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Weighted Polynomial Approximation of Some Entire Functions on the Real Line

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To Dr. Theodore J. Rivlin on his 70th Birthday (September 11, 1996)

In this paper, weighted uniform approximation by polynomials on the whole real line is considered. In particular, results are sharpened here of Akhiezer and Koosis, on the exact class of entire functions (with order one and type zero), which can be uniformly approximated on the whole real line with arbitrary accuracy by polynomials.

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The study of weighted approximation by polynomials on the whole real line is one of the widely investigated areas of approximation theory. This topic was initiated by Bernstein, who obtained some fundamental results in this area. His work was then continued by Akhiezer, Mergelian, Pollard, and others in the 1950's. For more recent contributions to this area, we refer the reader to the monographs of [4] and [6].

In the study of weighted approximation on $\mathbf{R} := (-\infty, \infty)$, the starting point is a weight function $w(x)$ which makes it possible to introduce the uniform norm on $\mathbf{R}$. As usual, we set $w(x) = e^{-Q(x)}$, where it is assumed that $Q(x)$ is positive, even, and

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continuous on $\mathbf{R}$, and for all sufficiently large $x > 0$, $Q(x)$ is increasing to $+\infty$ as $x \to +\infty$, with

$$\lim_{x\to+\infty} \frac{Q(x)}{\log x} = +\infty.$$

If $C(\mathbf{R})$ denotes the collection of all continuous functions on $\mathbf{R}$, then, given the weight w, we consider the class of functions

$$C_Q(\mathbf{R}) := \{f(x) \in C(\mathbf{R}) : \lim_{|x|\to\infty} f(x)e^{-Q(x)} = 0\}.$$

Then, for any $f \in C_Q(\mathbf{R})$, the following approximation problem is well defined:

$$E_n(f,Q) := \inf_{p\in\Pi_n} \|e^{-Q}(f-p)\|_{\mathbf{R}} \qquad (n \in \mathbf{Z}_+),$$

where Π_n is the set of real algebraic polynomials of degree at most n, and $\|\cdot\|_{\mathbf{R}}$ is the supremum norm over $\mathbf{R}$. Furthermore, let us introduce the set of functions

$$M_Q(\mathbf{R}) := \{f \in C_Q(\mathbf{R}) : \lim_{n\to\infty} E_n(f,Q) = 0\}.$$

Thus, $M_Q(\mathbf{R})$ consists of those functions which can be uniformly approximated, with arbitrary accuracy, by polynomials on the whole real line. This leads to the following fundamental problem studied by Bernstein and his followers: What are necessary and sufficient conditions so that $M_Q(\mathbf{R}) = C_Q(\mathbf{R})$? This question was answered in different fashions by Akhiezer, Mergelian, and Pollard (see [3] for detailed references). Essentially, the solution to the above problem can be formulated as follows: In order that $M_Q(\mathbf{R}) = C_Q(\mathbf{R})$, it is necessary and sufficient that

$$\int_{-\infty}^{\infty} \frac{Q(x)}{1+x^2}\,dx = \infty. \tag{1}$$

(It should be noted that the sufficiency of (1) is somewhat more delicate. Namely, it also requires some additional regularity of Q, but we do not include here such technical details.)

An interesting question then arises: describe the set $M_Q(\mathbf{R})$ in the case when (1) fails, i.e., when

$$\int_{-\infty}^{\infty} \frac{Q(x)}{1+x^2}\,dx < \infty. \tag{2}$$

It was noticed by Akhiezer [1] that if (2) holds, then $M_Q(\mathbf{R})$ consists of entire functions. Koosis [3] refined Akhiezer's argument in order to show, whenever (2) is satisfied, that

$$M_Q(\mathbf{R}) \subset \mathcal{E}_0,$$

where $\mathcal{E}_0$ denotes the set of all entire functions of order 1 and type 0, and all entire functions of order less than unity. In [3], one can also find an example of a weight e^{-Q} satisfying (2) for which $M_Q(\mathbf{R}) \neq \mathcal{E}_0$.

In this note, we shall investigate further the associated class $M_Q(\mathbf{R})$ when (2) is valid. It will turn out that, in order to describe $M_Q(\mathbf{R})$, a more delicate study of the rate of growth of functions in $M_Q(\mathbf{R})$ is needed. This rate of growth cannot be described only by the *order* and *type* of the entire functions; there is an intrinsic relation between the growth of functions in $M_Q(\mathbf{R})$ and the rate of growth of Q at infinity. In order to describe this growth, we shall use the following function:

$$I(t) := \int_t^\infty \frac{Q(x)}{x^2}\,dx \qquad (t > 0). \tag{3}$$

Clearly, as $Q(t)$ is positive on $\mathbf{R}_+$, then $I(t)$ is positive and decreasing on $(0, +\infty)$, with $I(t) \downarrow 0$ as $t \to \infty$, provided that (2) holds. Let us also note that $\lim_{t\to 0} tI(t) = Q(0)$, so that $tI(t)$ is well-defined at the origin.

Now, we can state:

Theorem .1
Let Q satisfy (2), *and consider an arbitrary $f \in M_Q(\mathbf{R})$ such that $\|e^{-Q}f\|_{\mathbf{R}} = 1$. Then, f is a restriction to $\mathbf{R}$ of an entire function satisfying*

$$|f(z)| \le ae^{5|z|\cdot I(2|z|)} \qquad (z \in \mathbf{C}), \tag{4}$$

where $a := a(Q)$ depends only on Q.

If $\mathcal{M}_f(r) := \max_{|z|=r} |f(z)|$ denotes the maximum modulus function for the entire function f in Theorem .1, note that (4) implies that

$$\log \mathcal{M}_f(r) \le \log a + 5r \cdot I(2r) \qquad (r > 0).$$

Hence as $I(t) \downarrow 0$ as $t \to \infty$ when (2) holds, we directly obtain (cf. Boas [2, p. 8]) that $f \in \mathcal{E}_0$, which gives us the following

Corollary .2
(Akhiezer–Koosis). If (2) *is satisfied, then $M_Q(\mathbf{R}) \subset \mathcal{E}_0$.*

Moreover, the estimate (4) also shows that the set $M_Q(\mathbf{R})$ is essentially *smaller* than $\mathcal{E}_0$. Indeed, we prove below the result of

Corollary .3
For every Q satisfying (2), *there exists an $f \in \mathcal{E}_0$ such that $f \notin M_Q(\mathbf{R})$.*

Note that in [3], the above statement is shown to be true only for a specially constructed Q, while Corollary .3 holds for *every* Q satisfying (2).

PROOF OF COROLLARY .3
Consider the function $S(t) := t\sqrt{I(t)}$. Then,

$$S'(t) = \sqrt{I(t)} + \frac{tI'(t)}{2\sqrt{I(t)}} = \frac{2I(t) - \frac{Q(t)}{t}}{2\sqrt{I(t)}}. \tag{5}$$

Since $Q(t)$ is increasing for, say, $t \geq t_0$, we have

$$I(t) \geq Q(t) \int_t^\infty \frac{dx}{x^2} = \frac{Q(t)}{t} \qquad (t \geq t_0). \tag{6}$$

Thus, we have from (5) and (6) that $S(t)$ is increasing (to ∞) for $t \geq t_0$. In addition, by (2) we obtain that $S(t) = o(t)$, as $t \to \infty$. Consider now the function

$$g(z) := \sum_{k=k_0}^{\infty} \left(\frac{ez}{R(k)} \right)^k := \sum_{k=k_0}^{\infty} c_k z^k,$$

where $R := S^{[-1]}$ is the inverse function of S, (i.e., R is the increasing function, on $[S(t_0), +\infty)$, which satisfies $S(R(u)) = u$ for all $u \geq S(t_0)$ and $R(S(t)) = t$ for all $t \geq t_0$), and where k_0 is chosen so that $k_0 \geq S(t_0)$. Since $S(t) = o(t)$ as $t \to \infty$, then

$$\lim_{k\to\infty} k c_k^{1/k} = e \lim_{k\to\infty} \frac{k}{R(k)} = e \lim_{k\to\infty} \frac{S(R(k))}{R(k)} = 0,$$

from which it follows (from Theorem 2.2.10 of [2]) that $g \in \mathcal{E}_0$. On the other hand, for all $x > 0$ sufficiently large, set $\hat{k} := \hat{k}(x) = [[S(x)]]$ (where $[[y]]$ denotes the integer part of any real number y), so that

$$S(x) - 1 \leq \hat{k} \leq S(x) \text{ and } R(\hat{k}) \leq R(S(x)) = x.$$

Then, as the Taylor coefficients c_k for g are all positive for $k \geq k_0$, we have

$$\mathcal{M}_g(x) = g(x) > \left(\frac{ex}{R(\hat{k})} \right)^{\hat{k}} \geq e^{\hat{k}} \geq e^{S(x)-1} = e^{x\sqrt{I(x)}-1}, \tag{7}$$

for all $x > 0$ sufficiently large. However, as $I(t) \downarrow 0$ as $t \to \infty$, the growth rate of g, from (7), is *incompatible* (larger) than the growth rate of (4), i.e., $g \notin M_Q(\mathbf{R})$. The proof of Corollary .3 is complete.

Evidently, if $f \in M_Q(\mathbf{R})$, with f not a polynomial, then

$$f_n(x) := \frac{f(x) - p_n(f,x)}{E_n(f,Q)} \in M_Q(\mathbf{R}) \qquad (n = 0, 1, \ldots),$$

where $p_n(f,x)$ is the best approximating polynomial of f from Π_n (with respect to the weight e^{-Q}). Thus, applying Theorem .1, with f_n replacing f, leads to

Corollary .4
If Q satisfies (2), *then for every $f \in M_Q(\mathbf{R})$,*

$$|f(z) - p_n(f,z)| \leq a E_n(f,Q) e^{5|z|\cdot I(2|z|)} \qquad (z \in \mathbf{C}).$$

The above statement shows, for functions from $M_Q(\mathbf{R})$, that convergence on the real line extends (with another weight) to convergence on the *whole* complex plane.

We can also give a partial converse to Theorem .1, provided that Q satisfies the following mild additional condition: for some for c with $0 < c < 1$, there exists a constant $\gamma(c)$ with $0 < \gamma(c) \leq 1$, such that

$$\gamma(c) := \liminf_{x\to\infty} \frac{Q(cx)}{Q(x)} > 0. \tag{8}$$

It can be easily shown that if (8) holds with *some* c in $0 < c < 1$, then it also holds for *every* c in $0 < c < 1$. We shall use this fact below by choosing $c = 1/e$.

Theorem .5
If $Q(x)$ satisfies (2) and (8), and if f is an entire function such that

$$|f(z)| \leq \nu e^{(1-\varepsilon)Q(|z|)} \qquad (z \in \mathbf{C}), \tag{9}$$

with some $\nu > 0$ and some ε satisfying $0 < \varepsilon < 1$, then $f \in M_Q(\mathbf{R})$. Moreover, we have in this case that

$$\limsup_{n\to\infty} \left(E_n(f,Q)\right)^{1/n} \leq e^{-\varepsilon\gamma(1/e)}. \tag{10}$$

Note that (10) gives the geometric convergence, of the errors $E_n(f,Q)$, to zero.

Let us compare our necessary condition of growth (4) with the sufficient condition (9). Since the weight Q must satisfy (2), we shall assume, for some $\delta > 0$, that $\frac{Q(x)\log^{1+\delta} x}{x}$ is decreasing (as $x \to \infty$). In our situation, this assumption may be considered "acceptable", since if $\frac{Q(x)\log x}{x}$ were increasing, it would contradict (2). Under the above assumption, we have

$$tI(2t) \leq tI(t) = t\int_t^\infty \frac{Q(x)}{x^2}\,dx \leq Q(t)\log^{1+\delta} t \int_t^\infty \frac{dx}{x\log^{1+\delta} x} = \frac{Q(t)\log t}{\delta}.$$

Hence in this case, with this added assumption, the corresponding necessary and sufficient conditions of (4) and (9) would differ only by a log factor. Moreover, if $Q(x)x^{-\beta}$ is decreasing (as $x \to \infty$) for some β with $0 < \beta < 1$, a similar calculation leads to

$$tI(2t) \leq tI(t) \leq Q(t)t^{1-\beta} \int_t^\infty x^{\beta-2}\,dx = \frac{Q(t)}{1-\beta}; \tag{11}$$

that is, even the log factor disappears! Moreover, as it can be verified that the assumption, that $Q(x)x^{-\beta}$ is decreasing (for all x sufficiently large) for some β with $0 < \beta < 1$, implies that both (2) and (8) are satisfied, then we obtain the following result. Assume that $Q(x)x^{-\beta}$ is decreasing, for all sufficiently large x, for some β with $0 < \beta < 1$, and assume that f is an entire function which satisfies

$$(12) \qquad |f(z)| \leq d_1 e^{d_2 Q(|z|)} \qquad (z \in \mathbf{C}),$$

where d_1 and d_2 are positive constants. Then, for $f \in M_Q(\mathbf{R})$, it is necessary that $d_2 \leq 5/(1-\beta)$, and sufficient that $d_2 < 1$. (These are direct consequences of (11) and Theorems .1 and .5.)

Thus, under the above conditions, the rate of growth (12) characterizes the class of functions $M_Q(\mathbf{R})$. Of course, the much more delicate question of giving the exact constants d_1 and d_2 in (12) remains open here. The authors feel that the rate of growth (12) might be in fact the general proper description of $M_Q(\mathbf{R})$ (that is, for every Q satisfying (2)).

Now, we turn to the proofs of Theorems .1 and .5.

PROOF OF THEOREM .1

Assume that $f \in M_Q(\mathbf{R})$, i.e., for the sequence of best approximating polynomials $p_n(f)$, we have

$$E_n(f,Q) = ||e^{-Q}(f - p_n(f))||_{\mathbf{R}} \to 0, \quad \text{as} \quad n \to \infty.$$

Since, by hypothesis, $||e^{-Q}f||_{\mathbf{R}} = 1$, we clearly have $||e^{-Q}p_n(f)||_{\mathbf{R}} \leq 2$, i.e., $|p_n(f,x)| \leq 2e^{Q(x)}$, for any $x \in \mathbf{R}$. By Theorem 6.5.4 in Boas [2], we have, for any $z = u + iv$ with $|z| = r$, that

$$\log \frac{|p_n(f,z)|}{2} \leq \frac{|v|}{\pi} \int_{-\infty}^{+\infty} \frac{\log(|p_n(f,t)|/2)dt}{(t-u)^2+v^2} \leq \frac{|v|}{\pi} \int_{-\infty}^{+\infty} \frac{Q(t)dt}{(t-u)^2+v^2},$$

or,

$$(13) \qquad \log \frac{|p_n(f,z)|}{2} \leq \frac{|v|}{\pi} \left\{ \int_{|t|\leq 2r} + \int_{|t|\geq 2r} \right\} \frac{Q(t)dt}{(t-u)^2+v^2}.$$

Recalling that $Q(t)$ is increasing (to ∞) for $t \geq t_0$, let $t_1 \geq t_0$ be such that

$$\max_{0\leq s\leq t} Q(s) = Q(t) \quad (t \geq t_1).$$

Then, the first integral in (13) can be bounded above, for $r \geq t_1/2$, by

$$\frac{|v|}{\pi} \int_{|t|\leq 2r} \frac{Q(t)dt}{(t-u)^2+v^2} \leq \frac{|v|\,Q(2r)}{\pi} \int_{-\infty}^{+\infty} \frac{dt}{(t-u)^2+v^2} \leq Q(2r),$$

and this is further bounded above by

$$Q(2r) \leq 2r \int_{2r}^{\infty} \frac{Q(t)dt}{t^2} = 2rI(2r) \qquad (r \geq t_1/2).$$

For the second integral of (13), we have, as $0 \leq |u|, |v| \leq r$, that

$$(t-u)^2 + v^2 \geq \frac{t^2}{4} \qquad (|t| \geq 2r),$$

from which it follows that

$$\frac{|v|}{\pi}\int_{|t|\geq 2r}\frac{Q(t)dt}{(t-u)^2+v^2}\leq\frac{8r}{\pi}\int_{2r}^{\infty}\frac{Q(t)dt}{t^2}=\frac{8rI(2r)}{\pi}.$$

Collecting these estimates and noting that $2+\frac{8}{\pi}<5$, we obtain

$$\log\frac{|p_n(f,z)|}{2}\leq 5|z|\cdot I(2|z|)\qquad\left(|z|\geq\frac{t_1}{2}\right).$$

By the maximum principle, we further have that

$$\log\frac{|p_n(f,z)|}{2}\leq 5\frac{t_1}{2}\cdot I(t_1)=:b\qquad\left(|z|\leq\frac{t_1}{2}\right),$$

where we note that b depends only on $Q(t)$. Thus, we have

$$\log\frac{|p_n(f,z)|}{2}\leq 5|z|\cdot I(2|z|)+b\qquad(z\in\mathbf{C}).\tag{14}$$

This yields, in particular, that the sequence $\{p_n(f,z)\}_{n=0}^{\infty}$ is bounded on compact subsets of $\mathbf{C}$. Therefore, by the Montel theorem ([5], p. 293), it possesses a subsequence converging uniformly in $\mathbf{C}$ (i.e., on compact subsets of $\mathbf{C}$) to some function F. Evidently, F must be an entire function, F coincides with f on $\mathbf{R}$, and in view of (14),

$$\log\frac{|F(z)|}{2}\leq 5|z|I(2|z|)+b.\qquad(z\in\mathbf{C}),$$

which is equivalent to (4) with $a:=2e^b$. This completes the proof of Theorem .1.

We note that in the case $Q(x)$ is increasing for every $x\geq 0$, we can take $a=1$ in Theorem .1.

PROOF OF THEOREM .5

Assuming that $Q(x)$ satisfies (2) and (8), let

$$f(z)=\sum_{k=0}^{\infty}a_kz^k$$

be an entire function such that (9) holds. Then, we obtain, by the Cauchy integral formula for any $r>0$, that

$$|a_k|=\frac{|f^{(k)}(0)|}{k!}=\frac{1}{2\pi}\left|\oint_{|z|=r}\frac{f(z)}{z^{k+1}}\,dz\right|\leq\frac{\nu e^{(1-\varepsilon)Q(r)}}{r^k}.$$

Using this estimate,

$$E_n(f,Q)\leq\sum_{k=n+1}^{\infty}|a_k|\cdot\|x^ke^{-Q(x)}\|_{\mathbf{R}}\leq\nu\sum_{k=n+1}^{\infty}e^{(1-\varepsilon)Q(r)}\frac{\|x^ke^{-Q(x)}\|_{\mathbf{R}}}{r^k}.\tag{15}$$

Suppose that

$$||x^k e^{-Q(x)}||_{\mathbf{R}} = x_k^k e^{-Q(x_k)} \qquad \text{(for some } x_k \in \mathbf{R}_+\text{)}.$$

We distinguish two cases.

Case 1: $x_k \leq Q^{[-1]}(k)/e$. (Here, $Q^{[-1]}$ stands for the inverse function of Q, where we assume that $k \geq n$ is large enough so that this inverse is well defined.) Setting $r := Q^{[-1]}(k)$, we have, as in the proof of Corollary .3, that

$$e^{(1-\varepsilon)Q(r)} \frac{||x^k e^{-Q(x)}||_{\mathbf{R}}}{r^k} \leq e^{-\varepsilon k - Q(x_k)} < e^{-\varepsilon k}.$$

Case 2: $x_k > Q^{[-1]}(k)/e$. Now we set $r := x_k$. Using (8), for arbitrary γ' (where $0 < \gamma' < \gamma(\frac{1}{e}) \leq 1$) and for all k sufficiently large, we have

$$Q(x_k) \geq \gamma' Q(ex_k) > \gamma' Q\left(Q^{[-1]}(k)\right) = \gamma' k.$$

Hence,

$$e^{(1-\varepsilon)Q(r)} \frac{||x^k e^{-Q(x)}||_{\mathbf{R}}}{r^k} = e^{-\varepsilon Q(x_k)} < e^{-\varepsilon \gamma' k}.$$

Since the estimate in Case 2 is weaker than in Case 1, we obtain from (15) that

$$E_n(f, Q) \leq \nu \sum_{k=n+1}^{\infty} e^{-\varepsilon \gamma' k} = O(e^{-\varepsilon \gamma' n}).$$

Hence, as $E_n(f,Q) \to 0$ as $n \to \infty$, then $f \in M_Q(\mathbf{R})$. Moreover, as γ' was any number satisfying $0 < \gamma' < \gamma(1/e)$, we thus obtain (10), completing the proof of Theorem .5.

Examples.

The following weights provide typical examples for which the above results apply:

$$Q(x) = x^\alpha \log^\beta (x+2),$$

where either $0 < \alpha < 1$, β arbitrary; or $\alpha = 1$, $\beta < -1$; or $\alpha = 0$, $\beta > 1$.

Remark 1

In [3], the author considers the class $M_Q^\star(\mathbf{R})$ consisting of those functions from $C_Q(\mathbf{R})$ which can be approximated by entire functions of arbitrarily small type (as above, the approximation is meant in uniform norm on $\mathbf{R}$ with the weight e^{-Q}). It is shown in [3] that $M_Q^\star(\mathbf{R})$ coincides with either $C_Q(\mathbf{R})$ or $E_0 \cap C_Q(\mathbf{R})$. Then, the author addresses the following question: Under what conditions on Q does the relation $M_Q(\mathbf{R}) = M_Q^\star(\mathbf{R})$ hold? In view of Corollary .3, it is clear that $M_Q(\mathbf{R})$ either coincides with $C_Q(\mathbf{R})$, or it is a *proper* subset of $\mathcal{E}_0$. Thus, $M_Q(\mathbf{R}) = M_Q^\star(\mathbf{R})$ holds essentially only when both of them equal $C_Q(\mathbf{R})$.

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References

[1] N. I. Akhiezer, On weighted approximation of continuous functions by polynomials on the whole real line, *Uspekhi Mat. Nauk,* **11**(1956), 3-43.

[2] R. P. Boas, *Entire Functions*, Academic Press (New York, 1954).

[3] P. Koosis, *The Logarithmic Integral I,* Cambridge Studies in Advanced Maths., Vol. 12, Cambridge Univ. Press (Cambridge, 1988).

[4] D. S. Lubinsky and E. B. Saff, *Strong Asymptotics for Extremal Polynomials*, Lecture Notes in Mathematics 1305, Springer-Verlag, New York, 1988.

[5] A. I. Markushevits, *The Theory of Analytic Functions*, GITL (Moscow, 1950) (in Russian).

[6] V. Totik, *Weighted Approximation with Varying Weights*, Lecture Notes in Mathematics 1569, Springer-Verlag, New York, 1994.