227

ELSEVIER Linear Algebra and its Applications 302–303 (1999) 33–43

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

Hardy's inequality and ultrametric matrices

John Todd [a,1], Richard S. Varga [b,*]

[a] *Department of Mathematics, California Institute of Technology, Pasadena, CA 91125-0001, USA*
[b] *Institute for Computational Mathematics, Kent State University, Kent, OH 44242-0001, USA*

Received 22 June 1998; accepted 4 August 1998

Submitted by U. Rothblum

Abstract

For any $p > 1$ and for any sequence $\{a_j\}_{j=1}^{\infty}$ of nonnegative numbers, a classical inequality of Hardy gives that

$$\sum_{k=1}^{n}\left(\frac{\sum_{i=1}^{k} a_i}{k}\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_{k=1}^{n} a_k^p \quad \text{for each } n \in \mathbb{N},$$

unless all $a_j = 0$, where the constant $[p/(p-1)]^p$ is best possible. Here, we investigate this inequality in the case $p = 2$, and show how it can be interpreted in terms of symmetric ultrametric matrices. From this, a generalization of Hardy's inequality, in the case $p = 2$, is derived.

1. Introduction

In 1920, Hardy [2] established the following inequality.

Theorem 1. *If $p > 1$ and if $\{a_j\}_{j=1}^{\infty}$ is any sequence of nonnegative numbers, then*

$$\sum_{k=1}^{n}\left(\frac{\sum_{i=1}^{k} a_i}{k}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{k=1}^{n} a_k^p, \quad \textit{for all } n \in \mathbb{N}, \tag{1.1}$$

unless all $a_j = 0$. The constant $[p/(p-1)]^p$ is best possible.

[*] Corresponding author. E-mail: varga@mcs.kent.edu
[1] E-mail: johntodd@cco.caltech.edu

PII: S0024-3795(98)10168-4

This inequality of Eq. (1.1) arose in the course of attempts to simplify the proof of Hilbert's Double Series Theorem. Initially, Hardy [2] was not able to fix the best constant, $[p/(p-1)]^p$, in Eq. (1.1), and this (called an "imperfection" in cf. Ref. [3], p. 240) was later determined by Landau [5]. A complete proof of Theorem 1, including the sharpness of the constant $[p/(p-1)]^p$, can be found in Ref. [3], pp. 240–242.

Our interest here is in linear algebra connections of Theorem 1 in the case $p=2$, as was recently considered by Wang and Yuan [12]. In this case, the inequality in Eq. (1.1), for any sequence $\{a_j\}_{j=1}^{\infty}$ of nonnegative numbers, reduces to

$$\sum_{k=1}^{n}\left(\frac{\sum_{i=1}^{k} a_i}{k}\right)^2 < 4\sum_{k=1}^{n} a_k^2, \quad \text{for all } n \in \mathbb{N}, \tag{1.2}$$

unless all $a_j = 0$, and this inequality of Eq. (1.2) can be interpreted, via matrix theory, as follows. For any $n \in \mathbb{N}$, consider the real symmetric matrix $B_n = [b_{i,j}(n)] \in \mathbb{R}^{n\times n}$, whose entries are defined by

$$b_{i,j}(n) := \sum_{k=\max(i,j)}^{n} \frac{1}{k^2} \quad (1 \leqslant i,\ j \leqslant n). \tag{1.3}$$

(The matrix B_n is given explicitly in Eq. (2.5) for the case $n=4$.) Then, it can be verified that if $\mathbf{a} = [a_1, a_2, \ldots, a_n]^{\mathrm{T}}$ is any vector in $\mathbb{R}^n$, the quadratic form $\mathbf{a}^{\mathrm{T}}B_n\mathbf{a}$ is given by

$$\mathbf{a}^{\mathrm{T}}B_n\mathbf{a} = \sum_{i,j=1}^{n} b_{i,j}(n)a_i a_j = \sum_{k=1}^{n}\left(\frac{\sum_{i=1}^{k} a_i}{k}\right)^2.$$

As $\mathbf{a}^{\mathrm{T}}\mathbf{a} = \sum_{j=1}^{n} a_j^2$, the inequality of Eq. (1.2) then reduces to

$$\mathbf{a}^{\mathrm{T}}B_n\mathbf{a} < 4\mathbf{a}^{\mathrm{T}}\mathbf{a}, \quad \text{for all } \mathbf{a} \neq 0 \text{ in } \mathbb{R}^n, \quad \text{all } n \in \mathbb{N},$$

or equivalently, the associated Rayleigh–Ritz quotient for the matrix B_n satisfies

$$\frac{\mathbf{a}^{\mathrm{T}}B_n\mathbf{a}}{\mathbf{a}^{\mathrm{T}}\mathbf{a}} < 4, \quad \text{for all } \mathbf{a} \neq 0 \text{ in } \mathbb{R}^n, \quad \text{all } n \in \mathbb{N}. \tag{1.4}$$

If, for a real symmetric matrix C, we use the notation

$$\sigma_{\max}(C) := \max\{\lambda_i\colon\ \lambda_i \text{ is an eigenvalue of } C\}, \tag{1.5}$$

then as Eq. (1.4) is valid for any $\mathbf{a} \neq \mathbf{0}$ in $\mathbb{R}^n$, it follows (cf. Ref. [4], p. 176) that

$$\sigma_{\max}(B_n) < 4, \quad \text{for all } n \in \mathbb{N}. \tag{1.6}$$

The real symmetric matrix B_n of Eq. (1.3) turns out to be nonsingular for any $n \in \mathbb{N}$, since its inverse can be verified, by induction, to be the following $n \times n$ tridiagonal matrix.

$$B_n^{-1} =: [\beta_{i,j}(n)] =$$

$$\begin{bmatrix} 1 & -1 & & & & \\ -1 & [1^2+2^2] & -2^2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & -(n-2)^2 & [(n-2)^2+(n-)^2] & -(n-1)^2 \\ & & & & -(n-1)^2 & [(n-1)^2+n^2] \end{bmatrix}. \tag{1.7}$$

It is then easy to see, using the following simple linear algebra properties from Eq. (1.7) for B_n^{-1}, that B_n^{-1}, and hence B_n, are real symmetric and positive definite for all $n \in \mathbb{N}$.

(i) B_n^{-1} is real and symmetric;

(ii) the Gerschgorin disks for B_n^{-1}, namely,

$\{z \in \mathbb{C} : |z - \beta_{i,i}(n)| \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{n} |\beta_{i,j}(n)|\}$, only intersect the nonnegative

real axis; more precisely, all eigenvalues of B_n^{-1} lie in the interval $[0, n^2 + 2(n-1)^2]$, and B_n^{-1} is thus nonnegative definite.

(iii) B_n^{-1} is irreducible (i.e., its directed graph is strongly connected (cf. Ref. [10], p. 19);

(iv) all the Gerschgorin disks for B_n^{-1} pass through $z = 0$, except for the final Gerschgorin disk. (1.8)

Of course, as B_n and B_n^{-1} both exist, then $z = 0$ cannot be an eigenvalue of B_n^{-1}, so that (cf. 1.8 ii) B_n^{-1}, and hence B_n, are both real symmetric and positive definite. This can also be deduced as follows. If $z = 0$ were an eigenvalue of B_n^{-1}, it would be a boundary point of the union of its Gerschgorin disks, and, by a famous result of Olga Taussky (cf. Refs. [9], [10, p. 20]), all the Gerschgorin circles would, because B_n^{-1} is irreducible from 1.8 iii, necessarily have to pass through $z = 0$. As this is not the case from 1.8 iv, then $z = 0$ is not an eigenvalue of B_n^{-1}, and B_n^{-1} and B_n are thus positive definite.

Actually, B_n^{-1} is a *Stieltjes matrix*, since $B_n^{-1} = [\beta_{i,j}(n)]$ is real symmetric and positive definite with (cf. Eq. (1.7)) $\beta_{i,j}(n) \leqslant 0$ for all $i \neq j$. We will make use of this in the next section.

To conclude this section, we first remark that there is a very rich literature on generalizations of Hardy's inequality (1.1), and this can be found in the books by Bennett [1], Hardy et al. [3], Opic and Kufner [7] and Wilf [13]. Most of the results on Hardy's inequality are established using real analysis techniques, and what caught our attention was the sole use of linear algebra techniques in the recent paper by Wang and Yuan [12]. In particular, it is shown in Ref. [12] that the constant 4 in Eq. (1.2) is best possible by equivalently showing that $B_n^{-1} - \frac{1}{4}I_n$ is symmetric and positive definite for each $n \in \mathbb{N}$, and that, for any $\lambda > \frac{1}{4}$, $B_n^{-1} - \lambda I_n$ fails to be positive definite for all $n \in \mathbb{N}$. (This can be used to show that, for each $n > 1$, there is a positive diagonal matrix X_n in $\mathbb{R}^{n\times n}$ such that (cf. 18 iv) the left real boundary point of each of the first $n-1$ Gerschgorin disks for $X_n B_n^{-1} X_n$ is the point $x = \frac{1}{4}$, while the left real boundary of the final Gerschgorin disk exceeds $\frac{1}{4}$.)

2. Ultrametric matrices

We begin with a definition from Ref. [11].

Definition 1. A real symmetric matrix $A = [a_{i,j}] \in \mathbb{R}^{n\times n}$ is called a *symmetric ultrametric matrix* if it can be represented, in terms of a binary rooted tree, as the following sum of rank-one matrices:

$$A = \sum_{\ell=1}^{2n-1} \tau_\ell \mathbf{u}_\ell \mathbf{u}_\ell^{\mathrm{T}}, \tag{2.1}$$

where the τ_ℓ's are nonnegative numbers, the vectors $\mathbf{u}_\ell$ in $\mathbb{R}^n$ have only components of zeros and ones, and

$$\operatorname{span}\{\mathbf{u}_\ell\colon\ \tau_\ell > 0\} = \mathbb{C}^n. \tag{2.2}$$

We remark that a symmetric ultrametric matrix of Definition 1 is a generalization of the original concept of a symmetric *strictly* ultrametric matrix of Martínez et al. [6].

A result of Ref. [11] is as follows.

Theorem 2. *Let $A = [a_{i,j}] \in \mathbb{R}^{n\times n}$ be a symmetric ultrametric matrix in the sense of Definition 1. Then, A is positive definite and its inverse, $A^{-1} := [\alpha_{i,j}] \in \mathbb{R}^{n\times n}$, is a diagonally dominant Stieltjes matrix, i.e., $\alpha_{i,j} \leqslant 0$ for all $i \neq j$ and*

$$\alpha_{i,i} \geqslant \sum_{\substack{j=1\\ j\neq i}}^{n} |\alpha_{i,j}| \quad \text{for all } 1 \leqslant i \leqslant n, \tag{2.3}$$

with strict inequality holding for at least one i. Moreover,

$$a_{i,j} = 0 \quad \textit{implies} \quad \alpha_{i,j} = 0 \quad (\textit{but not necessarily conversely}). \tag{2.4}$$

As an example of a symmetric ultrametric matrix, consider the matrix $B_n = [b_{i,j}(n)]$ of Eq. (1.3) for the special case $n = 4$.

$$B_4 = \left[\begin{array}{c|c|c|c} 1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16} & \frac{1}{4}+\frac{1}{9}+\frac{1}{16} & \frac{1}{9}+\frac{1}{16} & \frac{1}{16} \\ \hline \frac{1}{4}+\frac{1}{9}+\frac{1}{16} & \frac{1}{4}+\frac{1}{9}+\frac{1}{16} & \frac{1}{9}+\frac{1}{16} & \frac{1}{16} \\ \hline \frac{1}{9}+\frac{1}{16} & \frac{1}{9}+\frac{1}{16} & \frac{1}{9}+\frac{1}{16} & \frac{1}{16} \\ \hline \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} \end{array}\right]. \tag{2.5}$$

It can be verified that B_4 can be expressed, as in Eq. (2.1), as the sum $\sum_{\ell=1}^{7} \tau_\ell \mathbf{u}_\ell \mathbf{u}_\ell^{\mathrm{T}}$, where

$$\begin{array}{llll} \tau_1 = \dfrac{1}{16}; & \mathbf{u}_1 = [1,1,1,1]^{\mathrm{T}} & \tau_2 = 0; & \mathbf{u}_2 = [0,0,0,1]^{\mathrm{T}} \\ \tau_3 = \dfrac{1}{9}; & \mathbf{u}_3 = [1,1,1,0]^{\mathrm{T}} & \tau_4 = 0; & \mathbf{u}_4 = [0,0,1,0]^{\mathrm{T}} \\ \tau_5 = \dfrac{1}{4}; & \mathbf{u}_5 = [1,1,0,0]^{\mathrm{T}} & \tau_6 = 0; & \mathbf{u}_6 = [0,1,0,0]^{\mathrm{T}} \\ \tau_7 = 1; & \mathbf{u}_7 = [1,0,0,0]^{\mathrm{T}} & & \end{array} \tag{2.6}$$

and its associated binary rooted tree is shown in Fig. 1. (The *root* of this tree, is the vertex $\{1,2,3,4\}$, at the top of Fig. 1. If we call the vertices $\{1\}, \{2\}, \{3\}$, and $\{4\}$ the *leaves* of the tree, then the tree is a *binary* tree since each vertex, not a leaf, determines exactly two subsequent arcs in the graph.) It can also be seen

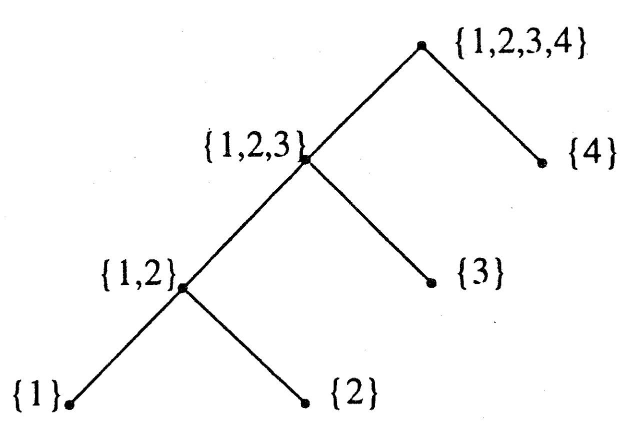

Fig. 1. Associated binary rooted tree for B_4 of Eq. (2.5).

from Eq. (2.6) that condition (2.2) of Definition 1 is valid. As such, Theorem 2 gives that B_4^{-1} is an irreducibly diagonally dominant Stieltjes matrix, with strict diagonal dominance holding for at least one row of B_4^{-1}. These properties B_4^{-1} can be directly seen from the explicit form of B_4^{-1} in Eq. (2.7).

$$B_4^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 5 & -4 & 0 \\ 0 & -4 & 13 & -9 \\ 0 & 0 & -9 & 25 \end{bmatrix}. \tag{2.7}$$

The example above explicitly shows that the matrix B_4 of Eq. (1.3) is a symmetric ultrametric matrix, in the sense of Definition 1, but this can be easily seen to be true for *all* $n \in \mathbb{N}$.

Theorem 3. *For each* $n \in \mathbb{N}$, *let* $B_n \in \mathbb{R}^{n\times n}$ *be the matrix defined in Eq.* (1.3). *Then, its inverse* B_n^{-1} *of Eq.* (1.7) *is a symmetric ultrametric matrix.*

Proof. This is the special case $\{r_k := 1/k^2\}_{k=1}^n$ of Theorem 4, to be given below. □

The actual eigenvalues of B_4 of Eq. (2.5) are 0.03337, 0.09580, 0.33543, and 1.61871, all truncated to five decimal digits, so that (cf. Eq. (1.6))

$$\sigma_{\max}(B_4) = 1.61871 < 4. \tag{2.8}$$

This last inequality is hardly sharp, and sharpness, it turns out, can be established only on letting n, the order of B_n, tend to infinity. We remark that it is evident from Eq. (1.4) and Eq. (1.6) that, for a *fixed* $n \in \mathbb{N}$, the sharpest inequality in Eq. (1.2), for this value of n, is evidently obtained by replacing the constant 4 in Eq. (1.2) by $\sigma_{\max}(B_n)$. But, this also raises the related question: can one similarly *add* a term to the left side of Hardy's inequality (1.1), for this value of n, so as to obtain a sharper inequality in (1.1)? Such a term is given in Theorem 2 of Wang and Yuan [12] for the case $p=2$, while a similar (but different) result appeared earlier in Ref. [8], p. 688, line-6 for the general case $1<p<\infty$.

For readers who may be interested in learning more about ultrametric matrices and their applications, we have added a list of papers on this topic, namely [14–19].

3. A generalization

Consider any n positive numbers $\{r_j\}_{j=1}^n$, i.e.,

$$r_j > 0 \quad (1 \leqslant j \leqslant n), \tag{3.1}$$

and consider its associated real symmetric matrix $A = [a_{i,j}(n)] \in \mathbb{R}^{n\times n}$, defined by

$$a_{i,j}(n) := \sum_{k=\max(i,j)}^{n} \frac{1}{r_k} \quad (1 \leqslant i,\ j \leqslant n). \tag{3.2}$$

Then, for the matrix $A = [a_{i,j}(n)] \in \mathbb{R}^{n\times n}$ defined in Eq. (3.2), its associated quadratic form, for $\mathbf{u} = [u_1,\ u_2, \ldots, u_n]^{\mathrm{T}} \in \mathbb{R}^n$, satisfies

$$\mathbf{u}^{\mathrm{T}} A \mathbf{u} = \sum_{k=1}^{n} \frac{\left(\sum_{i=1}^{k} u_i\right)^2}{r_k}, \tag{3.3}$$

so that the quadratic form, in Eq. (1.2) of Hardy's inequality, corresponds to the case $\{r_k := k^2\}_{k=1}^{n}$.

We then establish the following result.

Theorem 4. *With the assumption of Eq.* (3.1), *the associated $n \times n$ matrix $A = [a_{i,j}(n)]$ of Eq.* (3.2) *is a symmetric ultrametric matrix, in the sense of Definition* 1, *and is hence positive definite. Its inverse, A^{-1}, is a positive definite Stieltjes matrix which satisfies the conditions of Theorem* 2.

Proof. If suffices to show that the matrix A of Eq. (3.2) satisfies conditions (2.1) and (2.2) of Definition 1. For the expansion of Eq. (2.1), the following choices of nonzero τ_ℓ's and their associated vectors $\mathbf{u}_\ell$ (in $\mathbb{R}^n$), defined by

$$\begin{aligned} \tau_1 &= \frac{1}{r_n}, \quad \mathbf{u}_1 = [1, 1, \ldots, 1, 1]^{\mathrm{T}}; \\ \tau_2 &= \frac{1}{r_{n-1}}, \quad \mathbf{u}_2 = [1, 1, \ldots, 1, 0]^{\mathrm{T}}; \\ &\vdots \\ \tau_n &= \frac{1}{r_1}, \quad \mathbf{u}_n = [1, 0, \ldots, 0, 0]^{\mathrm{T}}, \end{aligned} \tag{3.4}$$

directly give Eq. (2.1). Then from Eq. (3.4), we see that

$$\operatorname{span}\{\mathbf{u}_\ell : \tau_\ell > 0\} = \operatorname{span}\{\mathbf{u}_1,\ \mathbf{u}_2,\ \ldots,\ \mathbf{u}_n\} = \mathbb{C}^n, \tag{3.5}$$

as required in Eq. (2.2) of Definition 1. □

For the matrix A of Eq. (3.2), we know that its inverse is a positive definite diagonally dominant Stieltjes matrix, with strict inequality, in the diagonal dominance, holding for at least one row of A^{-1}. But, this inverse, A^{-1}, has the explicit tridiagonal form

$$A^{-1} = \begin{bmatrix} r_1 & -r_1 & & & \\ -r_1 & (r_1+r_2) & -r_2 & & \\ & \ddots & \ddots & \ddots & \\ & & -r_{n-2} & (r_{n-2}+r_{n-1}) & -r_{n-1} \\ & & & -r_{n-1} & (r_{n-1}+r_n) \end{bmatrix}. \quad (3.6)$$

Note that the choice of $r_1 = r_2 = \cdots = r_n = 1$ in Eq. (3.1) gives in Eq. (3.6) one of the best known matrices in all of numerical analysis!

To extend the above results, consider now an infinite sequence $\{r_j\}_{j=1}^{\infty}$ of positive real numbers, i.e., (cf. Eq. (3.1))

$$r_j > 0 \quad (\text{all} \quad j \in \mathbb{N}). \quad (3.7)$$

We seek now a generalization of the special case $p = 2$ of Hardy's inequality (1.2), which similarly holds, as in Eq. (1.2) for *all* $n \in \mathbb{N}$. For each $n \geqslant 1$, let A_n denote the $n \times n$ matrix of Eq. (3.2) for the first n terms of the sequence $\{r_j\}_{j=1}^{\infty}$ and consider the associated sequence $\{A_n\}_{n=1}^{\infty}$ of matrices. From Theorem 4, each A_n is a symmetric ultrametric matrix, and each is therefore positive definite. Next, observe from Eq. (3.2) that, for any $n > 1$, we can express A_n in terms of A_{n-1}, using bordered matrices, by means of

$$A_n = \left[\begin{array}{ccc|c} & & & 0 \\ & A_{n-1} & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right] + \frac{1}{r_n}\xi_n\xi_n^{\mathrm{T}}, \quad \text{where } \xi_n := [1, 1, \ldots, 1]^{\mathrm{T}} \in \mathbb{R}^n. \quad (3.8)$$

The matrix $\xi_n\xi_n^{\mathrm{T}}$, the rank-one matrix in $\mathbb{R}^{n\times n}$ having all its entries unity, has all eigenvalues zero, except for one eigenvalue n. Thus, the matrix $(1/r_n)\xi_n\xi_n^{\mathrm{T}}$ is real symmetric and nonnegative definite, since $r_n > 0$ from Eq. (3.7). If we denote the bordered $n \times n$ matrix in Eq. (3.8) by $\tilde{A}_n$, then $A_n = \tilde{A}_n + (1/r_n)\xi_n\xi_n^{\mathrm{T}}$ is the sum of two real symmetric and nonnegative definite matrices. Assuming that the eigenvalues of $\tilde{A}_n$, $(1/r_n)\xi_n\xi_n^{\mathrm{T}}$, and A_n, respectively called $\{\lambda_j(\tilde{A}_n)\}_{j=1}^{n}$, $\{\lambda_j(1/r_n)\xi_n\xi_n^{\mathrm{T}})\}_{j=1}^{n}$ and $\{\lambda_j(A_n)\}_{j=1}^{n}$, are all arranged in increasing order, then it follows from Weyl's Theorem (cf. Ref. [4], p. 181) that

$$\lambda_n(\tilde{A}_n) + \lambda_1\left(\frac{1}{r_n}\xi_n\xi_n^{\mathrm{T}}\right) \leqslant \lambda_n(A_n) \leqslant \lambda_n(\tilde{A}_n) + \lambda_n\left(\frac{1}{r_n}\xi_n\xi_n^{\mathrm{T}}\right).$$

Thus, as $\lambda_1((1/r_n)\xi_n\xi_n^{\mathrm{T}}) = 0$ and $\lambda_n((1/r_n)\xi_n\xi_n^{\mathrm{T}}) = n/r_n$ we have, with the definition of Eq. (1.5), that

$$\sigma_{\max}(A_{n-1}) \leqslant \sigma_{\max}(A_n) \leqslant \sigma_{\max}(A_{n-1}) + \frac{n}{r_n}. \quad (3.9)$$

The first inequality of Eq. (3.9) can be sharpened as follows. Note from Eq. (3.8) that

$$A_n := \left(\tilde{A}_n + \frac{1}{r_n} I_n\right) + B_n, \quad \text{where } B_n := \frac{1}{r_n}\xi_n\xi_n^{\mathrm{T}} - \frac{1}{r} I_n$$

is a nonnegative matrix with positive off-diagonal entries for all $n \geqslant 2$. It follows from the Perron–Frobenius Theorem (cf. Ref. [10], p. 22) that

$$\sigma_{\max}(A_{n-1}) + \frac{1}{r_n} < \sigma_{\max}(A_n) \quad \text{for all } n \geqslant 2.$$

Thus, Eq. (3.9) can be sharpened to

$$\sigma_{\max}(A_{n-1}) + \frac{1}{r_n} < \sigma_{\max}(A_n) \leqslant \sigma_{\max}(A_{n-1}) + \frac{n}{r_n}. \tag{3.10}$$

As $\sigma_{\max}(A_1) = 1/r_1$, it follows by induction from Eq. (3.10) that

$$\sum_{j=1}^{n} \frac{1}{r_j} < \sigma_{\max}(A_n) \leqslant \sum_{j=1}^{n} \frac{j}{r_j} \quad \text{for any } n \geqslant 2, \tag{3.11}$$

where equality holds throughout in Eq. (3.11) when $n = 1$.

Returning to Eq. (3.10), it is evident that there is a unique positive number s_n with $1 < s_n \leqslant n$, for each $n \geqslant 2$, such that

$$\sigma_{\max}(A_n) = \sigma_{\max}(A_{n-1}) + \frac{s_n}{r_n}, \tag{3.12}$$

and, with $s_1 := 1$, we deduce from Eq. (3.12) that

$$\sigma_{\max}(A_n) = \sum_{j=1}^{n} \frac{s_j}{r_j}, \quad \text{where } 1 < s_j \leqslant j \text{ for all } j \geqslant 2. \tag{3.13}$$

In particular, it follows that $\{\sigma_{\max}(A_n)\}_{n=1}^{\infty}$ is a strictly increasing sequence of positive numbers. Thus, set

$$\gamma := \lim_{n \to \infty} \sigma_{\max}(A_n), \tag{3.14}$$

so that either γ is finite and positive, or $\gamma = +\infty$. Note that when γ is finite and positive, it follows, from Eq. (3.14) and the definition of $\sigma_{\max}(A_n)$, not only that

$$\frac{\mathbf{a}^{\mathrm{T}} A_n \mathbf{a}}{\mathbf{a}^{\mathrm{T}} \mathbf{a}} < \gamma, \quad \text{for all } \mathbf{a} \neq \mathbf{0} \text{ in } \mathbb{R}^n, \text{ all } n \in \mathbb{N}, \tag{3.15}$$

but also that the above inequality is *sharp*, i.e., γ in Eq. (3.15) *cannot* be reduced because of Eq. (3.14).

This naturally brings us to the question of when the infinite sequence $\{r_n\}_{n=1}^{\infty}$ of positive numbers in Eq. (3.7) results in a finite and positive γ in Eq. (3.14). This is considered in

Theorem 5. *Let* $\{r_n\}_{n=1}^{\infty}$ *be any infinite sequence of positive numbers. Then,* γ *of Eq.* (3.14) *is finite if and only if* $\sum_{j=1}^{\infty} s_j/r_j$ *is convergent, where* s_j*, defined in Eq.* (3.12)*, satisfies* $1 < s_j \leqslant j$ *for all* $j \geqslant 2$. *In particular,* γ *is finite implies* $\sum_{j=1}^{\infty} 1/r_j < \infty$. *Conversely, if*

$$\sum_{j=1}^{\infty} \frac{j}{r_j} =: \omega < \infty,$$

then $\gamma \leqslant \omega$.

Proof. The first part follows directly from Eq. (3.13) and Eq. (3.14). If γ is finite, the $\sum_{j=1}^{\infty} s_j/r_j$ is convergent, and as $j \geqslant s_j > 1$, then $\sum_{j=1}^{\infty} 1/r_j$ is also convergent, and if $\sum_{j=1}^{\infty} j/r_j$ is convergent, then so is $\gamma = \sum_{j=1}^{\infty} s_j/r_j$. □

As a consequence of Eq. (3.15) and Theorem 5, we have the result of the following generalization of Hardy's inequality (1.2).

Theorem 6. *Let* $\{r_n\}_{n=1}^{\infty}$ *be any infinite sequence of positive numbers for which* $\sum_{j=1}^{\infty} s_j/r_j < \infty$. *Then,* γ *of Eq.* (3.14) *is finite and positive, and for any sequence* $\{u_j\}_{j=1}^{\infty}$ *of nonnegative numbers,*

$$\sum_{k=1}^{n} \frac{\left(\sum_{i=1}^{k} u_i\right)^2}{r_k} < \gamma \sum_{i=1}^{n} u_i^2 \quad \text{for all } n \in \mathbb{N}, \tag{3.16}$$

unless all $u_j = 0$. *The constant* γ *is best possible.*

References

[1] G. Bennett, Factorizing the classical inequalities, Memoirs of the Am. Math. Soc., number 567, Am. Math. Soc., Providence, RI, 1996.
[2] G.H. Hardy, Note on a theorem of Hilbert, Math. Z. 6 (1920) 314–317.
[3] G.H. Hardy, J.E. Littlewood, G. Pólya, Inequalities, second edition, Cambridge Univ. Press, Cambridge, 1952.
[4] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge Univ. Press, Cambridge, 1985.
[5] E. Landau, A note on a theorem concerning series with positive terms, J. London Math. Soc. 1 (1926) 38–39.
[6] S. Martínez, G. Michon, J. San Martín, Inverses of ultrametric matrices are of Stieltjes type, SIAM J. Matrix Anal. Appl. 15 (1994) 98–106.
[7] B. Opic, A. Kufner, Hardy-type Inequalities, Pitman Research Notes in Mathematics Series 219, Longman Scientific and Technical, Harlow, Essex, UK, 1990.
[8] R. Redheffer, Recurrent inequalities, Proc. London Math. Soc. 17 (1967) 683–699.
[9] O. Taussky, Bounds for the characteristic roots of matrices, Duke Math. J. 15 (1948) 1043–1044.
[10] R.S. Varga, Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1962.
[11] R.S. Varga, R. Nabben, On symmetric ultrametric matrices, in: L. Reichel, A. Ruttan, R.S. Varga (Eds.), Numerical Linear Algebra, Walter de Gruyter, New York, 1993, pp. 193–199.

[12] I. Wang, Y. Yuan, A stronger extension of the Hardy inequality, Linear Algebra Appl. 270 (1998) 275–296.

[13] H.S. Wilf, Finite Sections of Some Classical Inequalities, Springer, New York, 1970.

[14] R. Nabben, R.S. Varga, A linear algebra proof that the inverse of a strictly ultrametric matrix is a strictly diagonally dominant Stieltjes matrix, SIAM J. Matrix Anal. Appl. 15 (1994) 107–113.

[15] J.J. McDonald, M. Neumann, H. Schneider, M.J. Tsatsomeros, Inverse M-matrix inequalities and generalized ultrametric matrices, Linear Algebra Appl. 220 (1995) 329–349.

[16] R. Nabben, R.S. Varga, Generalized ultrametric matrices – a class of inverse M-matrices, Linear Algebra Appl. 220 (1995) 365–390.

[17] R.S. Varga, R. Nabben, An algorithm for determining if the inverse of a strictly diagonally dominant matrix is strictly ultrametric, Numer. Math. 65 (1993) 493–501.

[18] R. Nabben, R.S. Varga, On classes of inverse Z-matrices, Linear Algebra Appl. 223/224 (1995) 521–552.

[19] C. Dellacherie, S. Martínez, J. San Martín, Ultrametric matrices and induced Markov chains, Adv. in Appl. Math. 17 (1996) 169–183.