that

$$T = \{1, 2, \ldots, \tilde{r}\} \text{ and } S = \{\tilde{r}+1, \tilde{r}+2, \ldots, n\}, \qquad \text{where} \quad 1 \leqslant \tilde{r} < n.$$

From (ii) of Definition 2.3, we have, for all $j \in T$ and $k \in S$, that

$$\tau(A) = a_{1,k} \geqslant \min\{a_{1,j}; a_{j,k}\}.$$

But since $a_{1,j} > \tau(A)$ for all $j \in T$, the above relation yields

$$a_{j,k} = \tau(A) \qquad \text{for all} \quad j \in T \text{ and } k \in S.$$

Moreover, we may assume that $a_{i,j} > \tau(A)$ for all i and j in T, i.e., $\tau(A_{1,1}) > \tau(A)$, for if there were an entry $a_{i,j} = \tau(A)$ for i and j in T with $i \neq j$, a suitable permutation of indices in T would bring this entry to the first row of A, and leave $a_{1,j} = \tau(A)$, for all $j \in S$, unchanged. Hence, we may assume that $\tilde{r}$ is *minimal* with respect to all such permutations. Next, we decompose the set S into two disjoint subsets R and Q such that

$$a_{i,1} = \min\{a_{j,1} : j \in S\} \qquad \text{for all} \quad i \in Q$$

and

$$a_{i,1} < a_{s,1} \qquad \text{for all} \quad i \in Q, \quad s \in R.$$

Thus, we have $S = R \cup Q$, where $R = \emptyset$ is possible. Again using suitable permutations, we can write

$$R = \{\tilde{r} + 1, \dots, p\} \text{ and } Q = \{p+1, p+2, \dots, n\}, \qquad \text{where} \quad \tilde{r} \leqslant p < n.$$

In other words, the elements of the partial first column of A, namely,

$$[a_{\tilde{r}+1,1}, a_{\tilde{r}+2,1}, \dots, a_{n,1}]^T,$$

are reordered (by a permutation of the rows and columns from S) so that the minimal (equal) entries (corresponding to the set Q) all appear at the *bottom* of this partial first column of A.

Using these sets of indices T, R, and Q, we obtain the following partition of A:

$$A = \begin{bmatrix} \tilde{A}_{1,1} & \tilde{A}_{1,2} & \tilde{A}_{1,3} \\ \tilde{A}_{2,1} & \tilde{A}_{2,2} & \tilde{A}_{2,3} \\ \tilde{A}_{3,1} & \tilde{A}_{3,2} & \tilde{A}_{3,3} \end{bmatrix}. \tag{3.4}$$

In the following we shall show, for $R \neq \emptyset$ and for $R = \emptyset$, that the partition of A in (3.4), defined by the two distinct sets $T \cup R$ and Q, satisfies (3.1) and (3.2) of Theorem 3.1.

Now, we consider condition (v) of Definition 2.3 and its implications and restrictions on the structure of the matrix blocks given in (3.4).

(a) If $T = \{1\}$, i.e., $a_{1,1} > \tau(A)$, we obtain, using the definitions of the sets R and Q, that

$$\tilde{A}_{3,1} = \gamma \xi_{n-p}, \qquad \text{where} \quad \tau(A) \leqslant \gamma < \tau(\tilde{A}_{2,1}).$$

(b) Consider any triple $\{q, s, t\}$ in N^3 with, say, q and t in T and s in S. Since, by construction, $\tau(\tilde{A}_{1,1}) > \tau(A)$, it is evident from the second part of (v.i) of Definition 2.3 that the reordering $\{i, j, k\}$ of this triple is such that $k \in S$. But then, from the second part of (v.i) of Definition 2.3, it follows that the columns of

$$\begin{bmatrix} \tilde{A}_{2,1} \\ \tilde{A}_{3,1} \end{bmatrix}$$

are all necessarily equal. Hence,

$$\tilde{A}_{3,1} = \gamma \xi_{n-p} \xi_r^T, \qquad \text{where} \quad \tau(A) \leqslant \gamma < \tau(\tilde{A}_{2,1}). \tag{3.5}$$

Moreover, it can be verified from (v.ii) of Definition 2.3 that

$$\gamma \leqslant \omega(\tilde{A}_{1,1}). \tag{3.6}$$

(c) The case where the triple $\{q, s, t\}$ in N^3 has all entries in T yields no further information about the structure of the matrix blocks of (3.4). However, $\tilde{A}_{1,1}$ must satisfy (v) of Definition 2.3.

(d) Since $n \in S$, then $S \neq \emptyset$. However, if $S = \{n\}$, then $R = \emptyset$. In this case, the desired conclusion of (3.1) of Theorem 3.1 follows from (3.5) of case (b).

(e) Next, consider any triple $\{q, s, t\}$ in N^3 with two elements in S and one in T. Here, $\tau(\tilde{A}_{1,1}) > \tau(A)$ is no restriction for the ordering of $\{q, s, t\}$. [If $R = \emptyset$, we have already established the representation (3.1).]

(e1) In the following, we consider any triple $\{q, s, t\}$ in N^3 with one element in T, one in R, and one in Q. A reordering $\{i, j, k\}$ of the triple having $k \in T$ is not possible, since we would obtain from (v.i) of Definition 2.3 that each entry of $\tilde{A}_{2,1}$ is equal to each entry, $\gamma(A)$, of $\tilde{A}_{3,1}$, a contradiction to our previous constructions. Similarly, a reordering of the triple having $k \in R$ is also not possible, since (v.i) of Definition 2.3 would give that all entries of $\tilde{A}_{3,2}$ are $\tau(A)$, and that all entries of $\tilde{A}_{2,1}$ and $\tilde{A}_{2,3}$ are equal. But then, applying (v.ii) of Definition 2.3 would give $\gamma(A) \geqslant \mu(\tilde{A}_{2,1})$, again a contradiction to our previous constructions. However, the remaining orderings $\{i, j, k\}$ with $k \in Q$ imply, from (v.i) of Definition 2.3, that $\tilde{A}_{2,3}$ is a constant block with constant $\tau(A)$, and $\tilde{A}_{3,2}$ is a constant block with constant γ. With these restrictions, (v.ii) of Definition 2.3 is then satisfied.

(e2) For the case that the triple $\{q, s, t\}$ in N^3 has two elements in Q and one in T (which includes the case $R = \emptyset$) and for the case when this triple has two elements in R and one in T, (v.i) of Definition 2.3 gives no restriction on the reordering of $\{q, s, t\}$. However, for each ordering, (v.ii) of Definition 2.3 yields

$$\gamma \leqslant \omega\left(\tilde{A}_{2,2}\right) \quad \text{and} \quad \gamma \leqslant \omega\left(\tilde{A}_{3,3}\right); \tag{3.7}$$

whence combining (3.7) with inequalities above gives

$$\gamma = \omega(A). \tag{3.8}$$

(f) The case where the triple $\{q, s, t\}$ in N^3 has elements all in Q, gives no added structure to the blocks of (3.3). However, $\tilde{A}_{3,3} \in \boldsymbol{R}^{n-p,n-p}$ must satisfy (v) of Definition 2.3.

The above discussion shows that the partitioned matrix of (3.4) then has the form

$$A = \left[\begin{array}{cc|c} \tilde{A}_{1,1} & \tau(A) & \tau(A) \\ \tilde{A}_{2,1} & \tilde{A}_{2,2} & \tau(A) \\ \hline \gamma & \gamma & \tilde{A}_{3,3} \end{array}\right] \quad \left(=: \left[\begin{array}{c|c} B_{1,1} & \tau(A) \\ \hline \gamma & B_{2,2} \end{array}\right]\right),$$

where constant block submatrices are indicated by their constant values. Then, on forming the sets $T \cup R$ and Q to define a repartitioning of A, the matrix on the right above is obtained, and this matrix has the desired form of (3.1) of Theorem 3.1. Moreover, using (3.5)–(3.8), it follows that

$$\omega(B_{1,1}) \geqslant \gamma \quad \text{and} \quad \omega(B_{2,2}) \geqslant \gamma,$$

which gives, since $\gamma = \omega(A)$, the desired result of (3.2) of Theorem 3.1. (Note again that the principal submatrices $B_{1,1}$ and $B_{2,2}$ of A are automatically generalized ultrametric matrices, since A is.)

Conversely, if $C \in \boldsymbol{R}^{r,r}$ and if $D \in \boldsymbol{R}^{n-r,n-r}$, $1 \leqslant r < n$, are generalized ultrametric matrices and if $0 \leqslant \delta \leqslant \omega(C)$ and $0 \leqslant \delta \leqslant \omega(D)$, then it is easily verified from Definition 2.3 that the matrix

$$\begin{bmatrix} C & O \\ \delta \xi_{n-r} \xi_r^T & D \end{bmatrix} + \tau \xi_n \xi_n^T$$

is a generalized ultrametric matrix for each nonnegative real τ. ■

Since the matrices $C \in \boldsymbol{R}^{r,r}$ and $D \in \boldsymbol{R}^{n-r,n-r}$ are again generalized ultrametric matrices, the reduction process of Theorem 3.1 can be applied to each of the matrices C and D, provided that $r > 1$ and $n - r > 1$. This reduction process stops with 1×1 matrices. To further describe this process, we change the notation in (3.1) by writing

$$\tau_1 := \tau(A), \qquad \delta_1 := \delta(A),$$

$$u_1 := \xi_n, \qquad v_1 := \left(0, \dots, 0, \xi_{n-r}^T\right)^T \in \boldsymbol{R}^n, \qquad w_1 := \left(\xi_r^T, 0, \dots, 0\right)^T \in \boldsymbol{R}^n.$$

Then (3.1) of Theorem 3.1 gives the following representation of a generalized ultrametric matrix A in $\boldsymbol{R}^{n,n}$:

$$PAP^T = \begin{bmatrix} C & O \\ O & D \end{bmatrix} + \tau_1 u_1 u_1^T + \delta_1 v_1 w_1^T.$$

Using this notation for all submatrices, the procedure (or, in other words, the nested block structure of a generalized ultrametric matrix) can be expressed by a binary rooted tree with n leaves and $2n - 1$ vertices (where, in a binary rooted tree, each vertex has one "incoming" edge, and each vertex, not a leaf, has exactly two "outgoing" edges). This binary rooted tree determines the vectors u_i, v_i, and w_i and the scalars τ_i and δ_i, $i = 1, \ldots, 2n-1$. The following example illustrates this process.

EXAMPLE 3.1. The matrix A with

$$A = \begin{bmatrix} 5 & 1 & 1 & 1 & 1 \\ 5 & 5 & 1 & 1 & 1 \\ 3 & 3 & 4 & 1 & 1 \\ 3 & 3 & 4 & 5 & 4 \\ 3 & 3 & 4 & 5 & 5 \end{bmatrix}$$

leads to the tree in Figure 2, with

$$\begin{aligned}
u_1 &:= (1,1,1,1,1)^T, & v_1 &:= (0,0,1,1,1)^T, & w_1^T &:= (1,1,0,0,0), \\
u_2 &:= (1,1,0,0,0)^T, & v_2 &:= (0,1,0,0,0)^T, & w_2^T &:= (1,0,0,0,0), \\
u_3 &:= (0,0,1,1,1)^T, & v_3 &:= (0,0,0,1,1)^T, & w_3^T &:= (0,0,1,0,0), \\
u_4 &:= (1,0,0,0,0)^T, & v_4 &:= (0,0,0,0,0)^T, & w_4^T &:= (0,0,0,0,0), \\
u_5 &:= (0,1,0,0,0)^T, & v_5 &:= (0,0,0,0,0)^T, & w_5^T &:= (0,0,0,0,0), \\
u_6 &:= (0,0,1,0,0)^T, & v_6 &:= (0,0,0,0,0)^T, & w_6^T &:= (0,0,0,0,0), \\
u_7 &:= (0,0,0,1,1)^T, & v_7 &:= (0,0,0,0,1)^T, & w_7^T &:= (0,0,0,1,0), \\
u_8 &:= (0,0,0,1,0)^T, & v_8 &:= (0,0,0,0,0)^T, & w_8^T &:= (0,0,0,0,0), \\
u_9 &:= (0,0,0,0,1)^T, & v_9 &:= (0,0,0,0,0)^T, & w_9^T &:= (0,0,0,0,0),
\end{aligned}$$

and

$$(\tau_1, \ldots, \tau_9) = (1,0,0,4,4,3,3,1,1),$$

$$(\delta_1, \ldots, \delta_9) = (2,4,3,0,0,0,1,0,0).$$

Here, the vertices of the tree are numbered from 1 to 9, starting at the top level with the first vertex (the root), and at each subsequent level the vertices are consecutively numbered from left to right, until all leaves are reached. The vectors u_i, v_i, w_i $(i = 1, \ldots, 9)$ indicate the vectors which built the related generalized ultrametric submatrices.

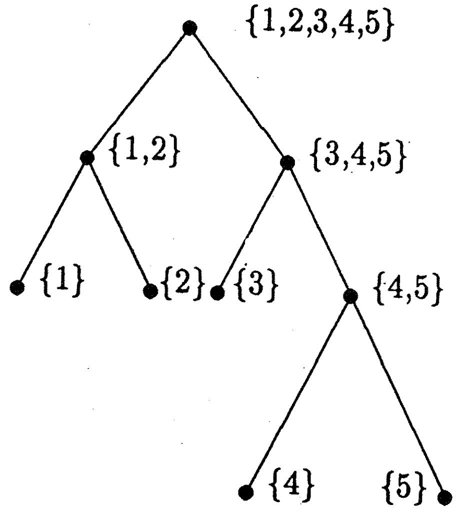

FIG. 2.

Thus, Theorem 3.1 gives a representation of a generalized ultrametric matrix $A \in \boldsymbol{R}^{n,n}$ as a sum of $2n-1$ at most rank-two matrices:

THEOREM 3.2. *Let $A = [a_{i,j}] \in \boldsymbol{R}^{n,n}$ be a generalized ultrametric matrix. Then there exists an associated binary rooted tree for $N = \{1, 2, \dots, n\}$, consisting of $2n-1$ vertices, such that*

$$A = \sum_{i=1}^{2n-1} \left(\tau_i u_i u_i^T + \delta_i v_i w_i^T\right), \tag{3.9}$$

where the (0, 1) vectors v_i, w_i, and u_i, determined from the vertices of the tree, are such that if there are edges in the tree from vertex i to vertex j and from vertex i to vertex $j+1$ with $i < j$, then $w_i = u_j$ and $v_i = u_{j+1}$, and $\delta_j + \tau_j \geqslant \delta_i$. Moreover, the scalars τ_i and δ_i in (3.9) are nonnegative, with $\delta_i = 0$ for every vertex corresponding to a leaf. Conversely, given any binary rooted tree for $N = \{1, \dots, n\}$, which determines the vectors $u_i, v_i, w_i \in \boldsymbol{R}^n$, and given any nonnegative constants $\{\tau_i\}_{i=1}^{2n-1}$ and $\{\delta_i\}_{i=1}^{2n-1}$, such that if any edge in the tree from vertex i to vertex j, with $i < j$, satisfies $\delta_j + \tau_j \geqslant \delta_i$, then

$$\sum_{i=1}^{2n-1} \left(\tau_i u_i u_i^T + \delta_i v_i w_i^T\right) \tag{3.10}$$

is a generalized ultrametric matrix.

We remark that the condition $\delta_j + \tau_j \geqslant \delta_i$, in both parts of Theorem 3.2, is just a manifestation of the inequalities of (3.2) of Theorem 3.1. We also remark that because $\delta_i = 0$ for any vertex corresponding to a leaf, the total number of parameters which actually play a role in the representation of the matrix in (3.9) is not $2(2n - 1)$, but $3n - 2$.

Note that the binary rooted tree used above illustrates the nested block structure of a generalized ultrametric matrix. The tree determines the vectors u_i, v_i, w_i and the scalars τ_i and δ_i, which are needed in the characterization of the generalized ultrametric matrix as the sum of rank-two matrices. However, this tree is closely related to the tree of Example 2.2 and to the way we constructed generalized ultrametric matrices in the previous section. Thus, we are now able to state and prove the following characterization of generalized ultrametric matrices using weighted trees and distance functions between the leaves of the tree.

THEOREM 3.3. *A nonnegative matrix A is a generalized ultrametric matrix if and only if there exists a rooted tree and weighting functions l and r such that the entries of A are given as indicated in* (2.4) *and* (2.5).

Proof. If A is a generalized ultrametric matrix, then with Theorem 3.2, there exists a associated binary rooted tree which determines the nonnegative numbers τ_i and δ_i. We introduce a new vertex, which becomes the new root, and we introduce a new edge from the new root to the old root. If we number the vertices as in Theorem 3.2 and if let 0 denote the new root, the desired weighting functions are given by

$$r(0,1) := \tau_1 \quad \text{and} \quad l(0,1) := \delta_1 + \tau_1$$

and, if there is an edge from vertex i to vertex j with $i < j$,

$$r(i,j) := \tau_j \quad \text{and} \quad l(i,j) := \delta_j + \tau_j - \delta_i.$$

[Note from Theorem 3.2 that $l(i, j) := \delta_j + \tau_j - \delta_i$ above is necessarily nonnegative.] Then it can be verified that the above weighting functions for the binary rooted tree, derived from the generalized ultrametric matrix $A = [a_{i,j}]$ in $\boldsymbol{R}^{n,n}$, do indeed *reproduce* the entries of A from the definitions of (2.4) and (2.5).

For the converse, consider any (not necessarily binary) rooted tree $G = (V, E)$ for which there are two nonnegative numbers (from the weighting functions l and r) assigned to each edge. Let $B \subset V$ denote the set of leaves, with $B = \{1, 2, \ldots, n\}$. We show below that the $n \times n$ matrix A, defined from (2.4) and (2.5), is a generalized ultrametric matrix.

Obviously, conditions (i), (iii), and (iv) of Definition 2.3 are satisfied. Now, consider any triple $\{q, s, t\} \in N^3$ and their related paths [i.e., $p(q)$, $p(s)$, and $p(t)$] from these leaves to the root of G.

We reorder the triple $\{q, s, t\}$ as the triple $\{i, j, k\}$ such that

$$|p(k) \cap p(i)| \leqslant |p(i) \cap p(j)| \quad \text{and} \quad |p(k) \cap p(j)| \leqslant |p(i) \cap p(j)|,$$

where $|p(k) \cap p(i)|$ denotes the cardinality of the common edges of $p(k)$ and $p(i)$. This means that there are exactly $|p(k) \cap p(i)|$ common consecutive distinct edges from the root to a certain vertex, say $\tilde{w}$, in both $p(i)$ and $p(k)$. If $|p(k) \cap p(i)| = 0$, then $\tilde{w}$ is just the root of G.

Thus, from the structure of the rooted tree, we see that

$$p(k) \cap p(j) = p(k) \cap p(i) \subseteq p(i) \cap p(j).$$

Hence, with the nonnegative numbers assigned to each edge of the tree, the triple $\{i, j, k\}$ satisfies conditions (v.i) and (v.ii) of Definition 2.3 and the statement (2.3), i.e.,

$$a_{k,i} = a_{k,j}, \qquad a_{i,k} = a_{j,k},$$

$$\max\{a_{i,j}, a_{j,i}\} \geqslant \max\{a_{i,k}, a_{k,i}\}, \qquad \min\{a_{i,j}, a_{j,i}\} \geqslant \min\{a_{i,k}, a_{k,i}\}.$$

Hence, using the statement preceding (2.3), it follows that (ii) of Definition 2.3 is also satisfied, and A is thus a generalized ultrametric matrix. ∎

It is evident that the null matrix in $\boldsymbol{R}^{n,n}$ satisfies Definition 2.3 and is thus a singular generalized ultrametric matrix. In the following, we derive a necessary and sufficient condition for a generalized ultrametric matrix to be *nonsingular*. Therefore, we consider the relation between A and $A + A^T$, where A is a generalized ultrametric matrix. With the representation (3.1) of A given in Theorem 3.1, (3.2) necessarily holds, i.e.,

$$\omega(A_{1,1}) = \omega(C) + \tau(A) \geqslant \omega(A), \qquad \omega(A_{2,2}) = \omega(D) + \tau(A) \geqslant \omega(A),$$

and we obtain

$$\tau(A) + \omega(A) = \tau(A + A^T) = 2\tau(A) + \delta(A).$$

This observation, when applied to the principal submatrices of A which arise in the reduction process leading to the representation of (3.9), shows that the binary rooted tree for A induces the *same* binary rooted tree for $A + A^T$. Consequently, $A + A^T$ has the representation

$$A + A^T = \sum_{i=1}^{2n-1} \hat{\tau}_i u_i u_i^T \tag{3.11}$$

where

$$\hat{\tau}_i := 2\tau_i + \delta_i \qquad (i = 1, 2, \ldots, 2n - 1).$$

Using the converse part of Theorem 3.2 [with $\hat{\tau}_i := 2\tau_i + \delta_i$ and $\hat{\delta}_i := 0$ in (3.10)], (3.11) gives that $A + A^T$ is then a *symmetric* generalized ultrametric matrix.

It was shown in [11] that a symmetric generalized ultrametric matrix A is nonsingular if and only if

$$\operatorname{span}\{u_i : \hat{\tau}_i > 0\} = \boldsymbol{C}^n, \tag{3.12}$$

and, moreover, A is positive definite if and only if (3.12) holds. Now for a generalized ultrametric matrix A, define the set

$$U_A := \{u_i : \tau_i + \delta_i > 0\},$$

where the vectors u_i and the nonnegative scalars τ_i and δ_i are taken from the representation (3.9) of A. As τ_i and δ_i are nonnegative, it is clear that $U_A = U_{A+A^T}$.

This leads to the following theorem, which gives a necessary and sufficient condition for a generalized ultrametric matrix to be nonsingular:

THEOREM 3.4. *Let $A = [a_{i,j}]$ in $\boldsymbol{R}^{n\times n}$ be a generalized ultrametric matrix. Then the following are equivalent*:

(I) A *is nonsingular*,
(II) span $U_A = \boldsymbol{C}^n$,
(III) span $U_{A+A^T} = \boldsymbol{C}^n$,
(IV) $A + A^T$ is nonsingular.

Proof. (I) implies (II): Assume that (II) does not hold, i.e., span $U_A \neq \boldsymbol{C}^n$. Then there exists a vector $x \in \boldsymbol{C}^n$, $x \neq 0$, with

$$u_i^T x = 0 \qquad \text{for all} \quad u_i \in U_A. \tag{3.13}$$

Using the representation (3.9) of Theorem 3.2, we obtain

$$Ax = \sum_{i=1}^{2n-1} \left(\tau_i u_i u_i^T x + \delta_i v_i w_i^T x \right). \tag{3.14}$$

If $u_i \notin U_A$, then $\tau_i + \delta_i = 0$, and, as the τ_i's and δ_i's are nonnegative, then $\tau_i = \delta_i = 0$, and its corresponding term in (3.14) is zero. If $u_i \in U_A$, then $\tau_i + \delta_i > 0$, and its corresponding term, because of (3.13), reduces to

$$\delta_i v_i w_i^T x.$$

If $\delta_i = 0$, the above term again vanishes, and if $\delta_i > 0$ (i.e., if v_i is not a leaf), then there is an edge, in the associated binary rooted tree, from vertex i to a vertex j with $j > i$. But, because of (3.2), it follows that $\tau_j + \delta_j \geqslant \delta_i > 0$, which gives that $u_j \in U_A$. But since $u_j = w_i$ (see the statement of Theorem 3.2), then from (3.13),

$$\delta_i v_i w_i^T x = \delta_i v_i u_j^T x = 0.$$

Thus, all terms of the sum in (3.14) vanish, and A is singular.

That (II) implies (III) follows immediately from the observation $U_A = U_{A+A^T}$, mentioned above.

That (III) implies (IV) is proved in Lemma 2.1 of [11]. However, for completeness, we repeat the proof. If there is an $x \in \boldsymbol{C}^n$ with $(A + A^T)x = 0$, we obtain

$$x^H (A + A^T) x = \sum_{i=1}^{2n-1} x^H \hat{\tau}_i u_i u_i^T x = \sum_{i=1}^{2n-1} \hat{\tau}_i \| u_i^T x \|_2^2 = 0.$$

Thus, $u_i^T x = 0$ for all $i = 1, \dots, 2n - 1$. Since span $U_{A+A^T} = \boldsymbol{C}^n$ by hypothesis, it follows that $x = 0$, and $A + A^T$ is nonsingular.

(IV) implies (I): Since $A + A^T$ is a nonsingular symmetric ultrametric matrix, $A + A^T$ is symmetric and positive definite. Thus, A is also positive definite, in the sense that $\mathrm{Re}(x^H Ax) = \frac{1}{2} x^H (A + A^T) x > 0$ for all $x \in \boldsymbol{C}^n$, $x \neq 0$. Therefore, A is nonsingular. ■

As a corollary of Theorem 3.4, we obtain:

COROLLARY 3.1. *Let* $A = [a_{i,j}]$ *in* $\boldsymbol{R}^{n \times n}$ *be a nonsingular generalized ultrametric matrix of the form*

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix},$$

where $A_{1,1}$ *and* $A_{2,2}$ *are square matrices,* $A_{1,2}$ *is a constant matrix with constant* $\tau(A)$, *and* $A_{2,1}$ *is a constant matrix with constant* $\omega(A) = \tau(A) + \delta(A)$, *with* $\delta(A) \geqslant 0$. *Then* $A_{1,1}$ *and* $A_{2,2}$ *are nonsingular generalized ultrametric matrices.*

Proof. Since A is a nonsingular generalized ultrametric matrix, we see from Theorem 3.4 that $A + A^T$ is a symmetric and positive definite generalized ultrametric matrix, and so are its principal submatrices, $A_{1,1} + A_{1,1}^T$ and $A_{2,2} + A_{2,2}^T$. Thus, again from Theorem 3.4, $A_{1,1}$ and $A_{2,2}$ are also nonsingular. ∎

We note, from the equivalence of (I) and (IV) in Theorem 3.4, that a *strictly* generalized ultrametric matrix [i.e., (iii) and (iv) of Definition 2.3 hold with strict inequality] is necessarily nonsingular. This will be used in our next result, Theorem 3.5, below.

THEOREM 3.5. *Let* $A = [a_{i,j}] \in \boldsymbol{R}^{n,n}$ *be a strictly generalized ultrametric matrix. Then A is nonsingular, and its inverse* $A^{-1} = [\alpha_{i,j}] \in \boldsymbol{R}^{n,n}$ *is a strictly row and strictly column diagonally dominant M-matrix with the additional properties*

$$a_{i,j} = 0 \quad \textit{if and only if} \quad \alpha_{i,j} = 0 \tag{3.15}$$

and, if $n > 1$,

$$\omega(A)\,\xi_n^T A^{-1} \xi_n < 1, \tag{3.16}$$

$$\mu(A)\,\xi_n^T A^{-1} \xi_n > 1. \tag{3.17}$$

Proof. The proof is an induction on the dimension n. The case $n = 1$ is obviously true. Now, let us assume that Theorem 3.5 is true for all dimensions $\tilde{n}$ with $1 \leqslant \tilde{n} < n$. With Theorem 3.1, A has, up to a suitable permutation, the following representation:

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix} := \begin{bmatrix} C & O \\ \delta(A)\xi_{n-r}\xi_r^T & D \end{bmatrix} + \tau(A)\xi_n\xi_n^T,$$

where

$$\delta(A) = \omega(A) - \tau(A) \geqslant 0.$$

Moreover, $C \in \boldsymbol{R}^{r,r}$ and $D \in \boldsymbol{R}^{n-r,n-r}$, with $1 \leqslant r < n$, are again strictly generalized ultrametric matrices, since A is a strictly generalized ultrametric matrix, with

$$\omega(A_{1,1}) \geqslant \omega(A) \quad \text{and} \quad \omega(A_{2,2}) \geqslant \omega(A). \tag{3.18}$$

Because the cases $r = 1$ and $n - r = 1$ can be similarly treated, assume $r > 1$ and $n - r > 1$. We have by induction that C^{-1} and D^{-1} are strictly row and strictly column diagonally dominant M-matrices which also satisfy (3.16) and (3.17). Thus,

$$\begin{gathered} \xi_r^T C^{-1} > 0, \qquad C^{-1}\xi_r > 0, \qquad \xi_{n-r}^T D^{-1} > 0, \qquad D^{-1}\xi_{n-r} > 0, \\ 0 \leqslant \omega(C)\xi_r^T C^{-1}\xi_r < 1, \qquad 0 \leqslant \omega(D)\xi_{n-r}^T D^{-1}\xi_{n-r} < 1. \end{gathered} \tag{3.19}$$

Let

$$M := \begin{bmatrix} C & O \\ \delta(A)\xi_{n-r}\xi_r^T & D \end{bmatrix}.$$

Then M is nonsingular, and M^{-1} is given by

$$M^{-1} = \begin{bmatrix} C^{-1} & O \\ -\delta(A)D^{-1}\xi_{n-r}\xi_r^T C^{-1} & D^{-1} \end{bmatrix}.$$

Since $D^{-1}\xi_{n-r}$ and $\xi_r^T C^{-1}$ are positive vectors and $\delta(A) \geqslant 0$, the induction hypothesis gives that M^{-1} is a Z-matrix, i.e., a real matrix in $\boldsymbol{R}^{n,n}$ whose off-diagonal entries are all nonpositive.

In the following, we show that M^{-1} is a strictly row and strictly column diagonally dominant matrix. Set

$$p := M^{-1}\xi_n = \begin{bmatrix} C^{-1} & O \\ -\delta(A)D^{-1}\xi_{n-r}\xi_r^T C^{-1} & D^{-1} \end{bmatrix} \xi_n$$

$$= \begin{bmatrix} C^{-1}\xi_r \\ \left(1 - \delta(A)\xi_r^T C^{-1}\xi_r\right) D^{-1}\xi_{n-r} \end{bmatrix}.$$

Since $\omega(A_{1,1}) \geqslant \omega(A)$ from (3.18), we obtain

$$\omega(C) = \omega(A_{1,1}) - \tau(A) \geqslant \omega(A) - \tau(A) = \delta(A).$$

Thus, using the inductive hypotheses [i.e., $\omega(C)\xi_r^T C^{-1}\xi_r < 1$], we obtain $p = M^{-1}\xi_n > 0$. Hence, since M^{-1} is a Z-matrix, M^{-1} is a strictly row diagonally dominant matrix.

Now, consider $\xi_n^T M^{-1}$, and set

$$q^T := \xi_n^T M^{-1} = \begin{bmatrix} \left(1 - \delta(A)\xi_{n-r}^T D^{-1}\xi_{n-r}\right)\xi_r^T C^{-1} \\ \xi_{n-r}^T D^{-1} \end{bmatrix}^T.$$

Since $\omega(A_{2,2}) \geqslant \omega(A)$ from (3.18), we obtain

$$\omega(D) = \omega(A_{2,2}) - \tau(A) \geqslant \omega(A) - \tau(A) = \delta(A).$$

Thus, using the inductive hypotheses [i.e., $\omega(D)\xi_{n-r}^T C^{-1}\xi_{n-r} < 1$], we obtain that $q^T = \xi_{n-r}^T M^{-1} > 0$. Again, since M^{-1} is a Z-matrix, M^{-1} is a strictly column diagonally dominant matrix.

Using the Sherman-Morrison formula [2, p. 51], A^{-1} can be expressed as

$$A^{-1} = \left[M + \tau(A)\xi_n^T\xi_n\right]^{-1} = M^{-1} - \frac{\tau(A)}{1 + \tau(A)\xi_n^T M^{-1}\xi_n} M^{-1}\xi_n\xi_n^T M^{-1}$$

$$= M^{-1} - \frac{\tau(A)}{1 + \tau(A)\xi_n^T p} pq^T.$$

Since

$$\frac{\tau(A)}{1+\tau(A)\xi_n^T p} pq^T \geqslant 0,$$

it follows from the inductive hypothesis that A^{-1} is a Z-matrix. Thus, it suffices to show that A^{-1} is a strictly row and strictly column diagonally dominant matrix, and that (3.16) and (3.17) hold. We first consider the row sums of A. We have

$$A^{-1}\xi_n = p - \frac{\tau(A)}{1+\tau(A)q^T\xi_n} pq^T\xi_n = \frac{1}{1+\tau(A)q^T\xi_n} p > 0. \quad (3.20)$$

Analogously, since $q^T\xi_n = \xi_n^T p$, we obtain

$$\xi_n^T A^{-1} = \frac{1}{1+\tau(A)\xi_n^T p} q^T > 0.$$

Hence, A^{-1} is a strictly row and strictly column diagonally dominant Z-matrix, and is thus a nonsingular M-matrix.

We now establish (3.16), i.e., $\omega(A)\xi_n^T A^{-1}\xi_n < 1$. From (3.20), we have

$$\omega(A)\xi_n^T A^{-1}\xi_n = \frac{\omega(A)\xi_n^T p}{1+\tau(A)\xi_n^T p}.$$

Thus, it reduces to showing that

$$\{\omega(A) - \tau(A)\}\xi_n^T p < 1.$$

However, as $\omega(A) - \tau(A) = \delta(A)$,

$$\{\omega(A) - \tau(A)\}\xi_n^T p$$
$$= [\delta(A)(\xi_r^T C^{-1}\xi_r)] + [1 - \delta(A)\xi_r^T C^{-1}\xi_r][\delta(A)\xi_{n-r}^T D^{-1}\xi_{n-r}]. \quad (3.21)$$

On using the inductive hypothesis, i.e.,

$$0 \leqslant \delta(A)\xi_r^T C^{-1}\xi_r < 1 \quad \text{and} \quad 0 \leqslant \delta(A)\xi_{n-r}^T D^{-1}\xi_{n-r} < 1,$$

all quantities in brackets in (3.21) are nonnegative, with the middle bracketed quantity being positive. Increasing the last factor in brackets to unity in (3.21) then gives

$$\{\omega(A) - \tau(A)\}\xi_n^T p < \delta(A)\xi_r^T C^{-1}\xi_r + [1 - \delta(A)\xi_r^T C^{-1}\xi_r] = 1,$$

the desired result.

Similarly, we next establish (3.17), i.e., $\mu(A)\xi_n^T A^{-1}\xi_n > 1$. On using the results above, we have

$$\mu(A)\xi_n^T A^{-1}\xi_n = \frac{\mu(A)\xi_n^T p}{1 + \tau(A)\xi_n^T p}.$$

Thus to establish (3.17), it is sufficient to show that $\{\mu(A) - \tau(A)\}\xi_n^T p > 1$. Then,

$$\{\mu(A) - \tau(A)\}\xi_n^T p = [\{\mu(A) - \tau(A)\}\xi_r^T C^{-1}\xi_r]$$
$$+[1 - \delta(A)\xi_r^T C^{-1}\xi_r][\{\mu(A) - \tau(A)\}\xi_{n-r}^T D^{-1}\xi_{n-r}].$$

The middle quantity in brackets above is again positive. Since $\mu(A) - \tau(A) \geqslant \mu(D)$ and since $\mu(D)\xi_{n-r}^T D^{-1}\xi_{n-r} > 1$ (by the induction hypothesis), the last factor in brackets is greater than unity. Hence, decreasing the last factor in brackets to unity then gives

$$\{\mu(A) - \tau(A)\}\xi_n^T p > [\{\mu(A) - \tau(A)\}\xi_r^T C^{-1}\xi_r] + [1 - \delta(A)\xi_r^T C^{-1}\xi_r],$$

and as $\delta(A) = \omega(A) - \tau(A) \leqslant \mu(A) - \tau(A)$, then

$$\{\mu(A) - \tau(A)\}\xi_n^T p > 1,$$

the desired result.

Finally, that the zero entries of A and A^{-1} are the same [cf. (3.15)] follows easily along the lines of the proof of [10], in the symmetric case, and is omitted. ■

Here, we mention that the Sherman-Morrison formula, which is another main tool in considering these kinds of matrices, was already used in [10] for the symmetric case. Later, this formula was used by Johnson [4] to show that the class of inverse M-matrices is invariant under a certain rank-one update. This update is similar to that used in [10] and also that used in the proof of Theorem 3.5.

THEOREM 3.6. *Let $A = [a_{i,j}] \in \boldsymbol{R}^{n,n}$ be a generalized ultrametric matrix. If A is nonsingular, then its inverse $A^{-1} = [\alpha_{i,j}] \in \boldsymbol{R}^{n,n}$ is a row and column diagonally dominant M-matrix with the additional properties*

$$a_{i,j} = 0 \quad \textit{implies} \quad \alpha_{i,j} = 0.$$

and, if $n > 1$,

$$\omega(A)\,\xi_n^T A^{-1} \xi_n \leqslant 1,$$

$$\mu(A)\,\xi_n^T A^{-1} \xi_n \geqslant 1.$$

Proof. The proof follows immediately from Theorem 3.5, since each generalized ultrametric matrix is a limit of a sequence of strictly generalized ultrametric matrices. ■

In considering the converse of Theorem 3.6, it can be easily verified that any nonsingular row and column diagonally dominant M-matrix in $\boldsymbol{R}^{1,1}$ or $\boldsymbol{R}^{2,2}$ has an inverse which is a (nonsingular) generalized ultrametric matrix. Examples however show that the above statement fails in general to be true in $\boldsymbol{R}^{n,n}$ for all $n > 2$. A heuristic explanation for this follows. As remarked after Theorem 3.2, the total number of parameters in the representation of the $n \times n$ matrix in (3.9) is $3n - 2$, while the total number of entries of a general matrix in $\boldsymbol{R}^{n,n}$ is n^2. But $3n - 2 = n^2$ is valid only if $n = 1$ or $n = 2$.

All results of this paper were presented by the first author at the Workshop on Nonnegative Matrices, Applications and Generalizations, held in Haifa, Israel, 30 May–4 June 1993. There, generalized ultrametric matrices were called pre-ultrametric matrices, and nonsingular generalized ultrametric *matrices were called ultrametric matrices. To avoid confusion, we changed these names.*

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REFERENCES

1 M. Fiedler and V. Pták, Diagonally dominant matrices, *Czechoslovak Math. J.* 17:420–433 (1967).

2 G. H. Golub and C. F. Van Loan, *Matrix Computations*, 2nd ed., Johns Hopkins U.P., Baltimore, 1989.

3 C. R. Johnson, Inverse M-matrices, *Linear Algebra Appl.* 47:195–216 (1982).

4 C. R. Johnson, A sufficient condition for inverse M-matrices, submitted for publication.

5 I. Koltracht and M. Neumann, On the inverse M-matrix problems for real symmetric positive-definite Toeplitz matrices, *SIAM J. Matrix Anal. Appl.* 12:310–320 (1991).

6 M. Lewin and M. Neumann, On the inverse M-matrix problem for (0, 1)-matrices, *Linear Algebra Appl.* 30:41–50 (1980).

7 T. L. Markham, Nonnegative matrices whose inverses are M-matrices, *Proc. Amer. Math. Soc.* 36:326–330 (1972).

8 S. Martínez, G. Michon, and J. San Martín, Inverses of ultrametric matrices are of Stieltjes type, *SIAM J. Matrix Anal. Appl.* 15:98–106 (1994).

9 J. J. McDonald, M. Neumann, H. Schneider, and M. J. Tsatsomeros, Inverse M-matrix inequalities and generalized ultrametric matrices, *Linear Algebra Appl.* 220:329–349 (1995).

10 R. Nabben and R. S. Varga, A linear algebra proof that the inverse of strictly ultrametric matrix is a strictly diagonally dominant Stieltjes matrix, *SIAM J. Matrix Anal. Appl.* 15:107–113 (1994).

11 R. S. Varga and R. Nabben, On symmetric ultrametric matrices, in *Numerical Linear Algebra* (L. Reichel, A. Ruttan, and R. S. Varga, Eds.), de Gruyter, New York, 1993, pp. 193–199.

12 R. A. Willoughby, The inverse M-matrix problem, *Linear Algebra Appl.* 18:75–94 (1977).

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