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CONSTRUCTIVE APPROXIMATION

On Zeros of Polynomials Orthogonal over a Convex Domain

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Abstract. We establish a discrepancy theorem for signed measures, with a given positive part, which are supported on an arbitrary convex curve. As a main application, we obtain a result concerning the distribution of zeros of polynomials orthogonal on a convex domain.

1. Introduction and Main Results

Let $G \subset \mathbf{C}$ be a bounded Jordan domain, and let $h(z)$ be a weight function on G, i.e., a function which is positive and measurable on G. Next, let $Q_n(z) = Q_n(h, z) = \lambda_n z^n + \cdots$, $\lambda_n > 0, n = 0, 1, \dots$, be the sequence of polynomials orthogonal in G with respect to the weight function $h(z)$, that is,

$$\int_G Q_k(z)\overline{Q_l(z)}h(z)\,dm(z) = \begin{cases} 1 & \text{if } k = l, \\ 0 & \text{if } k \neq l, \end{cases}$$

where $dm(z)$ denotes two-dimensional Lebesgue measure (area).

With L denoting the boundary of G, we assume that

$$h(z) \geq c(\operatorname{dist}(z, L))^m, \qquad z \in G, \tag{1.1}$$

for some constants $m > 0, c > 0$.

Recently, Eiermann and Stahl [9] made computations and raised some conjectures about the distribution of the zeros of the orthogonal polynomials $\tilde{Q}_n(z) := Q_n(h, z)$, in the special case where $h(z) \equiv 1$, on convex domains G having polygonal boundaries. In particular, N-gons G_N, $N = 3, 4, \dots$, which have their vertices at the Nth roots of unity, were also considered in [9]. It was previously shown in [5] that for *some* G and *some* n, the distribution, of zeros of the associated orthogonal polynomials $\tilde{Q}_n$, is governed by the equilibrium measure $\mu_{\overline{G}}$ of $\overline{G}$. The main purpose of this paper is to prove a discrepancy theorem for a special measure τ_n, which is closely connected with zeros of Q_n and $\mu_{\overline{G}}$, for *all* convex domains G and $n \in \mathbf{N}$.

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In what follows, we assume that $G \subset \mathbf{C}$ is always convex. It is known (see Stahl and Totik [21, p. 31]) that the zeros $z_{n,1}, \ldots, z_{n,n}$ of Q_n belong to $\overline{G}$, for any $n \in \mathbf{N}$.

Let $\omega(z, J, G)$, $z \in G$, and $J \subset L := \partial G$ be the harmonic measure of J at z with respect to G. We extend this notion to the boundary points $z \in L$, by setting

$$\omega(z, J, G) := \begin{cases} 1, & z \in J, \\ 0, & z \notin J. \end{cases}$$

Next, we associate with Q_n the measure

$$\tau_n(J) := \frac{1}{n} \sum_{j=1}^{n} \omega(z_{n,j}, J, G), \qquad n \in \mathbf{N}.$$

We will compare τ_n with the equilibrium measure $\mu = \mu_{\overline{G}}$ of $\overline{G}$ (see [19]), which has a simple interpretation using the conformal mapping Φ of $\Omega := \overline{\mathbf{C}} \backslash \overline{G}$ onto $\Delta := \{w : |w| > 1\}$, normalized by the conditions

$$\Phi(\infty) = \infty \qquad \text{and} \qquad \Phi'(\infty) := \lim_{z \to \infty} \frac{\Phi(z)}{z} > 0,$$

where we define $\Psi := \Phi^{-1}$. Namely, Φ can be extended to a homeomorphism $\Phi : \overline{\Omega} \to \overline{\Delta}$ and, for any subarc $J \subset L$:

$$\mu(J) = \frac{1}{2\pi} |\Phi(J)|,$$

where $|\gamma|$ denotes the length of $\gamma \subset \mathbf{C}$.

Remark. It is known that the measures τ_n converge to $\mu_{\overline{G}}$ in the weak* topology, as $n \to \infty$, for any Jordan domain G (see Theorem 2.2.1 of [21, p. 42] and its proof).

We define the discrepancy of a signed (Borel) measure σ, supported on L, by

$$D[\sigma] := \sup |\sigma(J)|,$$

where the supremum is taken over all subarcs $J \subset L$. With this definition, our new result, for the asymptotic zero distribution of polynomials orthogonal over a general convex domain, is stated as:

Theorem 1. *Let G be a bounded convex domain, and let $h(z)$ satisfy* (1.1). *Then for each $n = 2, 3, \ldots$:*

$$D[\mu_{\overline{G}} - \tau_n] \leq c \sqrt{\frac{\log n}{n}},$$

for some constant $c > 0$, which is independent of n.

The main idea of the proof of Theorem 1 is in its potential theoretical interpretation. Namely, let cap $\overline{G}$ be the (logarithmic) capacity of $\overline{G}$. We consider the logarithmic potentials of μ and τ_n in Ω:

$$U(\mu, z) := -\int \log|z-\zeta|\, d\mu(\zeta) = -\log|\Phi(z)| - \log(\operatorname{cap}\overline{G}),$$

$$U(\tau_n, z) := -\int \log|z-\zeta|\, d\tau_n(\zeta) = -\int \log|z-\zeta|\, d\nu_{Q_n}(\zeta) = -\frac{1}{n}\log\frac{|Q_n(z)|}{\lambda_n},$$

(where we have used the fact that τ_n is the balayage of the zero-counting measure ν_{Q_n} which associates the mass $1/n$ with each zero of Q_n according to its multiplicity), and their difference

$$U(\mu-\tau_n, z) := U(\mu, z) - U(\tau_n, z) = \frac{1}{n}\log\frac{|Q_n(z)|}{\lambda_n(\operatorname{cap}\overline{G})^n|\Phi(z)|^n}.$$

It is proved in [5] that the inequalities

$$\|Q_n\|_{\overline{G}} := \sup_{z\in\overline{G}}|Q_n(z)| \le c_1 n^{c_2}, \tag{1.2}$$

$$\lambda_n\, (\operatorname{cap}\overline{G})^n \ge c_3\, n^{-2}, \tag{1.3}$$

hold for some constants $c_j > 0$, $j = 1, 2, 3$, which are independent of n. This implies that, for any $n \ge 2$:

$$U(\mu-\tau_n, z) \le c_4\frac{\log n}{n}, \qquad z\in\Omega, \quad c_4 > 0,$$

where c_4 is also independent of n.

Theorem 1 is actually a consequence of our result given below, which is a new Erdős–Turán-type theorem (its proof will be given in subsequent sections).

Theorem 2. *Let $G \subset \mathbf{C}$ be a bounded convex domain, and let τ be a unit Borel measure supported on $L := \partial G$. If*

$$\varepsilon = \varepsilon(\tau) := \sup_{z\in\Omega} U(\mu_{\overline{G}} - \tau, z) \quad (\ge 0),$$

then

$$D[\mu_{\overline{G}} - \tau] \le c\sqrt{\varepsilon}, \tag{1.4}$$

for some constant $c > 0$, independent of τ.

For $G = \mathbf{D} := \{z : |z| < 1\}$, the result of Theorem 2 is due to Ganelius [11], which in turn generalized results of Erdős and Turán [10], concerning the distribution of zeros of polynomials with given uniform norms on the unit disk. Further results and bibliographies of papers devoted to this subject can be found in [7], [8], [23], [3], and [19].

The following example shows the sharpness of Theorem 2:

Example 1. Let $G = \mathbf{D}$ and let μ_δ, $0 < \delta \leq 1$, be the equilibrium measure of $V_\delta := \overline{\mathbf{D}} \cup [1, 1+\delta]$. Consider the measure τ_δ, supported on the unit circle $\mathbf{T} := \partial\mathbf{D}$, which is defined for any Borel set $B \subset \mathbf{T}$ by the formula

$$\tau_\delta(B) := \mu_\delta(\{z \in \mathbf{C}\backslash\{0\} : z/|z| \in B\}).$$

It is easy to see that

$$\operatorname{cap} V_\delta = \frac{1}{4}\left(3 + \delta + \frac{1}{1+\delta}\right) = 1 + \frac{\delta^2}{4(1+\delta)}.$$

Therefore, for $z \in \mathbf{T}$, we have

$$U(\mu - \tau_\delta, z) \leq U(\mu - \mu_\delta, z) = \log \operatorname{cap} V_\delta \leq \frac{\delta^2}{4}.$$

At the same time an elementary computation, involving the transformation $z \to (z + 1/z)/2$, shows that

$$D[\mu - \tau_\delta] \geq |(\mu - \tau_\delta)(1)| = \mu_\delta([1, 1+\delta]) \geq \frac{\delta}{3\pi}.$$

This implies that

$$D[\mu - \tau_\delta] \geq \frac{2}{3\pi}\sqrt{\varepsilon(\tau_\delta)},$$

which shows the sharpness of Theorem 2.

Note that statements similar to Theorem 1 can also be proved (by making use of Theorem 2) for other systems of polynomials. All that is needed for this purpose is to establish the analogues of (1.2), (1.3) and to have the property that

(1.5) *all zeros of the corresponding polynomials belong to* $\overline{G}$.

We cite three examples of well-known polynomials suited for such applications of Theorem 2. In all of them, G is a convex domain and $n \in \mathbf{N}$.

Example 2. Let $F_n(z) := (\operatorname{cap} \overline{G})^{-n} z^n + \cdots$ be the nth Faber polynomial for $\overline{G}$ (see [20]). Then, (1.5) is valid by [14, Theorem 2]. In addition, we have, by the same Theorem 2 of [14], that

$$\|F_n\|_{\overline{G}} \leq 2, \qquad n \in \mathbf{N}.$$

Example 3. Consider the derivatives $F'_{n+1}(z)$ of the above Faber polynomials. For these polynomials, condition (1.5) is then proved in [24]. At the same time, by the Markov-type inequality for complex polynomials, which is a simple consequence of Löwner's distortion theorem (see, e.g., [2, p. 58]), there holds

$$\|F'_{n+1}\|_{\overline{G}} \leq c(n+1)^2, \qquad c = c(G) > 0.$$

Example 4. Let $T_n(z) = z^n + \cdots$, be the nth normalized Chebyshev polynomial for $\overline{G}$. Condition (1.5) is then well known (see [20]). The corresponding estimate for the uniform norm on $\overline{G}$ follows from the extremal property of the Chebyshev polynomial

$$\|T_n\|_{\overline{G}} \le (\operatorname{cap}\overline{G})^n \|F_n\|_{\overline{G}} \le 2(\operatorname{cap}\overline{G})^n.$$

In what follows, we denote by $c, c_1, \ldots$ positive constants, and by $\varepsilon_0, \varepsilon_1, \ldots$ sufficiently small positive constants (different each time, in general), that are either absolute or depend on parameters not essential for the arguments; sometimes such a dependence will be indicated. For $a > 0$ and $b > 0$ we use the expression $a \preceq b$ (order inequality) if $a \le cb$ for some $c > 0$. The expression $a \asymp b$ means that $a \preceq b$ and $b \preceq a$ hold simultaneously.

2. Some Facts from Geometric Function Theory

Each convex curve is known to be quasiconformal (see [15, pp. 63, 87]). It is further known (see [1, Chap. IV]) that the conformal mapping Φ can be extended, in this case, to a quasiconformal mapping of the whole plane onto itself. We keep the same notation for this extension. Note that the inverse function $\Psi := \Phi^{-1}$ will be quasiconformal too.

The following result is useful in the study of metric properties of the mappings Φ and Ψ:

Lemma 1 ([2, p. 97]). *Let $w = F(\zeta)$ be a K-quasiconformal mapping of $\overline{\mathbf{C}}$ onto itself with $F(\infty) = \infty$, $\zeta_j \in \mathbf{C}$, $w_j := F(\zeta_j)$, $j = 1, 2, 3$, and $|w_1 - w_2| \le c_1 |w_1 - w_3|$. Then $|\zeta_1 - \zeta_2| \le c_2 |\zeta_1 - \zeta_3|$ and, in addition,*

$$\left|\frac{\zeta_1 - \zeta_3}{\zeta_1 - \zeta_2}\right| \le c_3 \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^K,$$

where $c_j = c_j(c_1, K)$, $j = 2, 3$.

The convexity of G implies some special distortion properties of the function Φ.

Lemma 2. *Let $z_1 \in L$, $z_2, z_3 \in \overline{\Omega}$, and $\tau_j := \Phi(z_j)$, $j = 1, 2, 3$. If $|\tau_1 - \tau_2| \le |\tau_1 - \tau_3| \le 1$, then the inequality*

$$\left|\frac{z_2 - z_1}{z_3 - z_1}\right| \le c_1 \left|\frac{\tau_2 - \tau_1}{\tau_3 - \tau_1}\right| \tag{2.1}$$

holds with $c_1 = c_1(G) > 0$.

Proof. Without loss of generality, we assume that

$$|z_2 - z_1| < |z_3 - z_1| < \tfrac{1}{2} \operatorname{diam} L$$

(otherwise (2.1) follows easily from Lemma 1). Next we introduce the following notations. Denote by $\gamma(x) = \gamma(z_1, x) \subset \Omega$, for $0 < x < \frac{1}{2} \operatorname{diam} L$, the subarc of the circle

$\{\xi : |\xi - z_1| = x\}$ that separates the point z_1 from ∞ in Ω. Let $Q(\delta, t) = Q(z_1, \delta, t)$, for $0 < \delta < t < \frac{1}{2}\operatorname{diam} L$, be the quadrilateral bounded by the arcs $\gamma(\delta)$, $\gamma(t)$ and the two subarcs of L joining their endpoints. Denote the family of all locally rectifiable arcs in $Q(\delta, t)$, which separate the sides $\gamma(\delta)$ and $\gamma(t)$, by $\Gamma(\delta, t)$, and the module of $\Gamma(\delta, z)$ by $m(\delta, t)$ (see [1], [16]). By the comparison principle

$$m(\delta, t) \le \frac{1}{\pi}\log\frac{t}{\delta}, \qquad 0 < \delta < t < \tfrac{1}{2}\operatorname{diam} L.$$

For any triplet of points $\xi_1, \xi_2, \xi_3 \in \overline{\Omega}$ with $|\xi_1 - \xi_2| = |\xi_1 - \xi_3|$, we have, by Lemma 1, that

$$|\Phi(\xi_1) - \Phi(\xi_2)| \asymp |\Phi(\xi_1) - \Phi(\xi_3)|.$$

Hence, according to [6] (see also [2, p. 36]):

$$\left|\frac{\tau_3 - \tau_1}{\tau_2 - \tau_1}\right| \asymp \exp(\pi m(|z_2 - z_1|, |z_3 - z_1|)) \le \left|\frac{z_3 - z_1}{z_2 - z_1}\right|. \qquad \blacksquare$$

Lemma 3. *The inequality*

$$\omega(z, l, \mathbf{D}) \le 8\,\frac{1 - |z|}{\operatorname{dist}(z, l)} \tag{2.2}$$

holds true for any $z \in \mathbf{D}$ and any arc $l \subset \mathbf{T}$.

Proof. Using a rotation with respect to the origin, we can reduce the situation to the case when $0 < z < 1$ and $l = \{e^{i\theta} : \theta_1 \le \theta \le \theta_2\}$, $0 < \theta_1 < \theta_2 < 2\pi + \theta_1$. Moreover, we can assume that $\theta_2 < 2\pi$ (since, in the other case, (2.2) is trivially valid). Set

$$l_1 := \{\zeta \in l : \operatorname{Im}\zeta \ge 0\}, \qquad l_2 := l \setminus l_1.$$

We assume that $l_1 \ne \emptyset$. A simple geometric reasoning shows that, for $\zeta = e^{i\theta} \in l_1$:

$$|\zeta - z| \ge \frac{1}{\pi}(\theta - \theta_1), \qquad |\zeta - z| \ge |z - z_1|, \qquad z_1 := e^{i\theta_1}.$$

Therefore, by the Poisson formula,

$$\begin{aligned}
\omega(z, l_1, \mathbf{D}) &= \frac{1 - |z|^2}{2\pi}\int_{l_1}\frac{|d\zeta|}{|\zeta - z|^2} \le \frac{1}{\pi}(1 - |z|)\int_{l_1}\frac{|d\zeta|}{|\zeta - z|^2} \\
&\le 4\pi(1 - |z|)\int_{\theta_1}^{\pi}\frac{d\theta}{(\pi|z - z_1| + \theta - \theta_1)^2} \\
&\le \frac{4(1 - |z|)}{|z - z_1|} \le \frac{4(1 - |z|)}{\operatorname{dist}(z, l)}.
\end{aligned}$$

Writing the same estimate for $\omega(z, l_2, \mathbf{D})$, and taking their sum, we obtain (2.2). $\blacksquare$

3. Auxiliary Results

In this section we discuss the results needed in the proof of Theorem 2.

The concept of a regularized distance to an arbitrary compact set $E \subset \mathbf{R}^n$ is described in [22, pp. 170–171]. It is based on the decomposition of open sets into cubes and the partition of unity, which is due to Whitney. It is enough for our purposes to assume that E is a continuum in the complex plane, with the simply connected complement U. In this case, the notion of a regularized distance can be explained by making use of the properties of a conformal mapping of U onto the unit disk.

Namely, let $U \subset \mathbf{C}$ be a simply connected domain, $E := \overline{\mathbf{C}} \backslash U \neq \emptyset$, with $\infty \in E$. Denote the distance from z to E by $d(z) := d(z, E)$. This function is, in general, not smoother on U than what the obvious Lipschitz-condition-inequality

$$|d(z) - d(\zeta)| \leq |z - \zeta|, \qquad z, \zeta \in \mathbf{C},$$

indicates.

It is desirable for several applications to replace $d(z)$ by a regularized distance $\rho(z)$, which is infinitely differentiable for $z \in U$. In addition, this regularized distance should have essentially the same behavior as $d(z)$.

Let $g : U \to H_+ := \{w : \operatorname{Im} w > 0\}$ be a conformal mapping. Set $u(z) := \operatorname{Im} g(z)$. The function

$$\rho(z) := \frac{u(z)}{|g'(z)|}, \qquad z \in U, \tag{3.1}$$

is called a *regularized distance* from z to E.

Lemma 4 ([4, Lemma 1]). *For each $z \in U$, we have*

$$\frac{1}{4}\frac{u(z)}{d(z)} \leq |g'(z)| \leq 4\frac{u(z)}{d(z)}. \tag{3.2}$$

Moreover, if $|\xi - z| \leq d(z)/2$ then

$$\frac{1}{16}\frac{u(z)}{d(z)}|\xi - z| \leq |g(\xi) - g(z)| \leq 16\frac{u(z)}{d(z)}|\xi - z|. \tag{3.3}$$

Applying (3.2) we have

$$\tfrac{1}{4}d(z) \leq \rho(z) \leq 4d(z), \qquad z \in U.$$

We note the following fact about the smoothness properties of $\rho(z)$. Let $f(z)$, $z = x + iy$, be a nonvanishing analytic function in U. A simple calculation shows that, for any $z \in U$:

$$|f|'_x = |f|(\log|f|)'_x = |f| \operatorname{Re}(\log f)'_z = |f| \operatorname{Re}\frac{f'_z}{f}, \tag{3.4}$$

$$|f|'_y = |f|(\log|f|)'_y = |f| \operatorname{Re}(i \log f)'_z = -|f| \operatorname{Im}\frac{f'_z}{f}; \tag{3.5}$$

whence, we conclude that

$$\big||f|'_\xi\big| \le |f'_z| \qquad \text{with} \quad \xi = x \quad \text{or} \quad \xi = y. \tag{3.6}$$

Formulas (3.4) and (3.5) imply that $\rho(z) \in C^\infty(U)$. Differentiating them once more, we obtain for $j + k = 2,\ j, k \ge 0$, that

$$\left|\frac{\partial^2 |f|}{\partial x^j\, \partial y^k}\right| \le |f''_{zz}| + 2\frac{|f'_z|^2}{|f|}. \tag{3.7}$$

Next, we claim that for $z = x + iy \in U;\ j, k = 0, 1, 2;\ 1 \le j + k \le 2$:

$$\left|\frac{\partial^{j+k}}{\partial x^j\, \partial y^k}\, \rho(z)\right| \le c_1 \rho(z)^{1-j-k}, \tag{3.8}$$

for some absolute constant $c_1 > 0$.

Indeed, inequality (3.8) follows immediately from (3.6), (3.7), and (3.2) after a twice repeated differentiation of formula (3.1) with respect to $\xi_j = x$ or $\xi_j = y$, $j = 1, 2$:

$$\begin{aligned}
\frac{\partial \rho}{\partial \xi_1} &= \frac{1}{|g'_z|^2}(u'_{\xi_1}|g'_z| - u|g'_z|'_{\xi_1}),\\
\frac{\partial^2 \rho}{\partial \xi_1\, \partial \xi_2} &= \frac{1}{|g'_z|^4}\{(u''_{\xi_1\xi_2}|g'_z| + u'_{\xi_1}|g'_z|'_{\xi_2}\\
&\quad - u'_{\xi_2}\,|g'_z|'_{\xi_1} - u\,|g'_z|''_{\xi_1\xi_2})|g'_z|^2\\
&\quad - 2(u'_{\xi_1}\,|g'_z| - u\,|g'_z|'_{\xi_1})|g'_z|\,|g'_z|'_{\xi_2}\},
\end{aligned}$$

if we know that, for $k = 2, 3$:

$$|g^{(k)}(z)| \le c_2\, u(z)\, \rho(z)^{-k}, \qquad z \in U, \tag{3.9}$$

with an absolute constant $c_2 > 0$.

In order to prove (3.9), we put $d := d(z)/32$ and note that, by (3.3):

$$|g(\zeta) - g(z)| \le \tfrac{1}{2}u(z),$$

for any ζ with $|\zeta - z| = d$. Therefore, we have, according to (3.2), that

$$|g'(\zeta)| \le 4\frac{u(\zeta)}{d(\zeta)} \le 10\frac{u(z)}{d(z)},$$

for such ζ. Next, we apply Cauchy's formula and (3.2) to obtain that, for $k = 2, 3$:

$$\begin{aligned}
|g^{(k)}(z)| &= \frac{(k-1)!}{2\pi}\left|\int_{|\zeta - z| = d} \frac{g'(\zeta)}{(\zeta - z)^k}\, d\zeta\right|\\
&\le 10(k-1)!\, 32^{k-1}\, \frac{u(z)}{d^k(z)}.
\end{aligned}$$

This completes the proof of (3.9) and, consequently, of (3.8).

The second topic concerns the "body-contour" properties of harmonic functions. Let $G \subset \mathbf{C}$ be a bounded convex domain, and let $f(z)$ be a real-valued function, which is continuous on $\overline{G}$ and harmonic in G. Let $z \in L := \partial G$, $\zeta \in G$, and $\delta := |z - \zeta|$. We next estimate the quantity $|f(\zeta) - f(z)|$ in terms of the local modulus of continuity of f on L, that is,

$$\omega_{z,f,L}(t) := \sup_{\substack{\xi \in L \\ |\xi - z| \le t}} |f(\xi) - f(z)|, \qquad t > 0.$$

Let $z_0 \in G$ be a fixed point. We assume that $2\delta < \operatorname{dist}(z_0, L) =: d_0$. For $0 < t < d_0$, denote by $\gamma(t) = \gamma(z, t)$ a crosscut of G, i.e., an open Jordan arc in G with endpoints on L, which is a subarc of the circle $\{\xi : |\xi - z| = t\}$ and has nonempty intersection with the interval $[z, z_0]$. The endpoints of $\gamma(t)$ divide L into two subarcs. Denote the subarc containing z by $l(t)$.

Since L is quasiconformal, Ahlfors' geometric criterion (see [1]) gives the inequality

$$\min\{\operatorname{diam} L', \operatorname{diam} L''\} \le c\,|z_1 - z_2| \qquad \text{for any} \quad z_1, z_2 \in L, \tag{3.10}$$

with $c = c(L) \ge 1$, where L' and L'' are the associated two arcs of $L \backslash \{z_1, z_2\}$. Therefore, the quantity

$$M = M(z_0, L) := \sup_{z \in L} \sup_{0 < t < d_0} \frac{\operatorname{diam} l(t)}{t}$$

is finite. Moreover, it is easy to prove that $M \le M_0$, where M_0 depends only on the constant c from (3.10) and, consequently, only on the constant of quasiconformality of L.

Let

$$\nu(t) := \omega(\zeta, L \backslash l(t), G), \qquad 0 < t < d_0,$$

be the corresponding harmonic measure. Next we fix a number s, satisfying $2\delta < s \le d_0$, and define a natural number k such that

$$\frac{s}{2} \le 2^k \delta < s.$$

By the maximum principle for harmonic functions, we have

$$\begin{aligned} |f(\zeta) - f(z)| &\le \omega_{z,f,L}(M\delta) + \sum_{j=0}^{k-1} \omega_{z,f,L}(M 2^{j+1} \delta)\nu(2^j \delta) + 2\|f\|_L\, \nu\left(\frac{s}{2}\right) \\ &\le \omega_{z,f,L}(M\delta) + 2\int_\delta^s \frac{\omega_{z,f,L}(2Mt)}{t}\, \nu\left(\frac{t}{2}\right) dt + 2\|f\|_L\, \nu\left(\frac{s}{2}\right). \end{aligned}$$

Our next goal is to obtain effective estimates of the harmonic measure $\nu(t)$. Let $\Gamma = \Gamma(\zeta, l(t), G)$, $\delta < t < d_0$, be a family of all crosscuts of G that separate point ζ from $L \backslash l(t)$. We note that

$$m(\Gamma) \le \frac{1}{\pi} \log \frac{4}{\nu(t)}. \tag{3.11}$$

Indeed, taking into account that both module and harmonic measure are conformal invariants, we introduce the conformal mapping $g : G \to \mathbf{D}$ such that

$$g(\zeta) = 0, \qquad g(L \backslash l(t)) = \{e^{i\theta\pi} : -a \le \theta \le a\}, \qquad a := \nu(t).$$

According to [13, pp. 319–320] (see also [12, p. 6]), we have

$$m(\Gamma)^{-1} = m(g(\Gamma))^{-1} = 2T\left(\sin\frac{\pi}{2}(1-a)\right) = 2T\left(\cos\frac{\pi a}{2}\right),$$

where we set

$$T(k) := \frac{K((1-k^2)^{1/2})}{K(k)}$$

and

$$K(k) := \int_0^1 (1-x^2)^{-1/2}(1-k^2x^2)^{-1/2}\,dx,$$

for $0 < k < 1$. Hence

$$2m(\Gamma) = T\left(\sin\frac{\pi a}{2}\right).$$

By [16, p. 61]:

$$T\left(\sin\frac{\pi a}{2}\right) \le \frac{2}{\pi}\log\frac{4}{\sin \pi a/2} \le \frac{2}{\pi}\log\frac{4}{a}.$$

Thus we obtain (3.11) by comparing the last two equations.

On the other hand, comparing the families Γ and $\Gamma_1 := \{\gamma(u)\}_{\delta<u<t}$, we have

$$m(\Gamma) \ge m(\Gamma_1) \ge \frac{1}{\pi}\log\frac{t}{\delta}.$$

Therefore, it follows from (3.11) that

$$\nu(t) \le 4\,\frac{\delta}{t},$$

and that

$$|f(\zeta) - f(z)| \le \omega_{z,f,L}(M\delta) + 16\,\delta \int_\delta^s \frac{\omega_{z,f,L}(2Mt)}{t^2}dt + 16\|f\|_L\,\frac{\delta}{s}. \tag{3.12}$$

4. Proof of Theorem 2

Let $\sigma := \mu - \tau$. We can assume that $0 < \varepsilon < \varepsilon_0$, where $\varepsilon_0 = \varepsilon_0(G)$ is small enough for our constructions below. Let $J \subset L$ be an arbitrary subarc. In order to prove (1.4), it is sufficient to show that

$$-\sigma(J) \le c\sqrt{\varepsilon}, \tag{4.1}$$

for J small enough.

We set

$$\begin{aligned} \gamma &:= \Phi(J) = \{e^{i\theta} : \theta_1 \le \theta \le \theta_2\}, \\ \gamma(r) &:= \{e^{i\theta} : \theta_1 - r \le \theta \le \theta_2 + r\}, \qquad r > 0, \\ J(r) &:= \Psi(\gamma(r)), \qquad r > 0. \end{aligned}$$

Next, we introduce a curvilinear sector based on J. Let $z_0 \in G$ be a fixed point. Denote by $w = \varphi(z)$ the conformal mapping of G onto $\mathbf{D}$ with the normalization $\varphi(z_0) = 0$, $\varphi'(z_0) > 0$. Set $\psi := \varphi^{-1}$. Since L is quasiconformal, the functions φ and ψ can be extended to the quasiconformal mappings of the extended complex plane $\overline{\mathbf{C}}$ onto itself with ∞ as a fixed point (see [1, Chap. IV]), where we keep the same notations for these extensions.

Letting

$$\varphi(J) = \{e^{i\theta} : \tilde{\theta}_1 \le \theta \le \tilde{\theta}_2\},$$

we set

$$\begin{aligned} B(J) &:= \{\zeta \in \overline{\Omega} : \theta_1 \le \arg \Phi(\zeta) \le \theta_2\} \\ &\cup \{\zeta \in \overline{G} : \tilde{\theta}_1 \le \arg \varphi(\zeta) \le \tilde{\theta}_2\}. \end{aligned}$$

Set $t := \sqrt{\varepsilon}$ and consider the function

$$h(z) := \begin{cases} 1 & \text{if } z \in B(J(t)), \\ 0 & \text{otherwise in } \mathbf{C}. \end{cases}$$

Let $\rho(z) = \rho(z, B(J))$, $z \in \mathbf{C}$, be a regularized distance to $B(J)$ (see Section 3), i.e., a function with the following properties:

$$\tfrac{1}{4}\operatorname{dist}(z, B(J)) \le \rho(z) \le 4 \operatorname{dist}(z, B(J)), \qquad z \in \mathbf{C}, \tag{4.2}$$

$$\rho(z) \in C^\infty(\mathbf{C}), \tag{4.3}$$

$$\left| \frac{\partial^{j+k}}{\partial x^j \, \partial y^k} \rho(x+iy) \right| \le c\rho(x+iy)^{1-j-k}, \qquad j+k = 1, 2. \tag{4.4}$$

Next, we average the function h in the following way:

$$g(z) := \begin{cases} \dfrac{64}{\rho(z)^2} \displaystyle\int_{\mathbf{C}} h(\zeta) V\left(\frac{8(\zeta - z)}{\rho(z)}\right) dm(\zeta) & \text{if } z \in \mathbf{C}\backslash B(J), \\ 1 & \text{if } z \in B(J), \end{cases}$$

where $V(\zeta)$ is an arbitrary symmetric averaging kernel, i.e., $V(z) \in C^\infty(\mathbf{C})$,

$$\begin{aligned} V(z) = V(|z|) &\ge 0, \qquad z \in \mathbf{C}, \\ V(z) &= 0, \qquad |z| \ge 1, \\ \int V(z)\, dm(z) &= 1. \end{aligned}$$

Note that $g \in C^\infty(\mathbf{C})$ by virtue of (4.3). Set

$$L_\varepsilon := \{z \in \Omega : |\Phi(z)| = 1 + \varepsilon\},$$
$$z_L := \Psi(\Phi(z)/|\Phi(z)|), \qquad z \in \Omega\setminus\{\infty\}.$$

By Lemma 1, there exists a sufficiently small constant $\varepsilon_1 > 0$ such that

$$\operatorname{dist}(z, B(J)) < \operatorname{dist}(z, \mathbf{C}\setminus B(J(t)),$$

for $z \in L_\varepsilon$, with $z_L \in J(2\varepsilon_1 t)$. Therefore,

$$g(z) = 1, \quad z \in L_\varepsilon, \qquad z_L \in J(2\varepsilon_1 t),$$

according to (4.2). Further, by the same Lemma 1, there exists a sufficiently large constant $c_1 > 0$ such that

$$\operatorname{dist}(z, B(J)) \le 2 \operatorname{dist}(z, B(J(t))),$$

for $z \in L_\varepsilon$, with $z_L \in L\setminus J(c_1 t)$. Therefore, we have for such z that

$$\rho(z) \le 4 \operatorname{dist}(z, B(J)) \le 8 \operatorname{dist}(z, B(J(t))),$$

by (4.2), and we obtain

$$g(z) = 0, \quad z \in L_\varepsilon, \qquad z_L \in L\setminus J(c_1 t).$$

If $z = x + iy$ and $\xi = \tilde{x} + i\tilde{y} \in L_\varepsilon$ with $z_L, \xi_L \in L(\zeta_3, \zeta_1)$, where $\zeta_1 := \Psi(e^{i\theta_1})$, $\zeta_3 := \Psi(e^{i(\theta_1 - 3c_1 t)})$, and $L(\zeta_3, \zeta_1) := \{\zeta = \Psi(e^{i\theta}) : \theta_1 - 3c_1 t \le \theta \le \theta_1\}$, then we obtain by Taylor's formula that

$$g(z) = g(\xi) + A(\xi)(x - \tilde{x}) + B(\xi)(y - \tilde{y}) + r(z, \xi), \tag{4.5}$$

where we have

$$|A(\xi)| + |B(\xi)| \preceq |\zeta_1 - \zeta_3|^{-1} \tag{4.6}$$

and

$$|r(z, \xi)| \preceq \frac{|z - \xi|^2}{|\zeta_1 - \zeta_3|^2}, \tag{4.7}$$

according to (4.4).

The same relations are valid for $z, \xi \in L_\varepsilon$ with $z_L, \xi_L \in L(\zeta_2, \zeta_4)$, where $\zeta_2 := \Psi(e^{i\theta_2})$, $\zeta_4 := \Psi(e^{i(\theta_2 + 3c_1 t)})$, and $L(\zeta_2, \zeta_4) := \{\zeta = \Psi(e^{i\theta}) : \theta_2 \le \theta \le \theta_2 + 3c_1 t\}$.

We denote the harmonic extension of g from L_ε to $\overline{\mathbf{C}}\setminus L_\varepsilon$ by $f(z)$. Set

$$\tilde{f}(w) := f(\Psi(w)), \qquad w \in \overline{\Delta}.$$

Then the following estimate holds:

Lemma 5. *Let* $1 \le |w| \le 1 + 2\varepsilon$. *Then*

$$|\tilde{f}(w) - \tilde{f}(w_\varepsilon)| \le c_2 t, \qquad w_\varepsilon := \frac{w}{|w|}(1 + \varepsilon). \tag{4.8}$$

The proof of Lemma 5 will be given in the next section.

Further, we average the function $\tilde{f}$ in the following way. Let $V(z)$, $z \in \mathbf{C}$, be an averaging kernel as above. Consider the function

$$\tilde{u}(w) := \begin{cases} \dfrac{16}{\varepsilon^2} \displaystyle\int \tilde{f}(t) V\left(\dfrac{4(t-w)}{\varepsilon}\right) dm(t) & \text{if } 1 + \frac{3}{4}\varepsilon \le |w| \le 1 + \frac{5}{4}\varepsilon, \\ \tilde{f}(w) & \text{elsewhere in } \overline{\Delta}. \end{cases}$$

Note that $\tilde{u} \in C^\infty(\Delta)$:

$$0 \le \tilde{u}(w) \le 1, \qquad w \in \overline{\Delta}, \tag{4.9}$$

and that the Laplacian of $\tilde{u}$ satisfies

$$|\Delta \tilde{u}(w)| \preceq \frac{t}{\varepsilon^2}, \qquad 1 + \tfrac{3}{4}\varepsilon \le |w| \le 1 + \tfrac{5}{4}\varepsilon, \tag{4.10}$$

by (4.8). Let us introduce the function

$$u(z) := \begin{cases} \tilde{u}(\Phi(z)) & \text{if } z \in \Omega, \\ f(z) & \text{if } z \in \overline{G}, \end{cases}$$

which obviously belongs to the class $C^\infty(\mathbf{C})$. It follows that

$$\int \Delta u(z)\, dm(z) = 0, \tag{4.11}$$

by Green's formula. Applying the techniques of [7], we can establish the inequality

$$\left| \int u \, d\sigma \right| \preceq t. \tag{4.12}$$

Indeed, on setting

$$\tilde{U}(\sigma, w) := U(\sigma, \Psi(w)), \qquad w \in \Delta,$$

and, using the representation of the function u by means of Green's formula,

$$u(z) = u(\infty) + \frac{1}{2\pi} \int \Delta u(\zeta) \log|z - \zeta|\, dm(\zeta), \qquad z \in \mathbf{C},$$

we obtain that

$$\begin{aligned} \left| \int f \, d\sigma \right| &= \frac{1}{2\pi} \left| \int (\varepsilon - U(\sigma, \zeta)) \Delta u(\zeta)\, dm(\zeta) \right| \\ &\le \frac{1}{2\pi} \int (\varepsilon - \tilde{U}(\sigma, w)) |\Delta \tilde{u}(w)|\, dm(w) \preceq t, \end{aligned}$$

by (4.10) and (4.11) (see [7] for details).

Equations (4.8), (4.9), and (4.12) imply that

$$\begin{aligned} -\sigma(J) \le & -\int u \, d\sigma + \mu(J(c_1 t) \backslash J) + \int_{L \backslash J(c_1 t)} u \, d\mu \\ & + \int_J (1 - u)\, d\tau \preceq t, \end{aligned}$$

which is the assertion of (4.1). ∎

5. Proof of Lemma 5

Let $w = re^{i\theta}$. Applying Lemma 3, we easily obtain (4.8) for the case

$$1+\varepsilon < r < 1+2\varepsilon,$$

$$\theta_1 - \varepsilon_1 t \le \theta \le \theta_2 + \varepsilon_1 t \qquad \text{or} \qquad \theta_2 + 2c_1 t \le \theta \le 2\pi + \theta_1 - 2c_1 t.$$

If

$$1+\varepsilon < r < 1+2\,\varepsilon, \qquad \theta_1 - 2c_1 t \le \theta \le \theta_1 - \varepsilon_1 t,$$

we set $\xi = \zeta_\varepsilon := \Psi(w_\varepsilon)$ and write the function g in the form of (4.5).

Lemmas 1 and 2 imply that

$$\left|\frac{\zeta - \zeta_\varepsilon}{\zeta_1 - \zeta_3}\right| \preceq \left|\frac{\zeta_L - \zeta_\varepsilon}{\zeta_L - \zeta_1}\right| \preceq \frac{\varepsilon}{t} = t. \tag{5.1}$$

Define the harmonic extension of the function appearing in (4.5) to ext $L_\varepsilon \backslash \{\infty\}$ by the formula

$$r(z,\xi) := f(z) - g(\xi) - A(\xi)(x-\tilde{x}) - B(\xi)(y-\tilde{y}),$$

and set

$$\tilde{r}(\tau) := r(\Psi(\tau),\xi), \qquad |\tau| \ge 1+\varepsilon.$$

Note that for $z \in L_\varepsilon$ with $z_L \in L(\zeta_3, \zeta_1)$, we have that

$$\left|\frac{z-\xi}{\zeta_1-\zeta_3}\right| \preceq \frac{|\Phi(z) - w_\varepsilon|}{t}. \tag{5.2}$$

Indeed, without loss of generality, we assume that $|z-\zeta_1| \ge |z-\zeta_3|$, and therefore

$$|\zeta_1 - \zeta_3| \asymp |z-\zeta_1|,$$

$$\Phi(z) =: \tau = (1+\varepsilon)e^{i\eta}, \qquad |\theta_1 - \eta| \asymp t.$$

If $|\theta - \eta| \ge \varepsilon/32$, then (5.2) follows from Lemmas 1 and 2, because

$$\left|\frac{z-\xi}{\zeta_1-\zeta_3}\right| \asymp \left|\frac{z_L-\xi}{z_L-\zeta_1}\right| \preceq \left|\frac{\Phi(z_L)-\Phi(\xi)}{\Phi(z_L)-\Phi(\zeta_1)}\right| \asymp \frac{|\tau - w_\varepsilon|}{t}.$$

Now let $|\theta - \eta| < \varepsilon/32$. Then, by the analogue of Lemma 4 (see [4, Lemma 1]) for the conformal mapping Φ, we obtain that

$$|z-\xi| \le \tfrac{1}{2}\,\mathrm{dist}(z,L),$$

and, consequently,

$$\begin{aligned}\left|\frac{z-\xi}{\zeta_1-\zeta_3}\right| &\asymp \left|\frac{z_L - z}{z_L-\zeta_1}\right|\left|\frac{\xi - z}{z_L - z}\right| \\ &\preceq \frac{|\tau|-1}{t}\,\frac{|\tau - w_\varepsilon|}{|\tau|-1} = \frac{|\tau-w_\varepsilon|}{t}.\end{aligned}$$

Hence, (5.2) and (4.7) give

$$(5.3) \qquad |\tilde{r}(\tau)| \preceq \frac{|\tau - w_\varepsilon|^2}{t^2}, \qquad |\tau| = 1+\varepsilon, \quad |\tau - w_\varepsilon| \le c_2 t.$$

Relation (5.3) remains true for τ such that $|\tau| > 1+\varepsilon$, $|\tau - w_\varepsilon| = c_2 t$, by the definition of the function $\tilde{r}(\tau)$ and (4.5)–(4.7).

Further, a direct computation shows that

$$(5.4) \qquad |\tilde{r}(w)| \preceq t.$$

Indeed, let us introduce the auxiliary function $\tilde{R}(\tau)$, which we define to be the harmonic extension of the function

$$\tilde{R}(\tau) := \begin{cases} |\tilde{r}(\tau)| & \text{if } |\tau| = 1+\varepsilon,\ |\tau - w_\varepsilon| \le c_2 t, \\ c_3 & \text{otherwise for } |\tau| = 1+\varepsilon, \end{cases}$$

to $|\tau| \ge 1+\varepsilon$. It is clear that we have, for sufficiently large c_3:

$$|\tilde{r}(\tau)| \le \tilde{R}(\tau)$$

on the boundary of the domain

$$\{\tau : |\tau| > 1+\varepsilon,\ |\tau - w_\varepsilon| < c_2 t\}.$$

Therefore, by the maximum principle for harmonic functions, the Poisson formula, and (5.3), we obtain that

$$\begin{aligned} |\tilde{r}(w)| &\le \tilde{R}(w) = \tilde{R}(re^{i\theta}) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \tilde{R}((1+\varepsilon)e^{i\eta}) \frac{r^2 - (1+\varepsilon)^2}{r^2 - 2r(1+\varepsilon)\cos(\theta-\eta) + (1+\varepsilon)^2}\, d\eta \\ &\preceq \varepsilon\left(\frac{1}{t^2}\int_{\theta-c_2 t}^{\theta+c_2 t} d\eta + \int_{\theta+c_2 t}^{\theta+\pi} \frac{d\eta}{(\eta-\theta)^2} + \int_{\theta-\pi}^{\theta-c_2 t} \frac{d\eta}{(\eta-\theta)^2}\right) \preceq \frac{\varepsilon}{t} = t. \end{aligned}$$

Comparing (4.5), (4.6), (5.1), and (5.4), we get the desired inequality (4.8) by (2.1). The same reasoning gives an analogue of (4.8) for the case

$$1+\varepsilon < r < 1+2\varepsilon, \qquad \theta_2 + \varepsilon_1 t \le \theta \le \theta_2 + 2c_1 t.$$

Next we assume that

$$(5.5) \qquad 1 < r = |w| < 1+\varepsilon, \qquad \zeta = \Psi(w), \qquad \zeta_\varepsilon = \Psi(w_\varepsilon).$$

Note that L_ε is convex (see [17, p. 47]). Moreover, since Φ has a quasiconformal extension to $\overline{\mathbf{C}}$, each L_ε is K-quasiconformal with $K \ge 1$, independent of ε. Therefore, we have, by formula (3.12) for any $2|\zeta - \zeta_\varepsilon| < s < \varepsilon_2$ and any function $\kappa(z)$, continuous on $\overline{\text{int}\, L_\varepsilon}$ and harmonic in int L_ε, that

$$\begin{aligned} (5.6) \quad |\kappa(\zeta) - \kappa(\zeta_\varepsilon)| &\preceq \omega_{\zeta_\varepsilon,\kappa,L_\varepsilon}(c_4|\zeta - \zeta_\varepsilon|) \\ &\quad + |\zeta - \zeta_\varepsilon| \int_{|\zeta-\zeta_\varepsilon|}^{s} \frac{\omega_{\zeta_\varepsilon,\kappa,L_\varepsilon}(c_4 r)}{r^2}\, dr + \frac{|\zeta - \zeta_\varepsilon|}{s}\|\kappa\|_{L_\varepsilon}, \end{aligned}$$

where $c_4 > 0$ is independent of ζ and ε.

It is easy to prove (4.8) if, in addition to (5.5), $\zeta_L \notin J(2c_1 t)\backslash J(\varepsilon_1 t)$. Indeed, now let $\kappa := f$, $s := \varepsilon_3 |\zeta_L - \zeta_L^*|$, where $\zeta_L^* := \Psi(e^{it}\Phi(\zeta_L))$ and the sufficiently small constant ε_3 is chosen such that $\omega_{\zeta_\varepsilon,\kappa,L_\varepsilon}(c_4 s) = 0$. Therefore, we obtain (4.8) by (5.6), Lemma 1, and the obvious inequality

$$\left|\frac{\zeta_L - \zeta_\varepsilon}{\zeta_L - \zeta_L^*}\right| \preceq \frac{\varepsilon}{t} = t,$$

which follows from Lemma 2.

The situation is more complicated if, in addition to (5.5), $\zeta_L \in J(2c_1 t)\backslash J(\varepsilon_1 t)$. For definiteness, let $\zeta_L \in L(\zeta_3, \zeta_1)$. In this case, we represent the function g in the form of (4.5) with $\xi := \zeta_\varepsilon$, and set $\kappa(z) := r(z, \xi)$ (i.e., $\kappa(z)$ is the harmonic extension of $r(z, \xi)$ from L_ε to $\mathrm{int}L_\varepsilon$), $s := \varepsilon_4|\zeta_1 - \zeta_3|$, where ε_4 is chosen to be so small that the function $\kappa(z)$ satisfies (4.7) for $z \in l(s)$. Since

$$|\kappa(z)| \preceq 1, \qquad z \in \gamma(s),$$

by (4.5) and (4.6), we have, on setting $\delta := |\zeta - \zeta_\varepsilon|$, that

$$|r(\zeta, \xi)| \preceq \frac{\delta^2}{s^2} + \frac{\delta}{s^2}\int_\delta^s dr + \frac{\delta}{s} \asymp \frac{\delta}{s} \preceq \left|\frac{\zeta_L - \zeta_\varepsilon}{\zeta_1 - \zeta_3}\right|, \tag{5.7}$$

by (4.7) and (5.6). Comparing (5.7), (4.5), (4.6), and applying Lemma 2, we get

$$|f(\zeta) - f(\zeta_\varepsilon)| \preceq \left|\frac{\zeta_L - \zeta_\varepsilon}{\zeta_1 - \zeta_3}\right| \asymp \left|\frac{\zeta_L - \zeta_\varepsilon}{\zeta_L - \zeta_1}\right| \preceq \frac{\varepsilon}{t} = t. \qquad \blacksquare$$

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