$$\begin{pmatrix} \lambda_1 & & & \sim 10^{-9} \\ & \lambda_2 & & \\ & & \lambda_3 & \\ \sim 10^{-10} & & & \text{matrix of order 3 with eigenvalues} \\ & & & 2.5\times10^{-6},\ -2.5\times10^{-6},\ -10^{-10} \end{pmatrix}$$

4. Finally, a matrix of order 8 whose elements are given by the first 64 5-digit groups of π was tried. Here, the maximum subdiagonal element (with the exception of elements 4,3; 5,4; and 8,7) was:

43 000	after	1 step	1100	after	5 steps
30 000	„	2 steps	47	„	6 „
20 000	„	3 „	0.047	„	7 „
9 600	„	4 „	3×10^{-5}	„	8 „

and the approximate eigenvalues after 8 steps:

$$364\,635;\ -88\,257;\ 55\,298;\ 21\,957 \pm 43\,289i;$$
$$-43\,567;\ 5271 \pm 26\,688i.$$

This process, though time consuming, is almost foolproof. It should not be extended too far, but rather ended after about 15 steps (this is the mathematical equivalent of 32 768 ordinary QR-steps). If, then, the transformed matrix looks like

$$\begin{bmatrix} x & x & x & x & x & x & x & x \\ 0 & x & x & x & x & x & x & x \\ 0 & x & x & x & x & x & x & x \\ 0 & x & x & x & x & x & x & x \\ 0 & x & x & x & x & x & x & x \\ 0 & 0 & 0 & 0 & 0 & x & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & 0 & 0 & x & x \end{bmatrix},$$

this indicates that we have 4 eigenvalues of nearly equal moduli and another pair with nearly equal moduli. In such a case, the square submatrices of order greater than 2 along the diagonal are treated again by the same process, but this time with a shift of origin, the amount of shift being the common absolute value of the eigenvalues in the submatrix. The resulting Jacobi rotations are then applied again to the full matrix, thus breaking up the diagonal submatrices one by one into submatrices of order not greater then 2.

One may even obtain Schur's canonical form by computing in the complex plane only, in the last step. This, however, has not actually been carried out.

3.1 References

[1] Francis, J. G. F.: *The QR Transformation.* Computer Journal **4** No 3 (1961) 265–271; **4** No 4 (1962) 332–345.

[2] Iverson, K. E.: *The Zeros of the Partial Sums of e^z.* Math. Tables Aids Comp. **4** (1953) 165–168.

DISCUSSION

W. Givens (*USA*). In practical work do you advise a preliminary reduction to Hessenberg form? If so, which method would you use?

H. Rutishauser (*Switzerland*). For the ordinary QR-transformation, the methods of Givens or Householder, as proposed by Francis, are highly desirable and reduce the amount of computation from $O(n^3)$ to $O(n^2)$ since the Hessenberg form is retained by the process. The QR-Graeffe method, however, destroys the Hessenberg property, and therefore it has to be re-established after every step, which offsets any gain. In addition, I have no proof of the convergence of the process, in this case.

A. Newhouse (*USA*). Have you ever tried these methods with rational arithmetic?

H. Rutishauser (*Switzerland*). No, but I know that because of the root-squaring properties, it would produce extremely long numbers after a few steps.

A. S. Householder (*USA*). How is this quadratically convergent method related to that published by you and Bauer some years ago?

H. Rutishauser (*Switzerland*). The QR-transformation (ordinary or Graeffe-type) is superior to the LR-transformation (ordinary or Graeffe-type) because the former can easily be made foolproof with respect to such failures as division by zero, whereas the latter cannot (at least, not generally).

W. Borsch-Supan (*Germany*). Would you recommend the QR-process for matrices of Frobenius form?

H. Rutishauser (*Switzerland*). No, because the QR or QR-Graeffe process would fill out the matrix and the process would become uneconomical compared to the ordinary Graeffe process

4. ON VARIANTS OF SUCCESSIVE OVER-RELAXATION AND ALTERNATING DIRECTION. IMPLICIT METHODS

R. S. VARGA (*USA*)

The well known Peaceman-Rachford (1955) variant of the alternating direction methods is defined by

$$\left.\begin{aligned} (H+r_{i+1}I)\,\boldsymbol{x}^{(i+1/2)} &= (r_{i+1}I-V)\,\boldsymbol{x}^{(i)}+\boldsymbol{k}, \\ (V+r_{i+t}I)\,\boldsymbol{x}^{(i+1)} &= (r_{i+t}I-H)\,\boldsymbol{x}^{(i+1/2)}+\boldsymbol{k}, \end{aligned}\right\} \quad (1)$$

where H and V are Hermitian, positive definite matrices. Here, we are seeking to solve iteratively the matrix equation $(H+V)\,\boldsymbol{x}=\boldsymbol{k}$. Writing this as

$$\boldsymbol{x}^{(i+1)} = T_{r_{i+1}}\,\boldsymbol{x}^{(i)}+\boldsymbol{g}_i \quad i\geqslant 0, \quad (2)$$

the major object is to minimize the error vectors $\boldsymbol{\epsilon}_i = \boldsymbol{x}-\boldsymbol{x}^{(i)}$ which satisfy

$$\boldsymbol{\epsilon}_m = \Big(\prod_{i=1}^{m} T_{r_i}\Big)\,\boldsymbol{\epsilon}_0 \qquad m\geqslant 0. \quad (3)$$

Using spectral norms and the assumption that $HV=VH$, it is known that

$$\Big\|\prod_{i=1}^{m} T_{r_i}\Big\| \leqslant \Big(\max_{\alpha\leqslant\nu\leqslant\beta}\Big|\prod\frac{r_i-y}{r_i+y}\Big|\Big)^2, \quad (4)$$

where the eigenvalues τ_j of H and σ_l of V statisfy: $0 < \alpha \leqslant \tau_j, \sigma_l \leqslant \beta$.

This leads to the problem of selecting parameters $\tilde{r}_i(m)$ which minimize the right-hand side of (4), i.e., $\boldsymbol{r}(m)$ is any non-negative vector with m components. We seek

$$d_m[\alpha, \beta] = d_m[\alpha/\beta, 1] = \inf_{r(m) \geqslant 0} \left\{ \max_{\alpha \leqslant \nu \leqslant \beta} \prod_{i=1}^{m} \left| \frac{r_i(m) - \nu}{r_i(m) + \nu} \right| \right\}. \tag{5}$$

Wachspress [1]) has just published an elegant and simple algorithm based on geometric and arithmetic means for finding $\tilde{r}_i(m)$ when $m = 2^k$, $k \geqslant 0$.

Closely related to this is the work of de Boor and Rice [2]).

We want to determine the asymptotic behavior of $d_m[\alpha, \beta]$, based on Wachspress' results. Let $E_m[\alpha, \beta]$ be defined by

$$E_m[\alpha, \beta] = \frac{-2 \log_e d_m[\alpha, \beta]}{m} \leqslant -\log_e \frac{(\|\prod_{i=1}^{m} T_{r_i}\|)}{m} \tag{6}$$

the last inequality following from (4) and (5). It is easy to see that $E_{2k}[\alpha, \beta]$ is strictly increasing. More precisely [3])

$$0 < E_{2k+1} - E_{2k} < \frac{\log_e 2}{2^k}, \tag{7}$$

which directly establishes the existence of

$$\lim_{k \to \infty} E_{2_k}[\alpha, \beta] \equiv E_\infty[\alpha, \beta].$$

From this, it can be shown that

$$d_m[\alpha, \beta] \sim 2 \left(\frac{1}{\exp(\frac{1}{2} E_\infty[\alpha, \beta])} \right)^m, \quad m = 2^k, \ k \to \infty. \tag{8}$$

This asymptotic behavior was conjectured by de Boor and Rice [2]) for all $m \geqslant 0$.

Another variant of ADI methods has been given by Kellogg [4]), based on a suggestion in a paper by D'yakonov [5]). Instead of (1) we write

$$\left.\begin{aligned} (H + r_{i+1} I)\, \boldsymbol{x}^{(i+1/2)} &= (r_{i+1} I - H)\, \boldsymbol{x}^{(i)} + \boldsymbol{k}_1 \\ (V + r_{i+1} I)\, \boldsymbol{x}^{(i+1)} &= (r_{i+1} I - V)\, \boldsymbol{x}^{(i+1/2)} + \boldsymbol{k}_2, \end{aligned}\right\} \tag{9}$$

where $\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}$. For fixed $r > 0$, it can be easily seen that

$$\boldsymbol{x}^{(i+1/2)} \to \boldsymbol{y}; \quad \boldsymbol{x}^{(i+1)} \to \boldsymbol{z}, \quad i \to \infty, \tag{10}$$

so that

$$(\boldsymbol{x}^{(i+1/2)} + \boldsymbol{x}^{(i+1)}) \to \boldsymbol{x}, \tag{11}$$

where $(H + V)\, \boldsymbol{x} = \boldsymbol{k}$. This can be viewed as a form of Cesáro summability.

Hageman [6]) has done some recent work on the application of the theory of regular splittings to cyclically reduced matrix equations.

4.1 References

[1]) Wachspress, E. L.: *Optimum ADI Iteration Parameters for a Model Problem.* Trans. Soc. Indust. Appl. Math. (1962).

[2]) De Boor, C. and T. R. Rice: *Chebychev Approximation by $\alpha\pi(x - r_i)/(x + r_i)$ and Application to ADI Iteration.* To appear in the Trans. Soc. Indust. Appl. Math.

[3]) Varga, R. S.: *A Note on a Paper by de Boor and Rice.* To appear.

[4]) Kellogg, R. B.: *Another Alternating-Direction-Implicit Method.* To appear.

[5]) D'yakonov, E. G.: *On Some Difference Schemes for Solutions of Boundary Problems.* Z. Vyčisl. Mat. Fis. **2** (1962) 51–59.

[6]) Hageman, L. A.: Ph. D. Thesis, University of Pittsburgh (1962).

DISCUSSION

J. Douglas (*USA*). The alternating direction method of D'yakonov does not generalize directly to the decomposition of the basic matrix into the sum of three or more matrices. The process converges, but to the wrong answer. A modification of the process does converge to the correct answer at a very slow rate.

5. CALCULATION OF MATRIX EIGENVECTORS AND EIGENVALUES, ARISING IN THE APPROXIMATE SOLUTION OF EQUATIONS OF MATHEMATICAL PHYSICS

A. A. ABRAMOV (*USSR*)

Read by A. A. Dorodnicyn (*USSR*)

5.1 Introduction

In practical computation one has to deal with problems of linear algebra, which approximate problems in spaces of infinite dimensions.

Large algebraic problems of this type usually indicate an algebraic problem of smaller dimensions which gives the principal part of the solution of the large problem. Therefore, it is desirable to perform calculations in the following way. First, solve the small problem which approximates the large one, then introduce a small number of relatively easily calculated corrections, or indicate a speedily convergent iterative process, which requires at each of its steps the solution of a problem of small dimensions.

In this report, results obtained by a group of scientists of the Computing Centre of the USSR Academy of Sciences are given.

5.2 Problem 1

Let A be an Hermitian matrix of order N. Let the eigenvalues be numbered in an ascending order, i.e., $\Lambda_1 \leqslant \Lambda_2 \leqslant \ldots \leqslant \Lambda_n$. The eigenvalues Λ and corresponding eigenvectors z of A are required, where by definition, we have

$$Az = \Lambda z. \tag{1}$$

Let A and z be arranged in the block form as shown: