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J. Math. Pures Appl.,
78, 1999, *p.* 177-202

WEIGHTED RATIONAL APPROXIMATION IN THE COMPLEX PLANE *

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ABSTRACT. – Given a triple (G, W, γ) of an open bounded set G in the complex plane, a weight function $W(z)$ which is analytic and different from zero in G, and a number γ with $0 \leq \gamma \leq 1$, we consider the problem of locally uniform rational approximation of any function $f(z)$, which is analytic in G, by weighted rational functions $\{W^{m_i+n_i}(z)R_{m_i,n_i}(z)\}_{i=0}^{\infty}$, where $R_{m_i,n_i}(z) = P_{m_i}(z)/Q_{n_i}(z)$ with $\deg P_{m_i} \leq m_i$ and $\deg Q_{n_i} \leq n_i$ for all $i \geq 0$ and where $m_i + n_i \to \infty$ as $i \to \infty$ such that $\lim_{i\to\infty} m_i/[m_i + n_i] = \gamma$. Our main result is a necessary and sufficient condition for such an approximation to be valid. Applications of the result to various classical weights are also included. © Elsevier, Paris

1. Introduction and general results

In this paper, we shall develop the ideas of [11] and apply them to the study of the approximation of analytic functions in an open set G by weighted rationals $W^{m+n}(z)R_{m,n}(z)$. Specifically, we examine triples of the form

$$(G, W, \gamma), \tag{1.1}$$

where

$$\left\{\begin{array}{ll} i) & G \text{ is an open bounded set in the complex plane } \mathbb{C}, \text{ which can be represented as a finite or countable union of disjoint simply connected domains, i.e., } G = \bigcup_{\ell=1}^{\sigma} G_\ell \text{ (where } 1 \leq \sigma \leq \infty \text{)}, \\ ii) & W(z), \text{ the weight function, is analytic in } G \text{ with } W(z) \neq 0 \text{ for any } z \in G, \text{ and} \\ iii) & \gamma \text{ satisfies } 0 \leq \gamma \leq 1. \end{array}\right. \tag{1.2}$$

* This research was supported in part by the National Science Foundation, Grant No. DMS-9707359.

We say that the triple (G, W, γ) has the **rational approximation property** if:

$$(1.3)\quad \begin{cases} \text{for any } f(z) \text{ which is analytic in } G \text{ and for any compact} \\ \text{subset } E \text{ of } G, \text{ there exists a sequence of rational func-} \\ \text{tions } \{R_{m_i,n_i}(z)\}_{i=0}^{\infty}, \text{ where } R_{m_i,n_i}(z) = P_{m_i}(z)/Q_{n_i}(z), \text{ with} \\ \deg P_{m_i} \le m_i \text{ and } \deg Q_{n_i} \le n_i \text{ for all } i \ge 0, \text{ and where} \\ (m_i + n_i) \to \infty \text{ as } i \to \infty, \text{ such that} \\ i)\ \ \lim_{i\to\infty} \dfrac{m_i}{m_i + n_i} = \gamma, \\ \text{and} \\ ii)\ \ \lim_{i\to\infty} \|f - W^{m_i+n_i} R_{m_i,n_i}\|_E = 0, \end{cases}$$

where all norms throughout this paper are the uniform (Chebyshev) norms on the indicated sets.

Given a triple (G, W, γ), as in (1.1) which satisfies the conditions of (1.2), we state below our main result, Theorem 1.1, which gives a characterization, in terms of potential theory, for the triple (G, W, γ) to have the rational approximation property. Also, let $\mathcal{M}(E)$ be the set of all positive unit Borel measures on $\mathbb{C}$ which are supported on a compact set E, i.e., for any $\mu \in \mathcal{M}(E)$, we have $\mu(\mathbb{C}) = 1$ and $\operatorname{supp} \mu \subset E$. Also, ∂G denotes the boundary of the set G, and the logarithmic potential of an arbitrary compactly supported signed measure μ is defined (see Tsuji [18], p. 53) by

$$(1.4)\qquad U^{\mu}(z) := \int \log \frac{1}{|z-t|}\, d\mu(t).$$

THEOREM 1.1. – *A triple* (G, W, γ), *satisfying* (1.2), *has the rational approximation property* (1.3) *if and only if there exist a signed measure*

$$(1.5)\qquad \mu(G, W, \gamma) = \gamma\mu^{+}(G, W, \gamma) - (1-\gamma)\mu^{-}(G, W, \gamma),$$

with $\mu^{+}(G, W, \gamma), \mu^{-}(G, W, \gamma) \in \mathcal{M}(\partial G)$, *and a constant* $F(G, W, \gamma)$ *such that*

$$(1.6)\qquad U^{\mu(G,W,\gamma)}(z) - \log|W(z)| = F(G, W, \gamma), \quad \text{for any } z \in G.$$

Below, we state some consequences of Theorem 1.1, while in Section 2, we state applications of Theorem 1.1 in a number of specific cases. The proofs of all results in Sections 1 and 2 are given, respectively, in Sections 3 and 4.

Remark 1.2. – Results on weighted rational approximation of analytic functions in open sets with *multiply* connected components (as opposed, in (1.2i), to unions of simply connected domains) will be considered elsewhere.

Remark 1.3. – The condition in (1.2ii) that $W(z) \neq 0$ for all $z \in G$ cannot be dropped, for if $W(z_0) = 0$ for some $z_0 \in G_k$, where $G = \bigcup_{\ell=1}^{\sigma} G_\ell$, then the necessarily null sequence $\{W^{m_i+n_i}(z_0) R_{m_i,n_i}(z_0)\}_{i=0}^{\infty}$ trivially fails to converge to any $f(z)$, analytic in G, with $f(z_0) \neq 0$; whence, the rational approximation property fails.

COROLLARY 1.4. – *A triple* (G, W, γ), *satisfying* (1.2), *has the rational approximation property* (1.3) *if and only if* (1.3) *holds for* $f(z) \equiv 1$, *i.e., if and only if this single function* *is locally uniformly approximable on compact subsets of* G *by a corresponding sequence of the weighted rational functions.*

Remark 1.5. – The function $f(z) \equiv 1$ in Corollary 1.4 can be replaced by any function which is analytic in G and not equal identically to 0 in G.

COROLLARY 1.6. – *Given a triple* (G, W, γ), *which satisfies* (1.2) *with* σ *finite, assume that there exist a constant* F *and a signed measure* μ *with*

$$\text{supp } \mu \subset \partial G \quad \textit{and} \quad \mu(\mathbb{C}) = 2\gamma - 1, \tag{1.7}$$

such that

$$U^{\mu}(z) - \log|W(z)| = F, \quad \textit{for any } z \in G. \tag{1.8}$$

Then, the triple (G, W, γ) *has the rational approximation property* (1.3) *if and only if the signed measure* μ *can be decomposed as*

$$\mu = \gamma\mu^{+} - (1-\gamma)\mu^{-}, \tag{1.9}$$

with $\mu^{+}, \mu^{-} \in \mathcal{M}(\partial G)$.

Furthermore, let a Jordan decomposition of the signed measure μ, *satisfying* (1.7) *and* (1.8), *be given by*

$$\mu = \tau^{+} - \tau^{-}, \tag{1.10}$$

where τ^{+} *and* τ^{-} *are positive measures with*

$$\text{supp } \tau^{+},\ \text{supp } \tau^{-} \subset \partial G \quad \textit{and} \quad \mu(\text{supp } \tau^{+} \cap \text{supp } \tau^{-}) = 0. \tag{1.11}$$

Then, the triple (G, W, γ) *has the rational approximation property* (1.3) *if and only if*

$$\tau^{+}(\mathbb{C}) \leq \gamma. \tag{1.12}$$

If (1.9) *or* (1.12) *holds true for a signed measure* μ *satisfying* (1.7) *and* (1.8), *then*

$$\mu(G, W, \gamma) = \mu \quad \text{and} \quad F(G, W, \gamma) = F. \tag{1.13}$$

The study of weighted *rational* approximation has recently been introduced in papers by Borwein and Chen [1], Borwein, Rakhmanov and Saff [2], and Rakhmanov, Saff, and Simeonov [12]. The last two papers deal with weighted rational approximation only on the real line. Certain special cases of the triples (G, W, γ), in the notation of (1.1), were considered in the complex plane in [1], but that research did not attack the general question of necessary and sufficient conditions for (G, W, γ) to have the rational approximation property of (1.3), as in Theorem 1.1.

2. Applications

Finding the signed measure $\mu(G, W, \gamma)$ of Theorem 1.1, or verifying its existence, is a nontrivial problem in general. Since $U^{\mu(G,W,\gamma)}(z)$ is harmonic in $\mathbb{C} \setminus \operatorname{supp} \mu(G, W, \gamma)$ and, since it can be shown from (1.6), if $\log |W(z)|$ is continuous on $\overline{G}$ and if G is a *finite* union of $G_\ell, \ell = 1, 2, \ldots, \ell_0$, that $U^{\mu(G,W,\gamma)}(z)$ is equal to $\log |W(z)| + F(G, W, \gamma)$ on $\operatorname{supp} \mu(G, W, \gamma) \subset \partial G$, then $U^{\mu(G,W,\gamma)}(z)$ can be found as the solution of the corresponding Dirichlet problems. The signed measure $\mu(G, W, \gamma)$ can be recovered from its potential, using the Fourier method described in Section IV.2 of Saff and Totik [13].

However, we next consider a different method, dealing with specific weight functions, which allows us to deduce "explicit" expressions for the signed measure $\mu(G, W, \gamma)$ of Theorem 1.1. For simplicity, we assume throughout this section that G is given as in (1.2i), but with σ finite. We denote the unbounded component of $\overline{\mathbb{C}} \setminus \overline{G}$ by Ω. Let ν^+ and ν^- be two *positive* Borel measures on $\mathbb{C}$, with compact supports satisfying

$$\operatorname{supp} \nu^+ \subset \overline{\mathbb{C}} \setminus G \text{ and } \operatorname{supp} \nu^- \subset \overline{\mathbb{C}} \setminus G, \tag{2.1}$$

such that

$$\nu^+(\mathbb{C}) = \nu^-(\mathbb{C}) = 1. \tag{2.2}$$

For real numbers $\alpha \geq 0$ and $\beta \geq 0$, assume that the weight function $W(z)$, satisfying

$$\log |W(z)| = -(\alpha U^{\nu^+}(z) - \beta U^{\nu^-}(z)) = -U^{\nu}(z), \quad z \in G, \tag{2.3}$$

with $\nu := \alpha \nu^+ - \beta \nu^-$ being a signed measure, is analytic in G. Then, we state, as an application of Theorem 1.1, our next result as

THEOREM 2.1. – *Given any pair of real numbers $\alpha \geq 0$ and $\beta \geq 0$, given an open bounded set $G = \bigcup_{\ell=1}^{\sigma} G_\ell$, as in* (1.2i) *with σ finite, and given the weight function $W(z)$ of* (2.3), *then the triple (G, W, γ) has the rational approximation property* (1.3) *if and only if the signed measure*

$$\mu := (2\gamma - 1 + \alpha - \beta)\omega(\infty, \cdot, \Omega) - \alpha \hat{\nu}^+ + \beta \hat{\nu}^- \tag{2.4}$$

can be decomposed as

$$\mu = \gamma \mu^+ - (1 - \gamma)\mu^-, \tag{2.5}$$

where $\mu^+, \mu^- \in \mathcal{M}(\partial G)$. Here, $\omega(\infty, \cdot, \Omega)$ is the harmonic measure at ∞ with respect to Ω, and $\hat{\nu}^+$ and $\hat{\nu}^-$ are, respectively, the balayages of ν^+ and ν^- from $\overline{\mathbb{C}} \setminus \overline{G}$ to $\overline{G}$.

Furthermore, if μ of (2.4) *satisfies* (2.5), *then* (see Theorem 1.1)

$$\mu(G, W, \gamma) = \mu. \tag{2.6}$$

We point out that the harmonic measure $\omega(\infty, \cdot, \Omega)$ (defined in Nevanlinna [8] or Tsuji [18]) is the same as the equilibrium distribution measure for $\overline{G}$, in the sense of classical logarithmic potential theory [18]. For the notion of balayage of a measure, we refer the reader to Chapter IV of Landkof [6] or Section II.4 of Saff and Totik [13].

In the following series of subsections, we consider various classical weight functions and we find their corresponding signed measures, associated with the weighted rational approximation problem in G, as given in Theorem 1.1.

2.1. Incomplete rationals

With $\mathbb{N}_0$ and $\mathbb{N}$ denoting respectively the sets of nonnegative and positive integers, the *incomplete polynomials* of Lorentz [7] are a sequence of polynomials of the form

$$\{z^{m(i)}P_{n(i)}(z)\}_{i=0}^{\infty}, \quad \deg P_{n(i)} \leq n(i), \ (m(i), n(i) \in \mathbb{N}_0), \tag{2.7}$$

where it is assumed that $\lim\limits_{i\to\infty} \frac{m(i)}{n(i)} =: \alpha$, where $\alpha > 0$ is a real number. The question of the possibility of the approximation of functions by incomplete polynomials is closely connected to that of the approximation of functions by the weighted polynomials:

$$\{z^{\alpha n}P_n(z)\}_{n=0}^{\infty}, \quad \deg P_n \leq n. \tag{2.8}$$

The approximation question for the incomplete polynomials of (2.7) was completely settled, by Saff and Varga [14] and by v. Golitschek [4], for the real interval $[0,1]$ (see Totik [17] and Saff and Totik [13] for the associated history and later developments), and by the authors [11] in the complex plane. We consider now the analogous problem for *incomplete rational functions* in the complex plane. A special case of incomplete rational approximation in the complex plane was studied by Borwein and Chen in [1]. The latest such developments, on the real line, are in Borwein, Rakhmanov and Saff [2] and Rakhmanov, Saff and Simeonov [12].

Since the weight $W(z) := z^\alpha$ in (2.8) is multiple-valued in $\mathbb{C}$ if $\alpha \notin \mathbb{N}$, we then restrict ourselves to the slit domain $S_1 := \mathbb{C}\backslash(-\infty, 0]$ and the single-valued branch of $W(z)$ in S_1 satisfying $W(1) = 1$. Thus,

$$W(z) := z^\alpha, \quad z \in S_1 := \mathbb{C}\backslash(-\infty, 0], \tag{2.9}$$

where $\alpha > 0$ is a real number.

THEOREM 2.2. – *Given an open set G, as in* (1.2i) *with σ finite, such that $\overline{G} \subset S_1$, and given the weight function $W(z)$ of* (2.9), *then the triple (G, z^α, γ) has the rational approximation property* (1.3) *if and only if the signed measure*

$$\mu = (2\gamma - 1 + \alpha)\omega(\infty, \cdot, \Omega) - \alpha\omega(0, \cdot, \Omega) \tag{2.10}$$

can be decomposed as

$$\mu = \gamma\mu^+ - (1-\gamma)\mu^-,$$

where $\mu^+, \mu^- \in \mathcal{M}(\partial G)$. Here, $\omega(\infty, \cdot, \Omega)$ and $\omega(0, \cdot, \Omega)$ are the harmonic measures with respect to the unbounded component Ω of $\overline{\mathbb{C}}\backslash\overline{G}$, respectively, at $z = \infty$ and at $z = 0$.

In special cases where the geometric shape of G is given explicitly, it is possible to determine the explicit form of the signed measure in (2.10). As a simple example, we consider below the special case of a disk and $\gamma = 1/2$.

COROLLARY 2.3. – *Given the disk $D_r(a) := \{z \in \mathbb{C} : |z - a| < r\}$, where $a \in (0, +\infty)$ and where $\overline{D}_r(a) \subset S_1 = \mathbb{C}\backslash(-\infty, 0]$, i.e., $r < a$, and given the weight function of* (2.9), *then the triple $(D_r(a), z^\alpha, 1/2)$ has the rational approximation property* (1.3) *if and only if*

$$r \leq r_{\max}(a,\ \alpha) := \begin{cases} a, \ \alpha \in (0, 1/2], \\ a \sin\dfrac{\pi}{4\alpha}, \ \alpha \in (1/2, +\infty). \end{cases} \tag{2.11}$$

Furthermore, if (2.11) *is satisfied, then the associated signed measure* $\mu(D_r(a), z^\alpha, 1/2)$(see Theorem 1.1) *is given by*

$$d\mu(D_r(a), z^\alpha, 1/2) = \frac{\alpha}{2\pi r}\left(1 - \frac{a^2 - r^2}{|z|^2}\right) ds, \tag{2.12}$$

where ds *is the arclength measure on the circle* $|z - a| = r$.

Remark 2.4. – More generally, it is possible to show that the triple $(D_r(a),\ z^\alpha,\ \gamma)$, as in Corollary 2.3 but with *any* $\gamma \in [0, 1]$, has the approximation property (1.3) if and only if

$$r \leq \begin{cases} a & , \quad \alpha + \gamma \leq 1, \\ au_0 & , \quad \alpha + \gamma > 1, \end{cases}$$

where $u_0 \in (0, 1]$ is the largest solution of the equation

$$(2\gamma - 1)\arccos\left[\frac{1-2\gamma-(2\gamma-1+2\alpha)u^2}{2u(2\gamma - 1 + \alpha)}\right] + 2\alpha \arccos\left[\frac{(2\gamma-1+2\alpha)\sqrt{1-u^2}}{2\sqrt{\alpha(2\gamma-1+\alpha)}}\right] = \gamma\pi,$$

in the interval $(0, 1]$.

2.2. Exponential weight

Consider the weight function

$$W(z) := e^{-z}, \quad z \in \mathbb{C}. \tag{2.13}$$

This section is devoted to the study of weighted rational approximation, with respect to the exponential weight of (2.13), in disks centered at the origin and in certain domains, arising in connection with Padé approximations of the exponential function. Our next result treats the case of disks.

THEOREM 2.5. – *Given* $D_r(0) := \{z \in \mathbb{C} : |z| < r\}$ *and given the weight* $W(z)$ *of* (2.13), *then the triple* $(D_r(0), e^{-z}, \gamma)$ *has the rational approximation property* (1.3) *if and only if*

$$r \leq r_{\max}(\gamma), \quad 0 \leq \gamma \leq 1, \tag{2.14}$$

where $r_{\max}(\gamma)$ *is the unique positive solution, for* r *in the interval* $[|\gamma - \frac{1}{2}|, +\infty)$, *of the following equation:*

$$\sqrt{r^2 - \left(\gamma - \frac{1}{2}\right)^2} - \left(\gamma - \frac{1}{2}\right)\arccos\left(\frac{\gamma - \frac{1}{2}}{r}\right) = \frac{\pi}{2}(1 - \gamma). \tag{2.15}$$

Moreover, if (2.14) *holds, then the associated signed measure* $\mu(D_r(0), e^{-z}, \gamma)$ *is given by*

$$d\mu(D_r(0), e^{-z}, \gamma) = (2\gamma - 1 - 2r\cos\theta)\frac{d\theta}{2\pi}, \tag{2.16}$$

where $d\theta$ *is the angular measure on* $|z| = r$ *and where* $z = re^{i\theta}$.

In particular, $r_{\max}(1) = \frac{1}{2}$ (see also Theorem 3.8 of [10]), $r_{\max}(\frac{1}{2}) = \frac{\pi}{4}$ *and* $r_{\max}(0) = \frac{1}{2}$.

We remark that the solution, $r_{\max}(\gamma)$ of (2.15), can be verified to be symmetric about $\gamma = \frac{1}{2}$, as a function of γ in the interval $[0, 1]$.

Next, we again consider the weight function $W(z) := e^{-z}$ of (2.13), but we now consider the triple $(G_\gamma,\ e^{-z},\ \gamma)$, where G_γ, a *generalized Szegő domain*, is defined below. To begin, first assume that $0 < \gamma < 1$. Then following [15], the two conjugate complex numbers, defined by,

$$z_\gamma^{\pm} := \exp\{\pm\, i \arccos(2\gamma - 1)\}, \tag{2.17}$$

have modulus unity, and we consider the complex plane $\mathbb{C}$ slit along the two rays

$$\mathbb{R}_\gamma := \left\{ z \in \mathbb{C} : z = z_\gamma^+ + i\tau \text{ or } z = z_\gamma^- - i\tau \text{ for any } \tau \geq 0 \right\}. \tag{2.18}$$

This is shown below in Figure 1. (For readers who are familiar with [15], the quantity $\sigma := \lim\limits_{i \to \infty} \frac{n_i}{m_i}$ in that paper and γ of (1.3i) are related through $\gamma = \frac{1}{1+\sigma}$.) Next, the function

$$\hat{g}_\gamma(z) := \sqrt{1 + z^2 - 2z(2\gamma - 1)} \tag{2.19}$$

has z_γ^+ and z_γ^- as branch points, which are the finite extremities of $\mathbb{R}_\gamma$. On taking the principal branch for the square root, i.e., on setting $\hat{g}_\gamma(0) = 1$ and extending g_γ analytically on the doubly slit domain $\mathbb{C}\backslash\mathbb{R}_\gamma$, then g_γ is analytic and single-valued on $\mathbb{C}\backslash\mathbb{R}_\gamma$. It can also be verified that $1 \pm z + \hat{g}_\gamma(z)$ does not vanish on $\mathbb{C}\backslash\mathbb{R}_\gamma$.

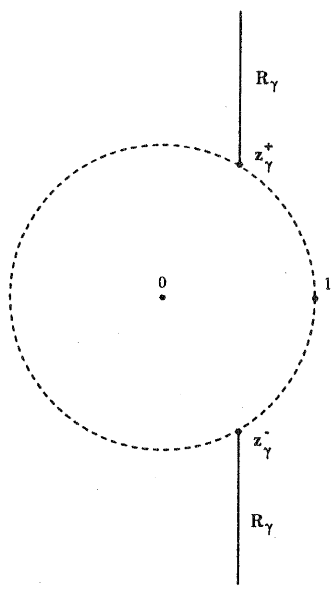

Fig. 1: The set $\mathbb{C}\backslash\mathbb{R}_\gamma$.

Next, we define the functions $(1 + z + \hat{g}_\gamma(z))^{2\gamma}$ and $(1 - z + \hat{g}_\gamma(z))^{2(1-\gamma)}$ so that their values at $z = 0$ are respectively, $2^{2\gamma}$ and $2^{2(1-\gamma)}$, with remaining values determined by analytic continuation. These functions are also analytic and single-valued on $\mathbb{C}\backslash\mathbb{R}_\gamma$. With these definitions, we then set:

$$w_\gamma(z) := \frac{4\gamma(\frac{1-\gamma}{\gamma})^{1-\gamma} z e^{\hat{g}_\gamma(z)}}{(1 + z + \hat{g}_\gamma(z))^{2\gamma}(1 - z + \hat{g}_\gamma(z))^{2(1-\gamma)}} \quad (0 < \gamma < 1), \tag{2.20}$$

and it follows that $w_\gamma(z)$ is also analytic and single-valued on $\mathbb{C}\backslash\mathbb{R}_\gamma$. For the omitted cases $\gamma = 0$ and $\gamma = 1$, it can be verified that $w_1(z) = \lim_{\gamma\to 1} w_\gamma(z)$ and $w_0(z) = \lim_{\gamma\to 0} w_\gamma(z)$ satisfy

$$\begin{cases} w_1(z) = ze^{1-z} & \text{for Re } z < 1, \text{ and} \\ w_0(z) = ze^{1+z} & \text{for Re } z > -1. \end{cases} \tag{2.21}$$

(Again, for those familiar with [15], the function $w_\gamma(z)$ of (2.20) is exactly the function $w_\sigma(z)$ in [15], eq. (2.5).)

It is known (see [15], [Lemma 4.1]) that $w_\gamma(z)$ is univalent in $|z| < 1$, for any γ with $0 \le \gamma \le 1$, and this allows us to define the domain

$$G_\gamma := \{z \in \mathbb{C} : |w_\gamma(z)| < 1 \text{ and } |z| < 1\}, \text{ for any } 0 \le \gamma \le 1. \tag{2.22}$$

Its boundary, ∂G_γ, is a well-defined Jordan curve which lies interior to the unit disk, except for its points $z_\gamma^\pm$ of (2.17). This is shown in Figure 2. We call G_γ an *extended Szegő domain*, as the special case $\gamma = 1$ corresponds to a domain originally treated by Szegő in [16] in 1924.

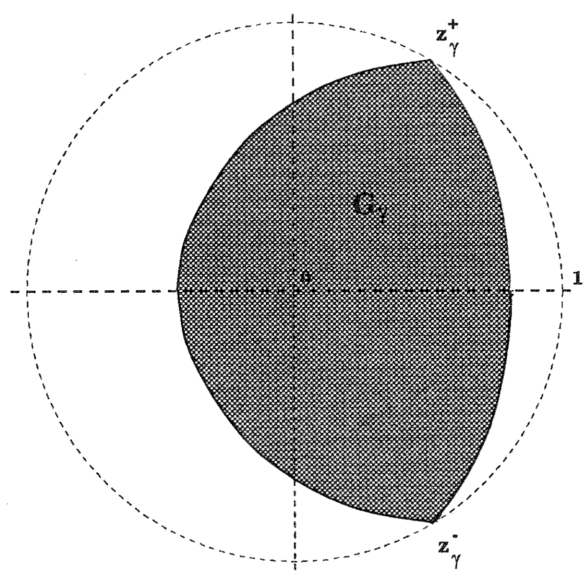

Fig. 2: The set G_γ.

We now have all the necessary definitions for the statement of our next result.

THEOREM 2.6. – *For any γ with $0 \le \gamma \le 1$, let G_γ be the domain of* (2.22), *and let $W(z) = e^{-z}$ be the weight function of* (2.13). *Then, the triple $(G_\gamma, e^{-z}, \gamma)$ has the rational approximation property* (1.3).

To conclude this section, we note that, except for the final result of Theorem 2.6, all preceding results stated in Sections 1 and 2 are of the "if and only if" type, i.e., these results are by definition *sharp*. The result of Theorem 2.6, however, leaves open the possibility that for a given γ with $0 \le \gamma \le 1$, there could be a *larger* domain H, with $G_\gamma \subset H$, such that the triple (H, e^{-z}, γ) has the rational approximation property (1.3), but we strongly doubt this.

Also of general interest is the extension of the results of this paper to triples (G, W, γ) of (1.1), where one has the *sharpened rational approximation property*, that is, for any $f(z)$, analytic in G and continuous in $\overline{G}$, there is a sequence of rational functions $\{R_{m_i,n_i}\}_{i=0}^{\infty}$ satisfying (1.3i), such that

$$\lim_{i\to\infty} \|f - W^{m_i+n_i} R_{m_i,n_i}\|_{\overline{G}} = 0.$$

For the essentially polynomial case of $\gamma = 0$ and $W(z) := e^{-z}$, this is treated in part in [10], Theorem 3.2. Some general results in weighted polynomial approximation on compact sets are obtained in [9]. To our knowledge, general results on the sharpened rational approximation property have not as yet been treated in the literature.

3. Proofs of results of Section 1

Proof of Theorem 1.1. – Assuming that a signed measure $\mu(G, W, \gamma)$ exists and satisfies the conditions (1.5) and (1.6) of Theorem 1.1, we first prove that the triple (G, W, γ) has the rational approximation property (1.3). To show this, we consider the following three cases:

Case $\gamma = 1$. – The hypothesis (1.5) with $\gamma = 1$ implies that $\mu(G, W, 1) = \mu^+(G, W, 1) \in \mathcal{M}(\partial G)$. As (1.6) is also valid by hypothesis, applying Theorem 1.1 of [11] gives that the pair (G, W) has the **polynomial approximation property**, i.e., for any $f(z)$, which is analytic in G, and for any compact subset E of G, there exists a sequence of polynomials $\{P_m(z)\}_{m=0}^{\infty}$, with deg $P_m \le m$ for all $m \ge 0$, such that

$$\lim_{m\to\infty} \|f - W^m P_m\|_E = 0. \tag{3.1}$$

On simply setting $n_m := 0$ and $Q_{n_m}(z) := 1$, the sequence of rational functions $\{R_{m,n_m}(z) := P_m(z)/Q_{n_m}(z) = P_m(z)\}_{m=0}^{\infty}$ clearly satisfies (1.3i) with $\gamma = 1$, and (3.1) shows that (1.3ii) is also valid, i.e., the triple $(G, W, 1)$ has the rational approximation property.

Case $\gamma = 0$. – Let $f(z)$ be any function analytic in G. If $f(z) \equiv 0$ in G, it suffices to define the sequence $\{R_{0,n}(z) := P_0(z)/Q_n(z)\}_{n=0}^{\infty}$, where $P_0(z) \equiv 0$, i.e., deg $P_0 = 0$, and for each $n \ge 0$, deg $Q_n \le n$. Clearly, (1.3i) is satisfied with $\gamma = 0$, and (1.3ii) is trivially satisfied for any compact subset E of G. If $f(z) \not\equiv 0$ in G, then for any given compact subset E of $G, f(z)$ has only a finite number of zeros, say $\{z_k(E)\}_{k=1}^{m}$, in E, where $m = m(E)$ is a *fixed* nonnegative integer depending on E. Then, we can write

$$f(z) = g(z) W^m(z) P_m(z), \text{ with } P_m(z) := \prod_{k=1}^{m} (z - z_k(E)), \tag{3.2}$$

where $g(z)$ is analytic and nonzero in E. Consequently, $1/g(z)$ is also analytic in E.

In this case $\gamma = 0$, hypothesis (1.5) implies that $\mu(G, W, 0) = -\mu^-(G, W, 0)$, where $\mu^-(G, W, 0) \in \mathcal{M}(\partial G)$, and similarly, hypothesis (1.6) implies, with the definition of (1.4), that

$$U^{\mu^-(G,W,0)}(z) - \log \frac{1}{|W(z)|} = -F(G, W, 0), \quad \text{for any } z \in G. \tag{3.3}$$

Because of the form of (3.3), it follows from Theorem 1.1 of [11] that the pair $(G, 1/W)$ now has the **polynomial approximation property**, and this can be applied to the analytic function $1/g(z)$ on E. Thus, there is a sequence of polynomials $\{Q_n(z)\}_{n=0}^{\infty}$, with deg $Q_n \leq n$ for all $n \geq 0$, such that

$$\lim_{n\to\infty} \left\| \frac{1}{g} - \frac{Q_n}{W^n} \right\|_E = 0. \tag{3.4}$$

Since $\{Q_n(z)/W^n(z)\}_{n=0}^{\infty}$ tends uniformly to $1/g(z)$ on E, it follows from Hurwitz's Theorem that $Q_n(z)$ has no zero in E, for, say, all $n \geq N_0$. Moreover, it also follows from (3.4) that

$$\lim_{n\to\infty} \left\| g - \frac{W^n}{Q_n} \right\|_E = 0. \tag{3.5}$$

Hence, using (3.2), we have

$$f(z) - W^{m+n}(z)P_m(z)/Q_n(z) = W^m(z)P_m(z)\left(g(z) - \frac{W^n(z)}{Q_n(z)}\right),$$

and as $m = m(E)$ is a fixed integer, (3.5) gives us that

$$\lim_{n\to\infty} \left\| f - W^{m+n}P_m/Q_n \right\|_E = 0. \tag{3.6}$$

Thus, (1.3i) is satisfied with $\gamma = 0$ and (3.6) shows that (1.3ii) is also satisfied, i.e., the triple $(G, W, 0)$ has the rational approximation property.

Case $0 < \gamma < 1$. – Recall from (1.2ii) that $G = \bigcup_{\ell=1}^{\sigma} G_\ell$ is a bounded open set where $\{G_\ell\}_{\ell=1}^{\sigma}$ are disjoint simply connected domains, and consider the Jordan domains $G_{\ell,j} \subset G_\ell$, $j \in \mathbb{N}$, which exhaust the domain G_ℓ, for each ℓ with $1 \leq \ell \leq \sigma$. A convenient way to define the sequence $\{G_{\ell,j}\}_{j=1}^{\infty}$ is to set:

$$G_{\ell,j} := \left\{ z \in \mathbb{C} : |\varphi_\ell(z)| < 1 - \frac{1}{2j} \right\}, \quad j \in \mathbb{N}, \tag{3.7}$$

where $\varphi_\ell : G_\ell \to D := \{w \in \mathbb{C} : |w| < 1\}$ is a canonical conformal map of domain G_ℓ onto the open unit disk D, where $1 \leq \ell \leq \sigma$. Thus, each $G_{\ell,j}$ is bounded by the analytic Jordan curve

$$\Gamma_{\ell,j} := \left\{ z \in \mathbb{C} : |\varphi_\ell(z)| = 1 - \frac{1}{2j} \right\},$$

which is a level curve of φ_ℓ. Let $E \subset G$ be an arbitrary compact set. Because E is compact, it is clear that E is contained in the finite union of $G_{\ell,j}, \ell = 1, 2, \ldots, \ell_0$, for some $\ell_0 \in \mathbb{N}$, provided that j is large enough. Set $H_j := \bigcup_{\ell=1}^{\ell_0} G_{\ell,j}$ and $\Gamma_j := \bigcup_{\ell=1}^{\ell_0} \Gamma_{\ell,j}$. Then, $\Gamma_j = \partial H_j$ and $\overline{H}_j \subset G$ for all $j \in \mathbb{N}$, and also $E \subset H_j$ for all sufficiently large $j \in \mathbb{N}$.

Introducing the domain $\Omega_j := \overline{\mathbb{C}} \backslash \overline{H}_j$, $j \in \mathbb{N}$, we observe, from the existence of the Borel measure $\mu^+(G, W, \gamma)$ of (1.5), that the balayage μ_j^+ of $\mu^+(G, W, \gamma)$, out of Ω_j

to $\partial\Omega_j = \partial H_j$, is such that, for each $j \in \mathbb{N}$, the following statements are true (see Theorem II.4.4 of [13]):

$$U^{\mu_j^+}(z) = U^{\mu^+(G,W,\gamma)}(z) + c_j, \quad z \in \overline{H}_j \tag{3.8}$$

and

$$U^{\mu_j^+}(z) \leq U^{\mu^+(G,W,\gamma)}(z) + c_j, \quad z \in \mathbb{C}, \tag{3.9}$$

where $\mu_j^+(\mathbb{C}) = 1$, $\operatorname{supp} \mu_j^+ \subset \partial H_j$ and $c_j > 0$. (We remark that equality in (3.8) holds on $\partial\Omega_j$ since each point of $\partial\Omega_j$ is regular (see [18], Theorem I.11).) As (1.6) holds by hypothesis for any $z \in G$ and as $\overline{H}_j \subset G$, then (3.8) and the hypothesis (1.5) give

$$\gamma U^{\mu_j^+}(z) - (1-\gamma)U^{\mu^-(G,W,\gamma)}(z) - \log|W(z)| = F_j, \quad z \in \overline{H}_j, \tag{3.10}$$

where

$$F_j := F(G,W,\gamma) + \gamma c_j, \tag{3.11}$$

for any $j \in \mathbb{N}$.

Fixing a sufficiently large $j \in \mathbb{N}$ so that $E \subset H_j$, consider the function

$$v(z) := U^{\mu_j^+}(z) - U^{\mu^+(G,W,\gamma)}(z), \quad z \in \overline{\mathbb{C}}, \tag{3.12}$$

which is subharmonic in Ω_j with $v(\infty) = 0$, and satisfies, by (3.9), the inequality

$$v(z) \leq c_j, \quad z \in \overline{\mathbb{C}}. \tag{3.13}$$

Observe that if we had equality in (3.13) for some $z_0 \in \Omega_j$, then, by the maximum principle for subharmonic functions and (3.8), this would give

$$v(z) \equiv c_j > 0, \quad z \in \Omega_j,$$

which is in contradiction with the fact that $v(\infty) = 0$. Thus, it follows from (3.12) that

$$U^{\mu_j^+}(z) < U^{\mu^+(G,W,\gamma)}(z) + c_j, \quad z \in \Omega_j.$$

Multiplying the above inequality by γ, adding $-(1-\gamma)U^{\mu^-(G,W,\gamma)}(z) - \log|W(z)|$ to both sides of it, and using (1.6) and (3.11), we obtain that

$$\gamma U^{\mu_j^+}(z) - (1-\gamma)U^{\mu^-(G,W,\gamma)}(z) - \log|W(z)| < F_j, \quad z \in G \cap \Omega_j. \tag{3.14}$$

Next, let $f(z)$ by any function analytic in G. To construct a sequence of weighted rational functions which is uniformly convergent to $f(z)$ on E, we interpolate the analytic function $W^{-(m+n)}(z)Q_n(z)f(z)$ by a polynomial $P_m(z)$, where the choice of interpolation points $\{z_k^{(m+1)}\}_{k=1}^{m+1} \subset \Gamma_j$ and the choice of the polynomial $Q_n(z)$ are described below.

It follows from Krein-Milman theorem that any measure $\mu \in \mathcal{M}(B)$, where $B \subset \mathbb{C}$ is a compact set, is a weak*-limit of discrete measures

$$\sum_{i=1}^{k} \alpha_i^{(k)} \delta_{z_i^{(k)}}, \quad \alpha_i^{(k)} > 0, \ \sum_{i=1}^{n} \alpha_i^{(k)} = 1,$$

where $\{z_i^{(k)}\}_{i=1}^k \subset B,\ k \in \mathbb{N}$ (see [21], pp. 362-363). Since every $\alpha_i^{(k)}$ can be approximated arbitrarily closely by rational numbers, there exists a sequence of points $\{\zeta_j^{(k)}\}_{j=1}^k \subset B,\ k \in \mathbb{N}$, which may not all be distinct, such that

$$\frac{1}{k}\sum_{j=1}^{k} \delta_{\zeta_j^{(k)}} \xrightarrow{*} \mu, \quad \text{as } k \to \infty.$$

We discretize the measures $\mu^-(G, W, \gamma)$ and μ_j^+ so that

$$\nu_n^- := \frac{1}{n}\sum_{k=1}^{n} \delta_{t_k^{(n)}} \xrightarrow{*} \mu^-(G, W, \gamma), \quad \text{as } n \to \infty, \tag{3.15}$$

where $\{t_k^{(n)}\}_{k=1}^n \subset \operatorname{supp} \mu^-(G, W, \gamma) \subset \partial G$ for any $n \in \mathbb{N}$, and

$$\nu_m^+ := \frac{1}{m}\sum_{k=1}^{m} \delta_{z_k^{(m)}} \xrightarrow{*} \mu_j^+, \quad \text{as } m \to \infty, \tag{3.16}$$

where $\{z_k^{(m)}\}_{k=1}^m \subset \operatorname{supp} \mu_j^+ \subset \Gamma_j$ for any $m \in \mathbb{N}$. Then, we define

$$Q_n(z) := \prod_{k=1}^{n} \left(z - t_k^{(n)}\right), \quad n \in \mathbb{N}, \tag{3.17}$$

so that $Q_n(z)$ is not zero in G for any n, and we define a basic polynomial of Lagrange interpolation

$$\omega_{m+1}(z) := \prod_{k=1}^{m+1} \left(z - z_k^{(m+1)}\right), \quad m \in \mathbb{N}. \tag{3.18}$$

On setting $L_j := \bigcup_{\ell=1}^{\ell_0} L_{\ell,j}$, where each $L_{\ell,j} \subset G_\ell \backslash \overline{G}_{\ell,j}$, $\ell = 1, 2, \cdots, \ell_0$, is a rectifiable Jordan curve containing $\overline{G}_{\ell,j}$ in its interior, the polynomial $P_m(z)$, which interpolates $W^{-(m+n)}(z)Q_n(z)f(z)$ in the points $\{z_k^{(m+1)}\}_{k=1}^{m+1}$ of Γ_j, is given by the Hermite interpolation formula (see [20], p. 50):

$$\begin{aligned} &W^{-(m+n)}(z)Q_n(z)f(z) - P_m(z) \\ &\quad = \frac{\omega_{m+1}(z)}{2\pi i}\int_{L_j} \frac{W^{-(m+n)}(t)Q_n(t)f(t)dt}{(t-z)\omega_{m+1}(t)}, \quad z \in E. \end{aligned} \tag{3.19}$$

Multiplying (3.19) by $W^{m+n}(z)/Q_n(z)$ gives

$$\begin{aligned} f(z) - W^{m+n}(z)\frac{P_m(z)}{Q_n(z)} \\ = \frac{W^{m+n}(z)\omega_{m+1}(z)/Q_n(z)}{2\pi i} \int_{L_j} \frac{f(t)dt}{(t-z)W^{m+n}(t)\omega_{m+1}(t)/Q_n(t)}, \end{aligned} \tag{3.20}$$

for $z \in E$. Using (3.15) and (3.17), we have that

$$\lim_{n\to\infty} |Q_n(z)|^{1/n} = \exp\left\{-U^{\mu^-(G,W,\gamma)}(z)\right\} \tag{3.21}$$

holds locally uniformly in $\mathbb{C}\backslash\partial G$, and similarly, using (3.16) and (3.18), we have that

$$\lim_{m\to\infty} |\omega_m(z)|^{1/m} = \exp\left\{-U^{\mu_j^+}(z)\right\} \tag{3.22}$$

holds locally uniformly in $\mathbb{C}\backslash\Gamma_j$. Next, choose any sequence $\{(m_i, n_i)\}_{i=0}^{\infty}$ of pairs of nonnegative integers such that $\lim_{i\to\infty}(m_i + n_i) = \infty$ and $\lim_{i\to\infty} m_i/(m_i + n_i) = \gamma$, and let $\{P_{m_i}(z)/Q_{n_i}(z)\}_{i=0}^{\infty}$ be the associated rational functions from (3.17) and (3.19). From (3.10)-(3.11), we obtain that

$$\lim_{i\to\infty} |W^{m_i+n_i}(z)\omega_{m_i+1}(z)/Q_{n_i}(z)|^{1/(m_i+n_i)} = e^{-F_j}, \tag{3.23}$$

uniformly on E. Also, by (3.14) and the compactness of L_j,

$$\min_{z\in L_j} \lim_{i\to\infty} |W^{m_i+n_i}(z)\omega_{m_i+1}(z)/Q_{n_i}(z)|^{1/(m_i+n_i)} > e^{-F_j}, \tag{3.24}$$

since $\gamma U^{\mu_j^+}(z) - (1-\gamma)U^{\mu^-(G,W,\gamma)}(z) - \log|W(z)|$ is harmonic in $G\bigcap\Omega_j$. Thus, from (3.20) and on using (3.23) and (3.24), it follows that

$$\begin{aligned} \limsup_{i\to\infty} \left\| f - W^{m_i+n_i}\frac{P_{m_i}}{Q_{n_i}} \right\|_E^{1/(m_i+n_i)} \\ \leq \limsup_{i\to\infty} \frac{\|W^{m_i+n_i}(z)\omega_{m_i+1}(z)/Q_{n_i}(z)\|_E^{1/(m_i+n_i)}}{\min_{z\in L_j}|W^{m_i+n_i}(z)\omega_{m_i+1}(z)/Q_{n_i}(z)|^{1/(m_i+n_i)}} < 1. \end{aligned}$$

Hence, the sequence $\{W^{m_i+n_i}(z)P_{m_i}(z)/Q_{n_i}(z)\}_{i=0}^{\infty}$ converges (geometrically) to $f(z)$, uniformly on E, and the sequence of rational functions $\{R_{m_i,n_i}(z) := P_{m_i}(z)/Q_{n_i}(z)\}_{i=0}^{\infty}$ satisfies (1.3i) and (1.3ii), i.e., (G, W, γ) has the rational approximation property. This completes the first part of the proof of Theorem 1.1.

Now, suppose that a triple (G, W, γ), satisfying the conditions of (1.2), has the rational approximation property (1.3). To show that a signed measure $\mu(G, W, \gamma)$, satisfying the conditions of Theorem 1.1, exists, let $\{P_{m_i}(z)/Q_{n_i}(z)\}_{i=0}^{\infty}$ be a sequence of rational functions such that $W^{m_i+n_i}(z)P_{m_i}(z)/Q_{n_i}(z)$ converges to the particular function $f(z) \equiv 1$, locally uniformly in G, and such that $\lim_{i\to\infty} m_i/(m_i + n_i) = \gamma$. We may assume, without loss of generality, that $\deg P_{m_i} = m_i$ and $\deg Q_{n_i} = n_i$. Otherwise, one may define new sequences of polynomials

$$\tilde{P}_{m_i}(z) := P_{m_i}(z) + a_{m_i}z^{m_i}, \quad i \in \mathbb{N},$$

and

$$\tilde{Q}_{n_i}(z) := Q_{n_i}(z) + b_{n_i} z^{n_i}, \quad i \in \mathbb{N},$$

in such a way that $W^{m_i+n_i}(z)\tilde{P}_{m_i}(z)/\tilde{Q}_{n_i}(z)$ also converges to $f(z) \equiv 1$, locally uniformly in G, with $a_{m_i} \neq 0$ and $b_{n_i} \neq 0$ for any $i \in \mathbb{N}$, by choosing $a_{m_i} \neq 0$ and $b_{n_i} \neq 0$ to be sufficiently small. (This will be used in (3.29) below.)

Let $a_{m_i} \neq 0$ be the leading coefficient of $P_{m_i}(z)$ and let

$$\nu_{m_i}^{+} := \frac{1}{m_i} \sum_{P_{m_i}(z_k)=0} \delta_{z_k} \tag{3.25}$$

be the *normalized zero counting measure* for $P_{m_i}(z)$, where δ_{z_k} is a unit point mass at z_k. We count all zeros of $P_{m_i}(z)$ in (3.25), according to their multiplicities, so that

$$\nu_{m_i}^{+}(\mathbb{C}) = 1, \quad i \in \mathbb{N}, \tag{3.26}$$

i.e., these measures are unit positive Borel measures. Analogously, we take $b_{n_i} \neq 0$ as the leading coefficient of $Q_{n_i}(z)$ and define

$$\nu_{n_i}^{-} := \frac{1}{n_i} \sum_{Q_{n_i}(z_k)=0} \delta_{z_k}, \tag{3.27}$$

so that

$$\nu_{n_i}^{-}(\mathbb{C}) = 1, \quad i \in \mathbb{N}. \tag{3.28}$$

Hence, as $W^{m_i+n_i}(z)P_{m_i}(z)/Q_{n_i}(z) \to 1$ locally uniformly in G, then taking logarithms and using the definitions of (1.4), (3.25), and (3.27), we have

$$\begin{aligned} &\frac{1}{m_i+n_i}\log|a_{m_i}| - \frac{m_i}{m_i+n_i}U^{\nu_{m_i}^{+}}(z) + \frac{n_i}{m_i+n_i}U^{\nu_{n_i}^{-}}(z) - \frac{1}{m_i+n_i}\log|b_{n_i}| \\ &\quad + \log|W(z)| = \frac{1}{m_i+n_i}\log\left|W^{m_i+n_i}(z)\frac{P_{m_i}(z)}{Q_{n_i}(z)}\right| \to 0, \quad \text{as } i \to \infty, \end{aligned} \tag{3.29}$$

locally uniformly in G.

If $\hat{\nu}_{m_i}^{+}$ denotes the balayage of $\nu_{m_i}^{+}$ out of the open set $\overline{\mathbb{C}}\backslash\overline{G}$ to $\overline{G}$ (note that the part of $\nu_{m_i}^{+}$ supported on $\overline{G}$ is kept fixed), then

$$U^{\hat{\nu}_{m_i}^{+}}(z) = U^{\nu_{m_i}^{+}}(z) + c_{m_i}, \quad z \in G, \tag{3.30}$$

where $c_{m_i} \geq 0$, supp $\hat{\nu}_{m_i}^{+} \subset \overline{G}$ and $\hat{\nu}_{m_i}^{+}(\mathbb{C}) = \nu_{m_i}^{+}(\mathbb{C}) = 1$ (see Theorem II.4.7 of [13]). Similarly, we have, for the balayage of $\nu_{n_i}^{-}$ from $\overline{\mathbb{C}}\backslash\overline{G}$ to $\overline{G}$, that

$$U^{\hat{\nu}_{n_i}^{-}}(z) = U^{\nu_{n_i}^{-}}(z) + d_{n_i}, \quad z \in G, \tag{3.31}$$

where $d_{n_i} \geq 0$, supp $\hat{\nu}_{n_i}^{-} \subset \overline{G}$ and $\hat{\nu}_{n_i}^{-}(\mathbb{C}) = \nu_{n_i}^{-}(\mathbb{C}) = 1$.

By Helley's Theorem (Theorem 0.1.2 of [13]), we have that the sequences $\{\hat{\nu}_{m_i}^+\}_{i=1}^\infty$ and $\{\hat{\nu}_{n_i}^-\}_{i=1}^\infty$ contain weak* convergent subsequences, so that

$$\hat{\nu}_{m_j}^+ \xrightarrow{*} \mu^+ \quad \text{and} \quad \hat{\nu}_{n_j}^- \xrightarrow{*} \mu^-, \quad \text{as } j \to \infty, \quad j \in J \subset \mathbb{N}, \tag{3.32}$$

where μ^+ and μ^- are positive Borel measures with $\mu^+(\mathbb{C}) = \mu^-(\mathbb{C}) = 1$. One can immediately see, from the locally uniform convergence in G of $W^{m_i+n_i}(z)P_{m_i}(z)/Q_{n_i}(z)$ to unity, that

$$\operatorname{supp}\mu^+ \subset \partial G \quad \text{and} \quad \operatorname{supp}\mu^- \subset \partial G. \tag{3.33}$$

Furthermore, by (3.32),

$$\lim_{\substack{j\to\infty \\ j\in J}} U^{\hat{\nu}_{m_j}^+}(z) = U^{\mu^+}(z) \quad \text{and} \quad \lim_{\substack{j\to\infty \\ j\in J}} U^{\hat{\nu}_{n_j}^-}(z) = U^{\mu^-}(z), \quad z \in G. \tag{3.34}$$

It follows from (3.29), (3.30) and (3.31) that

$$\begin{aligned} &\frac{1}{m_j+n_j}\log|a_{m_j}| + \frac{m_j}{m_j+n_j}c_{m_j} - \frac{m_j}{m_j+n_j}U^{\hat{\nu}_{m_j}^+}(z) + \frac{n_j}{m_j+n_j}U^{\hat{\nu}_{n_j}^-}(z) \\ &\quad - \frac{n_j}{m_j+n_j}d_{n_j} - \frac{1}{m_j+n_j}\log|b_{n_j}| + \log|W(z)| \to 0, \quad \text{as } j\to\infty, j\in J, \end{aligned} \tag{3.35}$$

for any $z \in G$. Consequently,

$$\frac{1}{m_j+n_j}\log|a_{m_j}| + \frac{m_j}{m_j+n_j}c_{m_j} - \frac{n_j}{m_j+n_j}d_{n_j} - \frac{1}{m_j+n_j}\log|b_{n_j}|, \ j \in J,$$

converges, as $j \to \infty$, to a finite limit by (3.34). On defining:

$$F := \lim_{\substack{j\to\infty \\ j\in J}} \left(\frac{1}{m_j+n_j}\log|a_{m_j}| + \frac{m_j}{m_j+n_j}c_{m_j} - \frac{n_j}{m_j+n_j}d_{n_j} - \frac{1}{m_j+n_j}\log|b_{n_j}|\right),$$

we obtain from (3.34) and (3.35) that

$$\gamma U^{\mu^+}(z) - (1-\gamma)U^{\mu^-}(z) - \log|W(z)| = F, \quad z \in G.$$

Finally, from the above equation and (3.33), we see that (1.6) of Theorem 1.1 is satisfied with

$$\mu(G, W, \gamma) := \gamma\mu^+ - (1-\gamma)\mu^-,$$

and with

$$F(G, W, \gamma) := F,$$

which completes the proof of Theorem 1.1. □

Proof of Corollary 1.4 *and Remark* 1.5. – If the rational approximation property (1.3) holds, then $f(z) \equiv 1$ is, in particular, locally uniformly approximable by weighted rational functions in G. On the other hand, the second part of proof of Theorem 1.1 shows that if $f(z) \equiv 1$ can be locally uniformly approximated by weighted rationals, then there exists a measure $\mu(G, W, \gamma)$ satisfying the conditions of Theorem 1.1. But this immediately implies the rational approximation property (1.3).

Concerning Remark 1.5, one can easily see that the second part of proof of Theorem 1.1 holds without change for any fixed $f(z)$, which is analytic and is not equal identically to 0 in G. $\square$

Proof of Corollary 1.6. – Note that if (1.9) holds true for a signed measure μ, satisfying (1.7) and (1.8), then the triple (G, W, γ) has the approximation property (1.3) by Theorem 1.1, so that (1.13) is valid.

Suppose now that the triple (G, W, γ) has the approximation property (1.3). Then by Theorem 1.1, there exists a signed measure $\mu(G, W, \gamma) = \gamma\mu^+(G, W, \gamma) - (1 - \gamma)\mu^-(G, W, \gamma)$, with $\mu^+(G, W, \gamma), \mu^-(G, W, \gamma) \in \mathcal{M}(\partial G)$, such that

$$U^{\mu(G,W,\gamma)}(z) - \log|W(z)| = F(G, W, \gamma), \quad z \in G. \tag{3.36}$$

It follows from (1.8) and (3.36) that

$$U^{\mu(G,W,\gamma)}(z) = U^{\mu}(z) + c, \quad z \in G, \tag{3.37}$$

where $c := F(G, W, \gamma) - F$ is a constant. Since potentials are continuous in the *fine topology* (see Section I.5 of [13]) and since the boundary of each G_ℓ, $\ell = 1, \ldots, \sigma$, in the fine topology is the same as the Euclidean boundary (see Corollary I.5.6 of [13]), then (3.37) also holds for any $z \in \partial G$. Thus,

$$u(z) := U^{\mu(G,W,\gamma)-\mu}(z) = c, \quad z \in \overline{G}. \tag{3.38}$$

Observe that $u(z)$ is harmonic in the unbounded component of $\overline{\mathbb{C}}\backslash\overline{G}$, denoted by Ω (including $z = \infty$) with $u(\infty) = 0$, and that $u(z) \equiv c$ on $\partial\Omega \subset \partial G$. Therefore,

$$u(z) \equiv 0, \quad z \in \Omega \cup \overline{G}, \tag{3.39}$$

by the minimum-maximum principle for harmonic functions and by the continuity of $u(z)$ in the fine topology. Applying a similar argument to the bounded components of $\overline{\mathbb{C}}\backslash\overline{G}$, we obtain from (3.39) that

$$u(z) = U^{\mu(G,W,\gamma)-\mu}(z) \equiv 0, \quad z \in \overline{\mathbb{C}}, \tag{3.40}$$

where

$$(\mu(G, W, \gamma) - \mu)(\mathbb{C}) = 0. \tag{3.41}$$

Integrating (3.40), the logarithmic energy of $\mu(G, W, \gamma) - \mu$ satisfies

$$I(\mu(G, W, \gamma) - \mu) = \int U^{\mu(G,W,\gamma)-\mu}(z) d(\mu(G, W, \gamma) - \mu)(z) = 0,$$

which implies by (3.41) and Theorem 1.16 of [6] that $\mu(G, W, \gamma) - \mu = 0$. Thus,

$$\mu = \mu(G, W, \gamma) = \gamma\mu^+(G, W, \gamma) - (1 - \gamma)\mu^-(G, W, \gamma), \tag{3.42}$$

and (1.9) is established.

Assume, in addition to (1.7) and (1.8), that μ also satisfies (1.10) and (1.11). Clearly,

$$\tau^+(\mathbb{C}) - \tau^-(\mathbb{C}) = \mu(\mathbb{C}) = 2\gamma - 1. \tag{3.43}$$

We shall show that (1.12) and (1.9) are equivalent. Indeed if (1.9) holds, then

$$\tau^+(\mathbb{C}) = \tau^+(\text{supp }\tau^+) = \mu(\text{supp }\tau^+) \le \gamma\mu^+(\text{supp }\tau^+) \le \gamma, \tag{3.44}$$

which gives (1.12). Conversely, if (1.12) is valid, then, for any measure $\omega \in \mathcal{M}(\partial G)$, we have by (1.10) and (3.43) that

$$\begin{aligned}\mu = \tau^+ - \tau^- &= \big(\tau^+ + (\gamma - \tau^+(\mathbb{C}))\omega\big) - \big(\tau^- + (\gamma - \tau^+(\mathbb{C}))\omega\big) = \\ &= \big(\tau^+ + (\gamma - \tau^+(\mathbb{C}))\omega\big) - \big(\tau^- + (1 - \gamma - \tau^-(\mathbb{C}))\omega\big).\end{aligned}$$

Thus, (1.9) holds, for $0 < \gamma < 1$, with

$$\mu^+ = \frac{1}{\gamma}\big(\tau^+ + (\gamma - \tau^+(\mathbb{C}))\omega\big) \quad \text{and} \quad \mu^- = \frac{1}{1-\gamma}\big(\tau^- + (1 - \gamma - \tau^-(\mathbb{C}))\omega\big).$$

Obviously, if $\gamma = 0$ then

$$\mu^+ = \tau^+ = 0 \quad \text{and} \quad \mu^- = \tau^-,$$

and if $\gamma = 1$ then

$$\mu^+ = \tau^+ \quad \text{and} \quad \mu^- = \tau^- = 0.$$

Hence, we conclude that the rational approximation property (1.3) holds for (G, W, γ) if and only if (1.12) is satisfied. □

4. Proofs of results of Section 2

Proof of Theorem 2.1. – First, we recall, by the results of Section IV.2 of [6] (see also Theorem II.4.7 of [13]), that the following are valid:

$$U^{\hat{\nu}^+}(z) = U^{\nu^+}(z) + \int g_\Omega(t,\infty)\,d\nu^+(t), \quad z \in G, \tag{4.1}$$

and

$$U^{\hat{\nu}^-}(z) = U^{\nu^-}(z) + \int g_\Omega(t,\infty)\,d\nu^-(t), \quad z \in G, \tag{4.2}$$

where $g_\Omega(t,\infty)$ is the Green function for Ω with pole at ∞. Using (2.3), (4.1), (4.2) and Frostman's theorem [18], p. 60, it follows, for the measure μ defined in (2.4) and for $z \in G$, that

$$\begin{aligned}&U^\mu(z) - \log|W(z)| = \\ &(2\gamma - 1 + \alpha - \beta)U^{\omega(\infty,\cdot,\Omega)}(z) - \alpha U^{\hat{\nu}^+}(z) + \beta U^{\hat{\nu}^-}(z) - \log|W(z)| = \\ &(2\gamma - 1 + \alpha - \beta)\log\frac{1}{\text{cap}\overline{G}} - \alpha\int g_\Omega(t,\infty)\,d\nu^+(t) + \beta\int g_\Omega(t,\infty)\,d\nu^-(t),\end{aligned} \tag{4.3}$$

where cap $\overline{G}$ denotes the logarithmic capacity of $\overline{G}$ (see [18], p. 55).

Observe that μ, defined by (2.4), satisfies (1.7). Thus, Theorem 2.1 follows from Corollary 1.6 and (4.3). □

Proof of Theorem 2.2. – It is clear that, for $W(z) = z^\alpha$, we have:

$$\log|W(z)| = -\alpha \log\frac{1}{|z|} = -\alpha U^{\delta_0}(z), \quad z \in \mathbb{C}\backslash(-\infty, 0], \tag{4.4}$$

where δ_0 is the unit point mass at $z = 0$ and $\alpha > 0$ is a real number. Since the balayage $\hat{\delta}_0$ of δ_0 out of Ω to $\overline{G}$ is given (see [6], p. 222) by

$$\hat{\delta}_0 = \omega(0, \cdot, \Omega), \tag{4.5}$$

then Theorem 2.2 is an immediate consequence of Theorem 2.1 with $\beta = 0$. □

Proof of Corollary 2.3. – We have already shown in the proofs of Theorems 2.1 and 2.2 that the measure μ of (2.10) satisfies (1.7) and (1.8) of Corollary 1.6, with $W(z) = z^\alpha$. Note that, for $\gamma = 1/2$, (2.10) reduces to

$$\mu = \alpha\omega(\infty, \cdot, \Omega) - \alpha\omega(0, \cdot, \Omega), \tag{4.6}$$

which can be explicitly determined for $\Omega = \overline{\mathbb{C}}\backslash\overline{D_r(a)}$. Indeed, by (4.46) and (4.47) of [11], we have

$$\frac{d\omega(0, \cdot, \Omega)}{ds}(z) = \frac{a^2 - r^2}{2\pi r|z|^2}, \quad |z - a| = r,$$

and

$$\frac{d\omega(\infty, \cdot, \Omega)}{ds}(z) = \frac{1}{2\pi r}, \quad |z - a| = r.$$

This gives, for μ in (4.6), that

$$\frac{d\mu}{ds}(z) = \frac{\alpha}{2\pi r}\left(1 - \frac{a^2 - r^2}{|z|^2}\right), \quad |z - a| = r. \tag{4.7}$$

Observe that

$$\frac{d\mu}{ds}(a + re^{i\theta_0}) = \frac{d\mu}{ds}(a + re^{-i\theta_0}) = 0,$$

where

$$\theta_0 := \frac{\pi}{2} + \arcsin\frac{r}{a}.$$

Also, one can immediately see that

$$\frac{d\mu}{ds}(a + re^{i\theta}) > 0 \text{ for } \theta \in (-\theta_0, \theta_0), \tag{4.8}$$

and that

$$\frac{d\mu}{ds}(a + re^{i\theta}) < 0 \text{ for } \theta \in (\theta_0, 2\pi - \theta_0). \tag{4.9}$$

As we next show, (4.8) and (4.9) give the desired Jordan decomposition (1.10) for μ of (4.7). Recall that, for any Borel set $B \subset \mathbb{C}$,

$$\omega(0, B, \Omega) = m(\Phi(B \cap \partial\Omega)),$$

where

$$dm = \frac{d\theta}{2\pi} \text{ on } \{w \in \mathbb{C} : |w| = 1\}$$

and where

$$\Phi(z) := \frac{r^2 + a(z-a)}{rz}$$

is the conformal mapping of $\Omega = \overline{\mathbb{C}} \backslash \overline{D_r(a)}$ onto $D' = \{w \in \mathbb{C} : |w| > 1\}$, with $\Phi(0) = \infty$ (see [8], p. 37). Since

$$\Phi(a + re^{i\theta_0}) = \exp\left(i\left(\frac{\pi}{2} - \arcsin\frac{r}{a}\right)\right)$$

and

$$\Phi(a + re^{-i\theta_0}) = \exp\left(i\left(\arcsin\frac{r}{a} - \frac{\pi}{2}\right)\right),$$

we obtain from (4.6) that

$$\tau^+(\mathbb{C}) = \mu\left(\{a + re^{i\theta} : \theta \in (-\theta_0, \theta_0)\}\right) =$$
$$\alpha\frac{2\theta_0}{2\pi} - \frac{\alpha}{2\pi}\left(\frac{\pi}{2} - \arcsin\frac{r}{a} - \left(\arcsin\frac{r}{a} - \frac{\pi}{2}\right)\right) = \frac{2\alpha}{\pi}\arcsin\frac{r}{a}.$$

It is obvious that the inequality,

$$\frac{2\alpha}{\pi}\arcsin\frac{r}{a} \leq \frac{1}{2},$$

which corresponds to (1.12), is equivalent to (2.11). Thus, Corollary 2.3 follows from Corollary 1.6. □

Proof of Theorem 2.5. – It was shown in the proof of Theorem 4.3 of [10] (see also the proof of Theorem 2.7 of [11]) that for the measure

$$d\mu_1 = \frac{1}{2\pi}(1 - 2r\cos\theta)d\theta, \quad |z| = r, \tag{4.10}$$

we have

$$U^{\mu_1}(z) - \log|e^{-z}| = U^{\mu_1}(z) + \operatorname{Re} z = \log\frac{1}{r}, \quad z \in \overline{D_r(0)}. \tag{4.11}$$

Thus, one can immediately see, for the measure μ of (2.16), that

$$\begin{aligned} U^{\mu}(z) - \log|e^{-z}| &= \log\frac{1}{r} + (2\gamma - 2)\int_0^{2\pi}\log\frac{1}{|z - re^{i\theta}|}\frac{d\theta}{2\pi} \\ &= \log\frac{1}{r} + (2\gamma - 2)\log\frac{1}{r} = (2\gamma - 1)\log\frac{1}{r}, \quad z \in \overline{D_r(0)}. \end{aligned} \tag{4.12}$$

Note that the density function of μ, given by

$$h(\theta) := \frac{1}{2\pi}(2\gamma - 1 - 2r\cos\theta), \quad \theta \in [0, 2\pi), \tag{4.13}$$

satisfies, for $r \geq |2\gamma - 1|/2$,

$$h(\theta) > 0,\ \theta_0 < \theta < 2\pi - \theta_0, \quad \text{and} \quad h(\theta) < 0,\ -\theta_0 < \theta < \theta_0, \tag{4.14}$$

where

$$\theta_0 := \arccos\left(\frac{2\gamma - 1}{2r}\right). \tag{4.15}$$

Therefore, the Jordan decomposition of μ is immediate from (4.14), with

$$\tau^+(\mathbb{C}) = \int_{\theta_0}^{2\pi-\theta_0} (2\gamma - 1 - 2r\cos\theta)\,\frac{d\theta}{2\pi} = (2\gamma - 1)\,\frac{\pi - \theta_0}{\pi} + \frac{2r}{\pi}\,\sin\,\theta_0.$$

The inequality (1.12) of Corollary 1.6 can be written in this case as

$$(2\gamma - 1)\frac{\pi - \theta_0}{\pi} + \frac{2r}{\pi}\sin\theta_0 \leq \gamma,$$

which simplifies, with the help of (4.15), to

$$\sqrt{4r^2 - (2\gamma - 1)^2} - (2\gamma - 1)\arccos\left(\frac{2\gamma - 1}{2r}\right) \leq \pi(1 - \gamma), \tag{4.16}$$

where $r \geq |2\gamma - 1|/2$. One can verify directly, on denoting the left side of (4.16) by $f_\gamma(r)$, that

$$f'_\gamma(r) = \frac{\sqrt{4r^2 - (2\gamma - 1)^2}}{r} > 0, \qquad \text{for all } r \in \left(\frac{|2\gamma - 1|}{2}, +\infty\right).$$

Hence, $f_\gamma(r)$ is strictly increasing on $(|2\gamma - 1|/2, +\infty)$, with

$$f_\gamma\left(\frac{|2\gamma - 1|}{2}\right) = \left\{\begin{matrix} 0, \text{ for } \quad 2\gamma - 1 \geq 0 \\ \\ \pi(1 - 2\gamma), \text{ for } \quad 2\gamma - 1 < 0 \end{matrix}\right\} < \pi(1 - \gamma),$$

and

$$\lim_{r \to +\infty} f_\gamma(r) = +\infty.$$

Therefore, the equation (2.15) has the unique solution $r_{\max}(\gamma)$, with

$$r_{\max}(\gamma) \geq \frac{|2\gamma - 1|}{2},$$

such that (4.16) holds if and only if $r \in [|2\gamma - 1|/2,\ r_{\max}(\gamma)]$. □

Proof of Theorem 2.6. – To begin, for any pair (m, n) of nonnegative integers, the (m, n)-th Padé rational approximation to e^z is the rational function

$$R_{m,n}(z) = \frac{P_{m,n}(z)}{Q_{m,n}(z)}, \tag{4.17}$$

where

$$\begin{cases} \text{i)} & \deg P_{m,n} \le m \text{ and } \deg Q_{m,n} \le n, \text{with } Q_{m,n}(0) = 1, \text{and} \\ \text{ii)} & e^z - R_{m,n}(z) = O(z^{m+n+1}), \text{ as } z \to 0. \end{cases} \tag{4.18}$$

It is well known that, for any pair (m, n) of nonnegative integers, these polynomials are given explicitly (see [15], p. 242) by

$$P_{m,n}(z) = \sum_{k=0}^{m} \frac{(m+n-k)!m!z^k}{(m+n)!k!(m-k)!},$$

and

$$Q_{m,n}(z) = \sum_{k=0}^{n} \frac{(m+n-k)!n!(-z)^k}{(m+n)!k!(n-k)!},$$

and these polynomials $P_{m,n}(z)$ and $Q_{m,n}(z)$ are called, respectively, the *Padé numerator* and *Padé denominator* of type (m, n) for e^z. It is further known (see [15], eq. (4.8)) that

$$(m+n)!e^z Q_{m,n}(z) = \int_{-z}^{\infty} e^{-t}(t+z)^m t^n dt, \quad \text{for any } z \in \mathbb{C}, \tag{4.19}$$

where the path of integration in (4.19) is the horizontal ray $-z + \mu$ for all $\mu \ge 0$, and similarly (see [15], eq. (4.9)) that

$$(m+n)!(e^z Q_{m,n}(z) - P_{m,n}(z)) = \int_{-z}^{0} e^{-t}(t+z)^m t^n dt \quad \text{for any } z \in \mathbb{C}, \tag{4.20}$$

where the path of integration in (4.20) is chosen to be the line segment from $-z$ to 0. Thus, on dividing the above two equations, we have, with (4.17), that

$$\frac{e^z Q_{m,n}(z) - P_{m,n}(z)}{e^z Q_{m,n}(z)} = 1 - e^{-z} R_{m,n}(z) = \frac{\displaystyle\int_{-z}^{0} e^{-t}(t+z)^m t^n dt}{\displaystyle\int_{-z}^{\infty} e^{-t}(t+z)^m t^n dt}, \tag{4.21}$$

for any $z \in \mathbb{C}$, provided that the $Q_{m,n}(z) \neq 0$. Replacing z and t, respectively, by $(m+n)z$ and $(m+n)t$ in (4.21) gives

$$1 - e^{-(m+n)z} R_{m,n}((m+n)z) = \frac{\displaystyle\int_{-z}^{0} e^{-(m+n)t}(t+z)^m t^n dt}{\displaystyle\int_{-z}^{\infty} e^{-(m+n)t}(t+z)^m t^n dt}. \tag{4.22}$$

Let γ be a fixed number with $0 \leq \gamma \leq 1$. As the treatment of the special cases $\gamma = 0$ and $\gamma = 1$ is similar (see the proof of Theorem 1.1), assume that $0 < \gamma < 1$, and assume that $\{(m_j, n_j)\}_{j=0}^{\infty}$ is an infinite sequence of pairs of nonnegative integers satisfying (1.3i), i.e.,

$$\gamma = \lim_{j\to\infty} \frac{m_j}{m_j + n_j}, \qquad \text{where } \lim_{j\to\infty} (m_j + n_j) = +\infty.$$

From [19], p. 182, it follows that the rational function $R_{m_j,n_j}((m_j+n_j)z)$ has no zeros and no poles in the closed set $\overline{G}_\gamma$ of (2.22), for all j sufficiently large. Consequently, from (4.22), the representation

$$(4.23) \qquad 1 - e^{-(m_j+n_j)z} R_{m_j,n_j}((m_j+n_j)z) = \frac{\displaystyle\int_{-z}^{0} e^{-(m_j+n_j)t}(t+z)^{m_j} t^{n_j}\, dt}{\displaystyle\int_{-z}^{\infty} e^{-(m_j+n_j)t}(t+z)^{m_j} t^{n_j}\, dt}$$

holds for all $z \in G_\gamma$, provided that j is sufficiently large. Noting that the integrands in the two integrals in (4.23) are the same, we set

$$(4.24) \quad h_j(t) = h_j(t;z) := -t + \left(\frac{m_j}{m_j+n_j}\right)\log(t+z) + \left(\frac{n_j}{m_j+n_j}\right)\log t \quad (j \geq 0),$$

so that (4.23) can be equivalently expressed, for all j sufficiently large, as

$$(4.25) \qquad 1 - e^{-(m_j+n_j)z} R_{m_j,n_j}((m_j+n_j)z) = \frac{\displaystyle\int_{-z}^{0} e^{(m_j+n_j)h_j(t)} dt}{\displaystyle\int_{-z}^{\infty} e^{(m_j+n_j)h_j(t)} dt},$$

when $z \in G_\gamma$. Note that if we similarly set

$$(4.26) \qquad \hat{h}_\gamma(t) = \hat{h}_\gamma(t;z) := -t + \gamma\log(t+z) + (1-\gamma)\log t,$$

it follows from (1.3i) that

$$(4.27) \qquad h_j(t;z) \to \hat{h}_\gamma(t;z), \text{ as } j \to \infty, \text{ for any } t \neq -z \text{ and } t \neq 0.$$

The point of the above construction is to prepare for an application of the *steepest descent method* to the two integrals in (4.25), as was earlier done in [15]. As

$$\hat{h}'_\gamma(t) = -1 + \frac{\gamma}{t+z} + \frac{(1-\gamma)}{t} \text{ and } \hat{h}''_\gamma(t) = -\frac{\gamma}{(t+z)^2} - \frac{(1-\gamma)}{t^2},$$

then for $z \in \mathbb{C}\backslash(\mathbb{R}_\gamma \cup \{0\})$, the only zeros of $\hat{h}'_\gamma$ can be verified, with the definition of (2.19), to be the two numbers

$$(4.28) \qquad \hat{t}_\gamma^{\pm}(z) := \frac{1}{2}(1 - z \pm \hat{g}_\gamma(z))$$

and it can be further verified that

$$(4.29) \qquad \hat{t}_\gamma^{+}(z) \neq \hat{t}_\gamma^{-}(z), \quad \text{for any } z \in \mathbb{C}\backslash(\mathbb{R}_\gamma \cup \{0\}),$$

and that

$$\hat{h}''_\gamma(\hat{t}^\pm_\gamma(z)) \neq 0, \text{ for any } z \in \mathbb{C}\backslash(\mathbb{R}_\gamma \cup \{0\}). \tag{4.30}$$

In a completely analogous fashion, the only zeros of $h'_j(t)$, for $z \in \mathbb{C}\backslash(\mathbb{R}_\gamma \cup \{0\})$, are the two distinct numbers (for all j sufficiently large)

$$t^\pm_j(z) := \frac{1}{2}(1 - z \pm g_j(z)), \tag{4.31}$$

where

$$g_j(z) := \sqrt{1 + z^2 - 2z\left(\frac{m_j - n_j}{m_j + n_j}\right)}, \tag{4.32}$$

and where

$$h''_j(t^\pm_j(z)) \neq 0. \tag{4.33}$$

(The excluded point above, $z = 0$, is exceptional in that $\hat{h}'_\gamma(t; 0) = 0$ holds only for the *single* point $t = 1$. On the other hand, the expression in (4.25) is clearly zero for $z = 0$ for every $j \geq 0$, which is ultimately what is needed in our quest in Theorem 2.6 to show that $f(z) \equiv 1$ can be uniformly approximated, on compact subsets of G_γ, by the weighted rational functions $e^{-(m_j+n_j)z} R_{m_j,n_j}((m_j + n_j)z)$.) To summarize, for $0 < \gamma < 1$ and for any $z \in \mathbb{C}\backslash(\mathbb{R}_\gamma \cup \{0\})$, $t^+_j(z)$ and $t^-_j(z)$ are distinct *saddle points* (of order one) of $h_j(t)$, for all j sufficiently large, and $\hat{t}^\pm_\gamma(z)$ are distinct *saddle points* (of order one) of $\hat{h}_\gamma(t)$. (We remark that the functions $g_j(z)$ in (4.32) require analogous cuts $\mathbb{R}_j$ in the z-plane, where in (2.18), the numbers $z^\pm_\gamma$ of (2.17) are replaced by $z^\pm_j := \exp\{\pm\ i \ \arccos\ (\frac{m_j - n_j}{m_j + n_j})\}$ for all j sufficiently large.)

Making use of the fact that $h'_j(t^\pm_j(z)) = 0$ and that $h''_j(t^\pm_j(z)) \neq 0$, for all sufficiently large j, the Taylor expansion of the function h_j about $t^\pm_j(z)$ shows that there exist real numbers $\theta^\pm_j(z)$ such that for ρ real and small,

$$h_j(t^\pm_j(z) + \rho e^{i\theta^\pm_j}) = h_j(t^\pm_j(z)) - \frac{\rho^2}{2}\left|h''_j(t^\pm_j(z))\right| + O(\rho^3), \tag{4.34}$$

as $\rho \to 0$. Then, this means that there is a *local descent path* $\Gamma^\pm_j$ through each of the points $t^\pm_j(z)$ such that (cf. [15], eq. (4.22)) with (4.24),

$$\begin{aligned} I^\pm_j(z) :=& \int_{\Gamma^\pm_j} e^{(m_j+n_j)h_j(t)} dt \\ =& e^{(m_j+n_j)h_j(t^\pm_j(z))} \left|\frac{2\pi}{(m_j + n_j)h''_j(t^\pm_j(z))}\right|^{1/2} e^{i\theta^\pm_j}\left\{1 + O\left(\frac{1}{m_j + n_j}\right)\right\} \\ =& e^{-(m_j+n_j)t^\pm_j(z)}\left[t^\pm_j(z) + z\right]^{m_j}\left[t^\pm_j(z)\right]^{n_j}\left|\frac{2\pi}{(m_j + n_j)h''_j(t^\pm_j(z))}\right|^{1/2} \\ & \times e^{i\theta^\pm_j}\left\{1 + O\left(\frac{1}{m_j + n_j}\right)\right\}, \end{aligned} \tag{4.35}$$

as $j \to \infty$, uniformly on any compact subset of $\mathbb{C}\backslash(\mathbb{R}_\gamma \cup \{0\})$.

The above expressions give the asymptotic behavior of the *local* descent path through the two saddle points $t_j^{\pm}(z)$, and, in the usual fashion, these local descent paths are then continued, beyond the saddle points $t_j^{\pm}(z)$ (on the suitably doubly-cut domain $\mathbb{R}_j$), along *descent paths* $\Gamma_j^{\pm}$, defined (cf. [15], eq. (4.19)) as points $t \in \mathbb{C}$ for which

$$\begin{cases} \text{Im } h_j(t) = \text{Im } h_j(t_j^{\pm}(z)), \text{ and} \\ \\ \text{Re } h_j(t) < \text{Re } h_j(t_j^{\pm}(z)), \quad \text{for } t \neq t_j^{\pm}(z). \end{cases}$$

These descent paths, from (4.24), can have endpoints only at $t = 0$, $t = -z$, or $t = \infty$. More specifically, we note that for z small and not zero, it can be verified that a descent path through $t_j^-(z)$ necessarily has endpoints $t = 0$ and $t = -z$. For the descent path through $t_j^+(z)$, one endpoint is at $t = \infty$, but the other endpoint can be either $t = 0$ or $t = -z$. To show that both of these cases can occur, consider the following two cases:

Case 1. $z > 0$ and z small. The steepest descent path through $t_j^+(z)$ in this case is that interval of the real axis which extends to the endpoints $t = 0$ and $t = \infty$, as shown below in Figure 3, where the arrows indicate the direction of *increasing* Re $h_j(t)$ along these paths.

Fig. 3: Descent paths for $z > 0$.

Case 2. $z < 0$ and small. The steepest descent path through $t_j^+(z)$ is that interval of the real axis which extends to the endpoints $t = -z$ and $t = \infty$, as shown below in Figure 4, where the arrows again indicate the direction of increasing Re $h_j(t)$ along these paths.

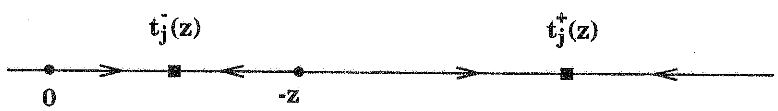

Fig. 4: Descent paths for $z < 0$.

We note, in Case 1, that on integrating from $t = -z$ to $t = \infty$, as is necessary for the denominator integral of (4.25), we pass through *two* saddle points, so that the asymptotic evaluation of this integral involves *both* contributions $I_j^{\pm}(z)$ from (4.35). In this case (and in all cases where the integration path, from $t = -z$ to $t = \infty$, through $t_j^+(z)$ passes through both saddle points $t_j^{\pm}$), the expression in (4.25) is of the form (cf. (4.35))

$$(4.36) \qquad 1 - e^{-(m_j+n_j)z} R_{m_j+n_j}((m_j+n_j)z) = \frac{I_j^-(z)}{I_j^-(z) + I_j^+(z)} = \frac{I_j^-(z)/I_j^+(z)}{1 + (I_j^-(z)/I_j^+(z))},$$

while in Case 2 (and all cases where the integration path from $t = -z$ to $t = \infty$ passes only through $t_j^+(z)$), the expression in (4.25) is of the form

$$(4.37) \qquad 1 - e^{-(m_j+n_j)z} R_{m_j+n_j}((m_j+n_j)z) = \frac{I_j^-(z)}{I_j^+(z)}.$$

As in [3], eqs. (9.21)-(9.22), we define the function:

$$N_j(z) := \frac{g_j(z) + 1 - z\left(\dfrac{m_j - n_j}{m_j + n_j}\right)}{z\sqrt{1 - \left(\dfrac{m_j - n_j}{m_j + n_j}\right)^2}}, \quad \text{for all } z \in \mathbb{C}\backslash(\mathbb{R}_j \cup \{0\}), \tag{4.38}$$

which is analytic and single-valued on $\mathbb{C}\backslash(\mathbb{R}_j \cup \{0\})$. Analogously,

$$N_\gamma(z) := \frac{g_\gamma(z) + 1 - z(2\gamma - 1)}{z\sqrt{1 - (2\gamma - 1)^2}}, \quad \text{for all } z \in \mathbb{C}\backslash(\mathbb{R}_\gamma \cup \{0\}), \tag{4.39}$$

is analytic and single-valued on $\mathbb{C}\backslash(\mathbb{R}_\gamma \cup \{0\})$. With (4.38), the modulus of the ratio $I_j^-(z)/I_j^+(z)$ can be expressed, with the definition of the function $w_j(z)$ of (2.20) (where γ has been replaced by $m_j/(m_j + n_j)$ for all sufficiently large j), as

$$|I_j^-(z)|/|I_j^+(z)| = |w_j(z)|^{m_j+n_j} \frac{\left(1 + O\left(\dfrac{1}{m_j + n_j}\right)\right)}{|N_j(z)|}, \tag{4.40}$$

as $j \to \infty$, uniformly on any compact subset of $\mathbb{C}\backslash(\mathbb{R}_j \cup \{0\})$. But, $1/N_j(z)$ and $1/N_\gamma(z)$ are both analytic in $|z| \le 1$ (cf. [3], Lemma 1) and since $1/N_j(z)$ converges to $1/N_\gamma(z)$ as $j \to \infty$, then $1/N_j(z)$ is locally uniformly bounded in $|z| \le 1$. Thus, consider any compact set E in G_γ. As this compact E is contained in some level curve $\Gamma_{\gamma,\mu} := \{z \in \mathbb{C} : |w_\gamma(z)| = \mu < 1\}$ of G_γ, for μ sufficiently close to unity, then on this set, it is evident that $w_j(z) \to w_\gamma(z)$, as $j \to \infty$, where $|w_\gamma(z)| \le \mu < 1$. Recalling that $R_{m_j,n_j}((m_j + n_j)z)$ has no zeros or poles in $\overline{G}_\gamma$ for any j sufficiently large, then the function $1 - e^{-(m_j+n_j)z} R_{m_j,n_j}((m_j + n_j)z)$ is then analytic in E for all j sufficiently large, and its maximum modulus on E, in either Case 1 or Case 2 of (4.36) or (4.37), is dominated above by $\mu^{m_j+n_j}$ from (4.40). Thus, we have:

$$\lim_{j \to \infty} \|1 - e^{-(m_j+n_j)z} R_{m_j,n_j}((m_j + n_j)z)\|_E = 0. \tag{4.41}$$

But this implies, from Corollary 1.4, that the triple $(G_\gamma, e^{-z}, \gamma)$ has the rational approximation property, which completes the proof of Theorem 2.6. □

REFERENCES

[1] P. B. Borwein and W. Chen, Incomplete rational approximation in the complex plane, *Constr. Approx.*, *11*, 1995, pp. 85–106.

[2] P. B. Borwein, E. A. Rakhmanov and E. B. Saff, Rational approximation with varying weights I, *Constr. Approx.*, *12*, 1996, pp. 223–240.

[3] M. G. de Bruin, E. B. Saff and R. S. Varga, On the zeros of generalized Bessel polynomials, I and II, *Nederl. Akad. Wetensch. Indag. Math.*, *23*, 1981, pp. 1–25.

[4] M. v. Golitschek, Approximation by incomplete polynomials, *J. Approx. Theory*, *28*, 1980, pp. 155–160.

[5] A. B. J. Kuijlaars, The role of the endpoint in weighted polynomial approximation with varying weights, *Constr. Approx.*, *12*, 1996, pp. 287–301.

[6] N. S. Landkof, Foundations of Modern Potential Theory, Springer-Verlag, Berlin, 1972.

[7] G. G. LORENTZ, Approximation by incomplete polynomials (problems and results), Padé and Rational Approximations: Theory and Applications (E. B. Saff and R. S. Varga, eds.), Academic Press, New York, 1977, pp. 289–302.
[8] R. NEVANLINNA, Analytic Functions, Springer-Verlag, New York, 1970.
[9] I. E. PRITSKER, Polynomial approximation with varying weights on compact sets of the complex plane, to appear in *Proc. Amer. Math. Soc.*
[10] I. E. PRITSKER and R. S. VARGA, The Szegő curve, zero distribution and weighted approximation, *Trans. Amer. Math. Soc.*, *349*, 1997, pp. 4085–4105.
[11] I. E. PRITSKER and R. S. VARGA, Weighted polynomial approximation in the complex plane, *Constr. Approx.*, *14*, 1998, pp. 475–492.
[12] E. A. RAKHMANOV, E. B. SAFF and P. C. SIMEONOV, *Rational approximation with varying weights II*, preprint.
[13] E. B. SAFF and V. TOTIK, Logarithmic Potentials with External Fields, Springer-Verlag, Heidelberg, 1997.
[14] E. B. SAFF and R. S. VARGA, On incomplete polynomials, Numerische Methoden der Approximationstheorie, (eds. L. Collatz, G. Meinardus, H. Werner), ISNM 42, Birkhäuser-Verlag, Basel, 1978, pp. 281–298.
[15] E. B. SAFF and R. S. VARGA, On the zeros and poles of Padé approximants to e^z. III, *Numer. Math.*, *30*, 1978, pp. 241–266.
[16] G. SZEGŐ, Über eine Eigenschaft der Exponentialreihe, *Sitzungsber. Berl. Math. Ges.*, *23*, 1924, pp. 50–64.
[17] V. TOTIK, Weighted Approximation with Varying Weights, *Lecture Notes in Math.*, *1569*, Springer-Verlag, Heidelberg, 1994.
[18] M. TSUJI, Potential Theory in Modern Function Theory, Maruzen, Tokyo, 1959.
[19] R. S. VARGA and A. J. CARPENTER, Asymptotics for zeros and poles of normalized Padé approximants to e^z, *Numer. Math.*, *68*, 1994, pp. 169–185.
[20] J. L. WALSH, Interpolation and Approximation by Rational Functions in the Complex Domain, Colloquium Publications, *20*, Amer. Math. Soc., Providence, 1969.
[21] K. YOSIDA, Functional Analysis, Springer-Verlag, Berlin, 1978.

(Manuscript received 16 February 1998.)

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