Numerical Algorithms **25:** 363–375, 2000.

Zeros of the partial sums of $\cos(z)$ and $\sin(z)$. I *

Richard S. Varga [a] and Amos J. Carpenter [b]

[a] *Institute for Computational Mathematics, Kent State University, Kent, OH 44242, USA*
E-mail: varga@mcs.kent.edu

[b] *Department of Mathematics and Computer Science, Butler University, Indianapolis, IN 46208, USA*
E-mail: acarpenter@butler.edu

Received 23 August 1999; accepted in revised form 10 April 2000

We extend results of Szegő (1924) and Kappert (1996) on the location of the zeros of the normalized partial sums of $\cos(z)$ and $\sin(z)$, and their rates of convergence to the associated Szegő curves.

Keywords: Szegő curve, Eneström–Kakeya theorem, Hurwitz theorem

AMS subject classification: 30C15

1. Introduction

With

$$s_n(z) := \sum_{j=0}^{n} \frac{z^j}{j!}, \quad n \geqslant 1, \tag{1.1}$$

denoting the familiar nth partial sum of the exponential function e^z, it was shown in 1924, in a remarkable paper by Szegő [5], that the zeros $\{z_{n,k}\}_{k=1}^{n}$ of the *normalized* partial sum $s_n(nz)$, tend, as $n \to \infty$, to the Jordan curve D_∞ in the closed unit disk, where

$$D_\infty := \left\{ z \in \mathbb{C}\colon \left|ze^{1-z}\right| = 1 \text{ and } |z| \leqslant 1 \right\}. \tag{1.2}$$

This Jordan curve is referred to in the literature as the *Szegő curve*. Now, it is known (cf. [2]), as a consequence of the Eneström–Kakeya theorem, that the zeros $\{z_{n,k}\}_{k=1}^{n}$ of $s_n(nz)$ all lie in the closed unit disk for every $n \geqslant 1$, and Szegő's result is, more precisely, that each accumulation point (in this closed unit disk) of all these zeros $\{z_{n,k}\}_{k=1,\ n=1}^{n\quad\ \infty}$ *must* lie on D_∞, and, conversely, *each* point of D_∞ is an accumulation point of these zeros!

Subsequently, the rate of convergence, as a function of n, of the zeros $\{z_{n,k}\}_{k=1}^{n}$ to the curve D_∞, a topic not considered in [5], was first studied by Buckholtz [1] who

* Research supported in part by the National Science Foundation grant DMS-9707359.

showed, with the notation

$$\operatorname{dist}\left[\{z_{n,k}\}_{k=1}^{n};\, D_\infty\right] := \max_{1\leqslant k\leqslant n}\left(\operatorname{dist}\,[z_{n,k};\, D_\infty]\right),$$

that

$$\operatorname{dist}\left[\{z_{n,k}\}_{k=1}^{n};\, D_\infty\right] \leqslant \frac{2\mathrm{e}}{\sqrt{n}}, \qquad \text{all } n \geqslant 1, \tag{1.3}$$

which of course implies that

$$\overline{\lim_{n\to\infty}}\left\{\sqrt{n}\cdot \operatorname{dist}\left[\{z_{n,k}\}_{k=1}^{n};\, D_\infty\right]\right\} \leqslant 2\mathrm{e} \doteq 5.43656. \tag{1.4}$$

To complement this result of (1.4), it was later shown in [2] that

$$\underline{\lim_{n\to\infty}}\left\{\sqrt{n}\cdot \operatorname{dist}\left[\{z_{n,k}\}_{k=1}^{n};\, D_\infty\right]\right\} \doteq 0.63665, \tag{1.5}$$

where the constant in (1.5) is related to the complex zero, in the upper half-plane, of the complementary error function $\operatorname{erfc}(w)$, which is closest to the origin. The result of (1.5) states that the exponent, $-\frac{1}{2}$, of n in (1.3), is thus *best possible.*

It was also shown in [2] that a quantitatively *faster* convergence, of the zeros of $s_n(nz)$ to D_∞, takes place if one stays uniformly away from the point $z = 1$. Specifically, with the notation

$$\Delta_\delta(a) := \left\{z \in \mathbb{C}\colon\ |z-a| < \delta\right\}, \qquad \text{any } a \in \mathbb{C},\ \text{any } \delta > 0, \tag{1.6}$$

then on covering the point $z = 1$ with the disk $\Delta_\delta(1)$, it was shown in [2, theorem 2] that, for each fixed δ with $0 < \delta < 1$,

$$\operatorname{dist}\left[\{z_{n,k}\}_{k=1}^{n}\backslash\Delta_\delta(1);\, D_\infty\right] = \mathrm{O}\left(\frac{\log n}{n}\right), \qquad n\to\infty, \tag{1.7}$$

where the constant, implicit in the big-O term, is dependent only on δ. In other words, the rate of convergence, as $n \to \infty$, of the zeros $\{z_{n,k}\}_{k=1}^{n}$ *not* in the disk $\Delta_\delta(1)$, to the Jordan curve D_∞, is $\mathrm{O}((\log n)/n)$. It was also shown in [2] that this convergence rate, $\mathrm{O}((\log n)/n)$ as $n\to\infty$, is *best possible.*

For a more precise location of the zeros $\{z_{n,k}\}_{k=1}^{n}$ of $s_n(nz)$, consider the arc D_n, defined by

$$D_n := \left\{z \in \mathbb{C}\colon\ \left|z\mathrm{e}^{1-z}\right|^n = \tau_n\sqrt{2\pi n}\left|\frac{1-z}{z}\right|,\ |z| \leqslant 1,\ \text{and}\right.$$
$$\left.|\arg z| \geqslant \cos^{-1}\left(\frac{n-2}{n}\right)\right\}, \tag{1.8}$$

for each $n \geqslant 1$, where τ_n, from Stirling's formula, is given by the asymptotic series (cf. [3, p. 377])

$$\tau_n := \frac{n!}{n^n\mathrm{e}^{-n}\sqrt{2\pi n}} \cong 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \cdots, \qquad n\to\infty. \tag{1.9}$$

It was shown in [2, proposition 3] that D_n is a well-defined Jordan arc in $\overline{\Delta}_1(0)$, for each $n \geqslant 1$, and it was also shown in [2, theorem 4] that, for each fixed δ with $0 < \delta < 1$,

$$\operatorname{dist}\left[\{z_{n,k}\}_{k=1}^{n} \backslash \Delta_\delta(1);\, D_n\right] = \mathrm{O}\left(\frac{1}{n^2}\right), \quad n \to \infty, \tag{1.10}$$

so that the arc D_n more closely approximates the zeros $\{z_{n,k}\}_{k=1}^{n}$ of $s_n(nz)$, than does the Szegő curve D_∞. It was shown in [2] that the rate of convergence in (1.10), namely, $\mathrm{O}(1/n^2)$ as $n \to \infty$, is also *best possible*.

For the reader's benefit, it may be useful to indicate how these curves D_∞ and D_n, of (1.2) and (1.8), arise naturally in the study of the zeros of $s_n(nz)$. Starting from the following identity of Szegő [5] (which can be verified by differentiation):

$$\mathrm{e}^{-z} s_n(z) = 1 - \frac{1}{n!}\int_0^z \zeta^n \mathrm{e}^{-\zeta}\,\mathrm{d}\zeta, \quad z \in \mathbb{C};\ n \in \mathbb{N}, \tag{1.11}$$

then replacing ζ by $n\zeta$ and z by nz gives, with (1.9), that

$$\mathrm{e}^{-nz} s_n(nz) = 1 - \frac{\sqrt{n}}{\tau_n\sqrt{2\pi}}\int_0^z \left(\zeta \mathrm{e}^{1-\zeta}\right)^n \mathrm{d}\zeta, \quad z \in \mathbb{C};\ n \in \mathbb{N}, \tag{1.12}$$

and it was shown in [2] that

$$\mathrm{e}^{-nz} s_n(nz) = 1 - \frac{(z\mathrm{e}^{1-z})^n}{\tau_n\sqrt{2\pi n}}\left(\frac{z}{1-z}\right)\left\{1 + \mathrm{O}\left(\frac{1}{n}\right)\right\}, \quad n \to \infty, \tag{1.13}$$

uniformly on any compact subset of $\overline{\Delta}_1(0)\backslash\{1\}$. (We remark that Szegő [5] had obtained (1.13) with the slightly weaker result where $\mathrm{O}(1/n)$ in (1.13) was replaced by $\mathrm{o}(1)$, as $n \to \infty$.) Recalling that the zeros of $s_n(nz)$ must all lie in the closed unit disk $\overline{\Delta}_1(0)$ for every $n \geqslant 1$, then on taking any convergent infinite sequence $\{z_{n_j}\}_{j=1}^{\infty}$ of zeros of $\{s_n(nz)\}_{n=1}^{\infty}$, not converging to $z = 1$, i.e.,

$$s_{n_j}(n_j z_{n_j}) = 0, \qquad \lim_{j\to\infty} z_{n_j} = \hat{z} \quad \text{with } \hat{z} \neq 1, \text{ and } \lim_{j\to\infty} n_j = \infty,$$

it is evident from (1.13) that $\hat{z}$ *must* satisfy $|\hat{z}\mathrm{e}^{1-\hat{z}}| = 1$, with $|\hat{z}| \leqslant 1$, which defines the curve D_∞ of (1.2). (The converse result of Szegő, i.e., that each point of D_∞ is an accumulation point of zeros of $\{s_n(nz)\}_{n\in\mathbb{N}}$ is more difficult, and depends on the use of conformal mapping theory.) Similarly, on deleting the term $\mathrm{O}(1/n)$ in (1.13) and on assuming that $s_n(nz) = 0$, one obtains that a solution of

$$\left(z\mathrm{e}^{1-z}\right)^n = \tau_n\sqrt{2\pi n}\left(\frac{1-z}{z}\right) \tag{1.14}$$

in $\overline{\Delta}_1(0)$ is an approximation to this zero, and on taking absolute values, this essentially gives the definition of the curves D_n in (1.8).

We now turn to the discussion of the behavior of the zeros of the normalized partial sums of $\cos(z)$ and $\sin(z)$, a topic also inaugurated by Szegő [5] and recently advanced by Kappert [4]. (With his results for the zeros of $s_n(nz)$, Szegő stated[1] in [5] that, for his study of the zeros of the normalized partial sums of $\cos(z)$ and $\sin(z)$, "hardly any further calculations are necessary".) For any *even* positive integer n (written $n \in 2\mathbb{N}$), let

$$\cos_n(z) := \sum_{k=0}^{n/2} \frac{(-1)^k z^{2k}}{(2k)!}, \quad n \in 2\mathbb{N}, \tag{1.15}$$

denote the nth partial sum of $\cos(z)$, and similarly, for any *odd* positive integer m (written $m \in 2\mathbb{N}-1$), let

$$\sin_m(z) := \sum_{k=0}^{(m-1)/2} \frac{(-1)^k z^{2k+1}}{(2k+1)!}, \quad m \in 2\mathbb{N}-1, \tag{1.16}$$

denote the mth partial sum of $\sin(z)$. Next, with

$$A_\infty^+ := \left\{z \in \mathbb{C}\colon \left|-\mathrm{i}z\mathrm{e}^{1+\mathrm{i}z}\right| = 1,\ |z| \leqslant 1,\ \operatorname{Im} z \geqslant 0\right\}, \tag{1.17}$$

i.e., A_∞^+ is just that part of the Szegő curve D_∞ of (1.2), when rotated by $\pi/2$, which lies in the upper half-plane, we set

$$A_\infty := A_\infty^+ \cup \left\{\bar{z} \in \mathbb{C}\colon z \in A_\infty^+\right\}. \tag{1.18}$$

Szegő [5] showed that the set of accumulation points of the zeros $\{\hat{z}_{n,k}\}_{k=1}^n$ of the normalized partial sum $\cos_n(nz)$, for all $n \in 2\mathbb{N}$, as well as the accumulation points of the zeros $\{w_{m,k}\}_{k=1}^m$ of the normalized partial sums $\sin_m(mz)$, for all $m \in 2\mathbb{N}-1$, is precisely the following set of points in the unit disk:

$$A_\infty \cup \left[-\frac{1}{\mathrm{e}}, +\frac{1}{\mathrm{e}}\right]. \tag{1.19}$$

(The zeros of $\cos_{60}(60z)$, shown as $\times$'s, are given in figure 1 along with A_∞, and the zeros of $\sin_{61}(61z)$ are similarly shown with A_∞ in figure 2.)

What is particularly interesting in considering the function $\cos(z)$ is that, unlike the case of e^z which has *no* zeros in the complex plane $\mathbb{C}$, $\cos(z)$ has infinitely many (real) zeros. Moreover, given any compact set G in $\mathbb{C}$, it follows, from the uniform convergence in G of $\{\cos_n(z)\}_{n\in 2\mathbb{N}}$ to $\cos(z)$ and Hurwitz's theorem (cf. [6, p. 119]), that $\cos_n(z)$ must now "inherit" zeros in G of $\cos(z)$, as $n \to \infty$, if G contains a sufficiently large real interval. (This is why the closed real interval $[-1/\mathrm{e}, 1/\mathrm{e}]$ is included in (1.19).)

For notation, given an ε with $0 < \varepsilon < 1$, define the rectangle R_ε by

$$R_\varepsilon := \{z \in \mathbb{C}\colon -1 < \operatorname{Re} z < 1 \text{ and } -\varepsilon < \operatorname{Im} z < \varepsilon\}. \tag{1.20}$$

Then, a result of Kappert [4], in a slightly different form, is

[1] "Hierbei sind kaum weitere Rechnungen nötig."

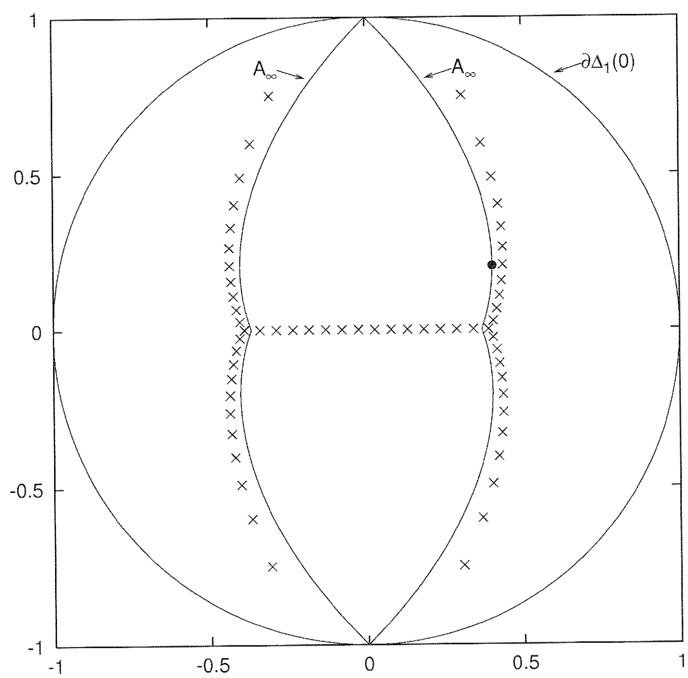

Figure 1. Zeros of $\cos_{60}(60z)$ and A_∞.

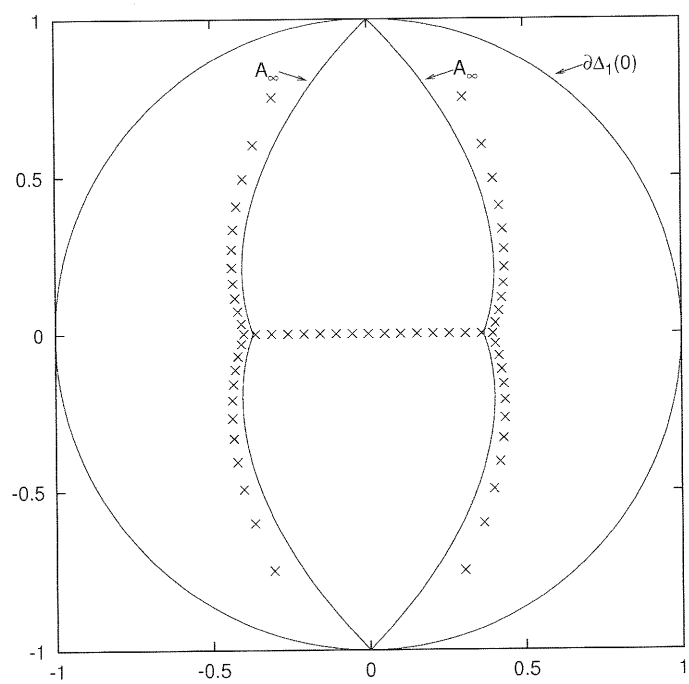

Figure 2. Zeros of $\sin_{61}(61z)$ and A_∞.

Theorem A. Let $\{\hat{z}_{n,k}\}_{k=1}^{n}$, $n \in 2\mathbb{N}$, denote the zeros of $\cos_n(nz)$ of (1.15). With the definitions of A_∞ of (1.18) and R_ε of (1.20) and with fixed positive ε and δ with $0 < \varepsilon + \delta < 1/8$, there holds

$$\operatorname{dist}\left[\{\hat{z}_{n,k}\}_{k=1}^{n}\backslash\left(\Delta_\delta(\mathrm{i}) \cup \Delta_\delta(-\mathrm{i}) \cup R_\varepsilon\right);\, A_\infty\right] = \mathrm{O}\left(\frac{\log n}{n}\right), \tag{1.21}$$

as $n \to \infty$. Similarly, if $\{w_{m,k}\}_{k=1}^{m}$, $m \in 2\mathbb{N}-1$, denotes the zeros of $\sin_m(mz)$ of (1.16), the result of (1.21) holds with m replacing n and $\{w_{m,k}\}_{k=1}^{m}$ replacing $\{\hat{z}_{n,k}\}_{k=1}^{n}$.

For any positive ε and δ with $0 < \varepsilon + \delta < 1/8$, it is a consequence of the results of Szegő [5, equation (8)] that the set of points $\{\hat{z}_{n,k}\}_{k=1}^{n}\backslash(\Delta_\delta(\mathrm{i}) \cup \Delta_\delta(-\mathrm{i}) \cup R_\varepsilon)$ in (1.21) is nonempty for *all* n sufficiently large ($n \in 2\mathbb{N}$), as is the set of points $\{w_{m,k}\}_{k=1}^{m}\backslash(\Delta_\delta(\mathrm{i}) \cup \Delta_\delta(-\mathrm{i}) \cup R_\varepsilon)$ for *all* m sufficiently large ($m \in 2\mathbb{N}-1$).

Next, as an analogue of the set D_n of (1.8), set, for any $n \in \mathbb{N}$,

$$\begin{aligned} A_n^+ := \Bigg\{ & z \in \mathbb{C}\colon \left|-\mathrm{i}z\mathrm{e}^{1+\mathrm{i}z}\right|^n = \tau_n\sqrt{2\pi n}\left|\frac{1+z^2}{-2z^2}\right|,\ |z| \leqslant 1,\ \operatorname{Im} z \geqslant 0,\ \text{and} \\ & \left|\frac{\pi}{2} - \arg z\right| \geqslant \frac{1}{2}\cos^{-1}\left(\frac{n-2}{n}\right),\ 0 \leqslant \arg z \leqslant \pi \Bigg\}. \end{aligned} \tag{1.22}$$

As shown in [4], A_n^+ consists of two well-defined (Jordan) arcs in $\operatorname{Im} z \geqslant 0$. Then on setting

$$A_n := A_n^+ \cup \left\{\bar{z} \in \mathbb{C}\colon z \in A_n^+\right\}, \tag{1.23}$$

another result of Kappert [4], again in a slightly different form, is

Theorem B. Let $\{\hat{z}_{n,k}\}_{k=1}^{n}$, $n \in 2\mathbb{N}$, denote the zeros of $\cos_n(nz)$ of (1.15). With the definitions of A_n of (1.23) and R_ε of (1.20) and with fixed positive ε and δ with $0 < \varepsilon + \delta < 1/8$, there holds

$$\operatorname{dist}\left[\{\hat{z}_{n,k}\}_{k=1}^{n}\backslash\left(\Delta_\delta(\mathrm{i}) \cup \Delta_\delta(-\mathrm{i}) \cup R_\varepsilon\right);\, A_n\right] = \mathrm{O}\left(\frac{1}{n^2}\right), \tag{1.24}$$

as $n \to \infty$. Similarly, if $\{w_{m,k}\}_{k=1}^{m}$, $m \in 2\mathbb{N}-1$, denotes the zeros of $\sin_m(mz)$ of (1.16), the result of (1.24) holds with m replacing n and $\{w_{m,k}\}_{k=1}^{m}$ replacing $\{\hat{z}_{n,k}\}_{k=1}^{n}$.

To illustrate the result of theorem B, we have included figure 3, showing the zeros of $\cos_{60}(z)$ and A_{60}, and figure 4, showing the zeros of $\sin_{61}(61z)$ and A_{61}.

One objective of this note is to first extend, in section 2, the results of theorems A and B to higher-order cases, much as was done for the partial sums of e^z, at the end of section 3 of [2]. With this extension, we also establish in section 3 that the results of theorems A and B are *best possible*.

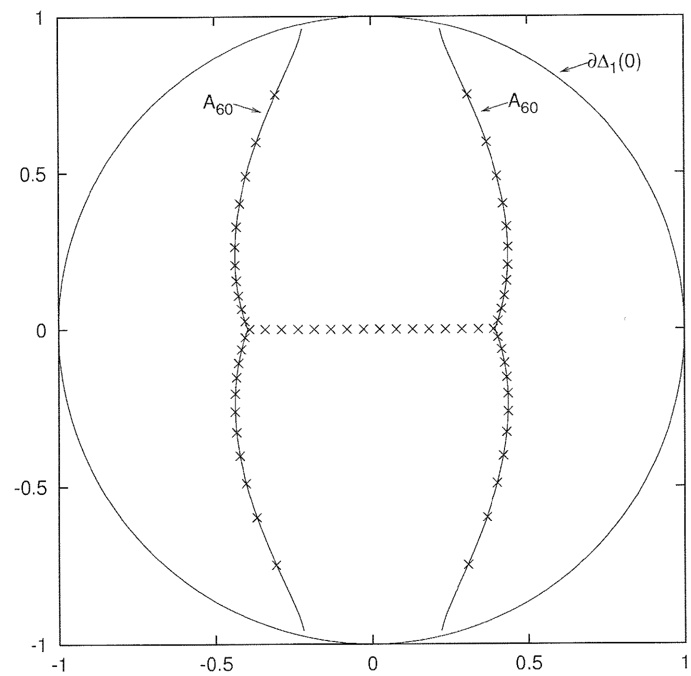

Figure 3. Zeros of $\cos_{60}(60z)$ and A_{60}.

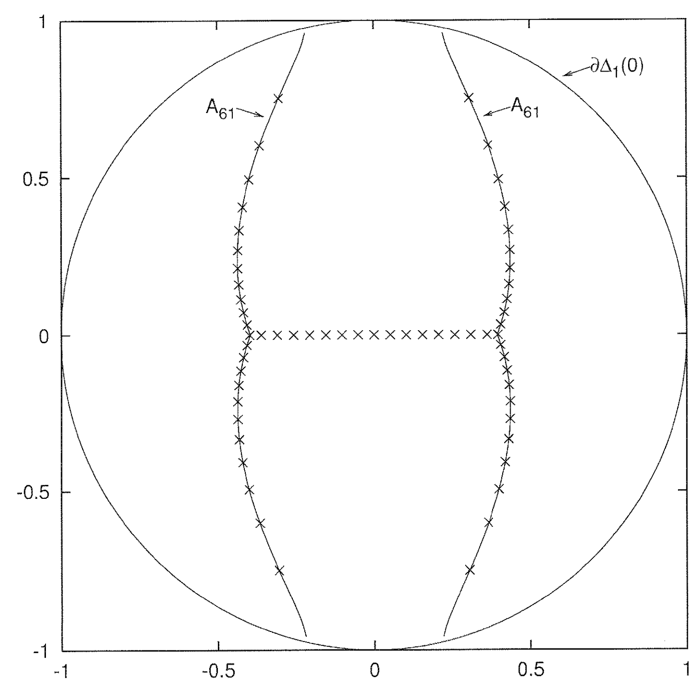

Figure 4. Zeros of $\sin_{61}(61z)$ and A_{61}.

2. The higher-order curves $D_n^{(j)}$

It was shown in [2, equation (2.13)] that $e^{-nz}s_n(nz)$ has the following representation for $z \neq 1$:

$$e^{-nz}s_n(nz) \cong 1 - \frac{(ze^{1-z})^n}{\tau_n\sqrt{2\pi n}}\left(\frac{z}{1-z}\right)\left\{1 - \frac{1}{(n+1)(1-z)^2} + \frac{z(4-2)}{(n+1)(n+2)(1-z)^4} - \frac{z^2(27-14z+2z^2)}{(n+1)(n+2)(n+3)(1-z)^6} + \cdots\right\}, \tag{2.1}$$

for any $n \to \infty$. Though this is not clearly stated in [2], the series representation in (2.1) is rigorously an *asymptotic series*, in the usual sense (cf. [3, chapter 11]) where one takes a finite Taylor's expansion of the integrand of the integral in (1.12), about the point z, and bounds the integral remainder term appropriately. This means that, on any compact subset T of $\overline{\Delta}_1(0)\backslash\{1\}$, we can write, on keeping the first two terms in braces in (2.1), that

$$e^{-nz}s_n(nz) = 1 - \frac{(ze^{1-z})^n}{\tau_n\sqrt{2\pi n}}\left(\frac{z}{1-z}\right)\left\{1 - \frac{1}{(n+1)(1-z)^2} + O\left(\frac{1}{n^2}\right)\right\}, \tag{2.2}$$

uniformly on T as $n \to \infty$. Thus, if z is a zero of $s_n(nz)$ of (1.1), then, on deleting the term $O(1/n^2)$ in (2.2) and on taking moduli, we arrive at the following new arc $D_n^{(2)}$ in $|z| \leqslant 1$, where

$$D_n^{(2)} := \left\{z \in \mathbb{C}\colon \frac{|ze^{1-z}|^n}{\tau_n\sqrt{2\pi n}}\left|\frac{z}{1-z}\right| \cdot \left|1 - \frac{1}{(n+1)(1-z)^2}\right| = 1, \text{ and } |z| \leqslant 1\right\}. \tag{2.3}$$

(The superscript (2) of D_n in (2.3) means that the first *two* terms in braces in (2.1) are kept, so that $D_n^{(1)}$ would then essentially correspond to D_n of (1.8).) We can similarly define high-order arcs $D_n^{(j)}$, $j > 2$, in $|z| \leqslant 1$, upon taking more terms in (2.1). The following was stated without proof in [2].

Theorem C. Let $\{z_{n,k}\}_{k=1}^n$, $n \in \mathbb{N}$, denote the zeros of $s_n(nz)$ of (1.1). With the definition of $D_n^{(j)}$ (cf. (2.3)) for any fixed positive integer j, and with any fixed δ with $0 < \delta < 1$, there holds

$$\operatorname{dist}\left[\{z_{n,k}\}_{k=1}^n\backslash\Delta_\delta(1);\, D_n^{(j)}\right] = O\left(\frac{1}{n^{j+1}}\right), \quad \text{as } n \to \infty. \tag{2.4}$$

3. Connections with $s_n(nz)$

From (1.15), it can be verified that

$$2\cos_n(nz) = s_n(inz) + s_n(-inz), \quad z \in \mathbb{C};\ n \in 2\mathbb{N}, \tag{3.1}$$

which we can express as

$$2\cos_n(nz) = \mathrm{e}^{inz}\left[\mathrm{e}^{-inz}s_n(inz)\right] + \mathrm{e}^{-inz}\left[\mathrm{e}^{inz}s_n(-inz)\right]. \tag{3.2}$$

Applying the result of (2.2) to each of the above bracketed terms, it follows that

$$2\cos_n(nz) = 2\cos(nz) - \frac{(-\mathrm{i}z\mathrm{e})^n}{\tau_n\sqrt{2\pi n}}\left(\frac{-2z^2}{1+z^2}\right)\left\{1 - \frac{3-z^2}{(n+1)(1+z^2)^2} + \mathrm{O}\left(\frac{1}{n^2}\right)\right\}, \tag{3.3}$$

as $n \to \infty$, uniformly on any compact subset of $\overline{\Delta}_1(0)\backslash(\{\mathrm{i}\} \cup \{-\mathrm{i}\})$. Next, as we also have that

$$2\cos(nz) = \mathrm{e}^{inz} + \mathrm{e}^{-inz}, \quad z \in \mathbb{C};\ n \in 2\mathbb{N},$$

whose only zeros lie on the real line $\mathbb{R}$, then dividing the expression in (3.3) by $2\cos(nz)$ gives, for any $z \notin \mathbb{R}$,

$$\frac{\cos_n(nz)}{\cos(nz)} = 1 - \frac{(-\mathrm{i}z\mathrm{e})^n}{\tau_n\sqrt{2\pi n}}\left(\frac{-2z^2}{1+z^2}\right)\frac{\{1 - \frac{3-z^2}{(n+1)(1+z^2)^2} + \mathrm{O}(\frac{1}{n^2})\}}{(\mathrm{e}^{inz} + \mathrm{e}^{-inz})}, \tag{3.4}$$

uniformly on any compact subset of $\overline{\Delta}_1(0)\backslash(\{\mathrm{i}\} \cup \{-\mathrm{i}\} \cup \mathbb{R})$. This can be equivalently expressed as

$$\frac{\cos_n(nz)}{\cos(nz)} = 1 - \frac{[-\mathrm{i}z\mathrm{e}^{1+\mathrm{i}z}]^n}{\tau_n\sqrt{2\pi n}}\left(\frac{-2z^2}{1+z^2}\right)\frac{\{1 - \frac{3-z^2}{(n+1)(1+z^2)^2} + \mathrm{O}(\frac{1}{n^2})\}}{(1 + \mathrm{e}^{2inz})}. \tag{3.5}$$

But if for $0 < \sigma < 1$, we define

$$\Delta_\sigma^+ := \{z = x + \mathrm{i}y\colon\ |z| \leqslant 1 \text{ and } y \geqslant \sigma > 0\} \quad \text{and}$$
$$\Delta^+ := \{z = x + \mathrm{i}y\colon\ |z| \leqslant 1 \text{ and } y > 0\}, \tag{3.6}$$

it follows that $z \in \Delta_\sigma^+$ implies that $|\mathrm{e}^{2inz}| \leqslant \mathrm{e}^{-2n\sigma}$, where $\mathrm{e}^{-2n\sigma} = \mathrm{o}(1/n)$, as $n \to \infty$, for any fixed $\sigma > 0$. Then (3.5) gives us that, for $n \in 2\mathbb{N}$,

$$\frac{\cos_n(nz)}{\cos(nz)} = 1 - \frac{[-\mathrm{i}z\mathrm{e}^{1+\mathrm{i}z}]^n}{\tau_n\sqrt{2\pi n}}\left(\frac{-2z^2}{1+z^2}\right)\left\{1 - \frac{3-z^2}{(n+1)(1+z^2)^2} + \mathrm{O}\left(\frac{1}{n^2}\right)\right\}, \tag{3.7}$$

as $n \to \infty$, uniformly on any compact subset of $\Delta^+\backslash\{\mathrm{i}\}$, with a similar statement holding for the reflection, in the real axis, of the set $\Delta^+\backslash\{\mathrm{i}\}$.

It is apparent from (3.7) that if there is a convergent infinite sequence $\{\hat{z}_{n_j}\}$ of zeros of $\{\cos_n(nz)\}_{n\in 2\mathbb{N}}$ in the upper half-plane, *not* converging to a real number or to i, i.e.,

$$\cos_{n_j}(n_j\hat{z}_{n_j}) = 0, \quad \operatorname{Im}\hat{z}_{n_j} > 0, \quad \lim_{j\to\infty}\hat{z}_{n_j} = \tilde{z}, \quad \text{with } \tilde{z} \text{ not real and } \tilde{z} \neq \mathrm{i},$$

and $\lim_{j\to\infty} n_j = \infty$, then it is evident from (3.7) that $|-\mathrm{i}\tilde{z}\mathrm{e}^{1+\mathrm{i}\tilde{z}}| = 1$, which defines the curve A_∞^+ of (1.17). As this argument also similarly applies in the lower half-plane, one sees that the conjugates of the points in A_∞^+ must also appear in the definition of A_∞ of (1.18), and, as previously mentioned, the closed interval $[-1/\mathrm{e}, +1/\mathrm{e}]$ in (1.19) is a consequence of the Hurwitz theorem. Then on deleting all but the constant term unity in the braces in (3.7) and assuming that $\cos_n(nz) = 0$, one obtains that a solution of

$$\left[-\mathrm{i}z\mathrm{e}^{1+\mathrm{i}z}\right]^n = \tau_n\sqrt{2\pi n}\left(\frac{1+z^2}{-2z^2}\right) \tag{3.8}$$

is an approximation of this zero z, and, on taking absolute values, this essentially gives the definition of the curves A_n^+ in (1.22).

The connection of the zeros of $\sin_m(mz)$ with the partial sums of e^z is very similar, as the zeros of $\sin_m(mz)$, again from the Eneström–Kakeya theorem, lie in $\Delta_1(0)$ for every $m \in 2\mathbb{N}-1$. Next, from (1.16), we similarly have (cf. (3.1)) that

$$2\mathrm{i}\sin_m(mz) = s_m(\mathrm{i}mz) - s_m(-\mathrm{i}mz), \quad z \in \mathbb{C},\ m \in 2\mathbb{N}-1, \tag{3.9}$$

and, on using (1.13) and (2.1), it can be verified (cf. (3.7)) that, for $m \in 2\mathbb{N}-1$,

$$\frac{\sin_m(mz)}{\sin(mz)} = 1 - \frac{[-\mathrm{i}z\mathrm{e}^{1+\mathrm{i}z}]^m}{\tau_m\sqrt{2\pi m}}\left(\frac{-2z^2}{1+z^2}\right)\left\{1 - \frac{3-z^2}{(m+1)(1+z^2)^2} + \mathrm{O}\left(\frac{1}{m^2}\right)\right\}, \tag{3.10}$$

as $m \to \infty$, uniformly on any compact subset of $\Delta^+\backslash\{\mathrm{i}\}$, with a similar statement holding for the reflection, in the real axis, of the set $\Delta^+\backslash\{\mathrm{i}\}$. Note that the right sides of (3.7) and (3.10) have the same form, except that (3.7) holds for $n \in 2\mathbb{N}$, while (3.10) holds for $m \in 2\mathbb{N}-1$.

It is evident from the derivations of (3.7) and (3.10) that a higher-order curve $A_n^{(2)}$ in $\overline{\Delta}_1(0)$, for approximating the zeros of $\cos_n(nz)$ or $\sin_m(mz)$, can be defined in an obvious way by taking the first *two* terms in the braces, of (3.7) or (3.10), and then taking absolute values. Other higher-order curves $A_n^{(j)}$, $j > 2$, can similarly be derived on using the first j terms in the braces of (2.1).) Moreover, based on the result of theorem C for $j = 2$, the following result can be established, in analogy with theorems A and B of Kappert. (This will be used below for theorem 3.)

Theorem 1. Let $\{\hat{z}_{n,k}\}_{k=1}^n$, $n \in 2\mathbb{N}$, denote the zeros of $\cos_n(nz)$ of (1.15). With the definition above of $A_n^{(2)}$ and of R_ε of (1.20), and with given fixed positive ε and δ with $0 < \varepsilon + \delta < 1/8$, there holds

$$\operatorname{dist}\left[\{\hat{z}_{n,k}\}_{k=1}^n\backslash\left(\Delta_\delta(\mathrm{i}) \cup \Delta_\delta(-\mathrm{i}) \cup R_\varepsilon\right);\ A_n^{(2)}\right] = \mathrm{O}\left(\frac{1}{n^3}\right), \tag{3.11}$$

as $n \to \infty$. Similarly, if $\{w_{m,k}\}_{k=1}^m$, $m \in 2\mathbb{N}-1$, denotes the zeros of $\sin_m(mz)$ of (1.16), the result of (3.11) holds with m replacing n and $\{w_{m,k}\}_{k=1}^m$ replacing $\{\hat{z}_{n,k}\}_{k=1}^n$.

To establish the sharpness of (1.21) of theorem A, consider the point $\check{z}$ of A_∞^+, in the first quadrant, where the tangent to A_∞^+ at this point is vertical. It can be easily verified from the definition of A_∞^+ in (1.17) that

$$\check{z} = \check{x} + \mathrm{i}\check{y}, \quad \text{where } \check{x} \doteq 0.40237 \text{ and } \check{y} \doteq 0.20319. \tag{3.12}$$

(The point $\check{z}$ of (3.12) is shown, for the reader's convenience, as a small solid disk in the first quadrant of figure 1.) Then for $n \in \mathbb{N}$, consider the unique point w_n of A_n^+, in the first quadrant, where

$$w_n = \check{z} + \delta_n, \quad \delta_n > 0. \tag{3.13}$$

Because A_∞^+ has a vertical tangent at $\check{z}$, it follows by definition that

$$\operatorname{dist}\big[w_n;\, A_\infty^+\big] = \delta_n > 0. \tag{3.14}$$

To determine δ_n, we have that $w_n \in A_n^+$, so that, from (1.22),

$$\left|-\mathrm{i}w_n \mathrm{e}^{1+\mathrm{i}w_n}\right|^n = \tau_n\sqrt{2\pi n}\left|\frac{1+w_n^2}{-2w_n^2}\right|.$$

Write $\check{z} = |\check{z}|\mathrm{e}^{\mathrm{i}\psi}$, where $\psi := \tan^{-1}(\check{y}/\check{x})$, so that $\cos\psi \doteq 0.89264$. Then because $\check{z} = \check{x} + \mathrm{i}\check{y} \in A_\infty^+$, the above expression reduces, with (3.13), to

$$\left|1 + \frac{\delta_n \mathrm{e}^{-\mathrm{i}\psi}}{|\check{z}|}\right|^n = \tau_n\sqrt{2\pi n}\,\frac{|1+(\check{z}+\delta_n)^2|}{2|\check{z}+\delta_n|^2}.$$

After some easy calculations, it can be verified that

$$\delta_n = \frac{|\check{z}|\log n}{2n\cos\psi} + \mathrm{O}\left(\frac{1}{n}\right), \quad \text{as } n \to \infty. \tag{3.15}$$

Next, for each n sufficiently large, $n \in 2\mathbb{N}$, let z_n be the closest zero (in the first quadrant) of $\cos_n(nz)$ to w_n of (3.13). We note that this zero z_n will, for all n sufficiently large, lie outside of the set $(\Delta_\delta(\mathrm{i}) \cup \Delta_\delta(-\mathrm{i}) \cup R_\varepsilon)$, so that (1.24) of theorem B is applicable. Thus, theorem B implies that

$$\operatorname{dist}[z_n;\, w_n] = \mathrm{O}\left(\frac{1}{n^2}\right), \quad \text{as } n \to \infty. \tag{3.16}$$

Hence, as the triangle inequality gives

$$\operatorname{dist}\big[w_n;\, A_\infty^+\big] - \operatorname{dist}\big[z_n;\, w_n\big] \leqslant \operatorname{dist}\big[z_n;\, A_\infty^+\big] \leqslant \operatorname{dist}\big[z_n;\, w_n\big] + \operatorname{dist}\big[w_n;\, A_\infty^+\big],$$

and as $\delta_n = \operatorname{dist}[w_n;\, A_\infty^+]$ is dominant from (3.15), over $\operatorname{dist}[z_n, w_n]$ of (3.16) for all n sufficiently large, then

$$\lim_{n\to\infty}\left\{\frac{n}{\log n}\cdot \operatorname{dist}\big[z_n;\, A_\infty^+\big]\right\} = \frac{|\check{z}|}{2\cos\psi} = \frac{|\check{z}|^2}{2\operatorname{Re}\check{z}} \doteq 0.25249. \tag{3.17}$$

Moreover, since $\operatorname{dist}[z_n;\, A_\infty^+] \leqslant \operatorname{dist}[\{\hat{z}_{n,k}\}_{k=1}^n \backslash (\Delta_\delta(\mathrm{i}) \cup \Delta_\delta(-\mathrm{i}) \cup R_\varepsilon\};\, A_\infty]$ for all n sufficiently large, we have the result of

Theorem 2. Let $\{\hat{z}_{n,k}\}_{k=1}^n$, $n \in 2\mathbb{N}$, denote the zeros of $\cos_n(nz)$ of (1.15). Then, with the definition of A_∞ of (1.18) and of R_ε of (1.20), the result of (1.21) of theorem A is best possible, in the sense, that, for given fixed positive ε and δ with $0 < \varepsilon + \delta < 1/8$, there holds

$$\varliminf_{n\to\infty}\left\{\frac{n}{\log n}\operatorname{dist}\left[\{\check{z}_{n,k}\}_{k=1}^{n}\backslash(\Delta_\delta(\mathrm{i})\cup\Delta_\delta(-\mathrm{i})\cup R_\varepsilon);\, A_\infty\right]\right\} \geqslant \frac{|\check{z}|}{2\cos\psi} = \frac{|\check{z}|^2}{2\operatorname{Re}\check{z}} \doteq 0.25249. \quad (3.18)$$

Similarly, if $\{w_{m,k}\}_{k=1}^{m}$, $m \in 2\mathbb{N}-1$ denotes the zeros of $\sin_m(mz)$ of (1.16), then (3.18) holds with m replacing n and $\{w_{m,k}\}_{k=1}^{m}$ replacing $\{\hat{z}_{n,k}\}_{k=1}^{n}$.

The key idea in the proof of theorem 2 was to use the result of the higher-order approximation of theorem B to obtain sharpness in theorem A. But this idea can be also applied to theorem B, where the higher-order approximation now comes from using the new curves $A_n^{(2)}$, on taking the first two terms in (3.7) or (3.10), to determine, from (3.11) of theorem 1, an $\mathrm{O}(1/n^3)$ approximation of the zeros of $\cos_n(nz)$, or an $\mathrm{O}(1/m^3)$ approximation of the zeros of $\sin_m(mz)$. Without going through all similar (and lengthy) details, we simply state the result of:

Theorem 3. Let $\{\hat{z}_{n,k}\}_{k=1}^{n}$, $n \in 2\mathbb{N}$, denote the zeros of $\cos_n(nz)$ of (1.15). Then, with the definitions of A_n of (1.23) and of R_ε of (1.20), the result of (1.24) of theorem B is best possible, in the sense that, given fixed positive ε and δ with $0 < \varepsilon + \delta < 1/8$, there is a positive constant β, defined from $\check{z}$ of (3.12) by

$$\beta := \operatorname{Re}\left\{\frac{3-(\check{z})^2}{(1+(\check{z})^2)^2}\right\}\cdot\frac{|\check{z}|^2}{\operatorname{Re}\check{z}} \doteq 1.06805, \quad (3.19)$$

such that

$$\varliminf_{n\to\infty}\left\{n^2\operatorname{dist}\left[\{z_{n,k}\}_{k=1}^{n}\backslash\left(\Delta_\delta(\mathrm{i})\cup\Delta_\delta(-\mathrm{i})\cup R_\varepsilon\right);\, A_n\right]\right\} \geqslant \beta. \quad (3.20)$$

Similarly, if $\{w_{m,k}\}_{k=1}^{m}$, $m \in 2\mathbb{N}-1$, denotes the zeros of $\sin_m(mz)$ of (1.16), then (3.19) holds with m replacing n and $\{w_{m,k}\}_{k=1}^{m}$ replacing $\{\hat{z}_{n,k}\}_{k=1}^{n}$.

We point out that the technique of proof for theorems 2 and 3 can similarly be applied to deduce the higher-order generalizations of theorems A and 2, and theorems B and 3. Starting with the asymptotic series for $\mathrm{e}^{-nz}s_n(nz)$ in (2.1), use (3.1) to generate an asymptotic series for $2\cos_n(nz)$, $n \in 2\mathbb{N}$, and divide this expression by $\cos(nz)$, for z not real. Then for a fixed positive integer j with $j \geqslant 1$, take the first j terms of this resulting asymptotic series, which, on taking absolute values, serves to define the curves $A_n^{(j)}$ in $|z| \leqslant 1$, for all $n \in 2\mathbb{N}$. (This can be seen in (3.7) for the case $j = 2$.) We remark that the same curves $A_n^{(j)}$ are obtained for $\sin_m(mz)/\sin(mz)$, $m \in 2\mathbb{N}-1$ and z not real, as can be seen from (3.10), again for the case $j = 2$. Without giving proofs, we state the final result of

Theorem 4. Let $\{\hat{z}_{n,k}\}_{k=1}^{n}$, $n \in 2\mathbb{N}$, denote the zeros of $\cos_n(nz)$ of (1.15). With the above definition of $A_n^{(j)}$, for any fixed $j \geqslant 1$, and of R_ε of (1.20), and with given fixed positive ε and δ with $0 < \varepsilon + \delta < 1/8$, there holds

$$\operatorname{dist}\left[\left\{\hat{z}_{n,k}\right\}_{k=1}^{n}\setminus\left(\Delta_\delta(\mathrm{i}) \cup \Delta_\delta(-\mathrm{i}) \cup R_\varepsilon\right);\ A_n^{(j)}\right] = \mathrm{O}\left(\frac{1}{n^{j+1}}\right), \tag{3.21}$$

as $n \to \infty$, and this result is best possible in the sense of the previous theorems. Similarly, if $\{w_{m,k}\}_{k=1}^{m}$, $m \in 2\mathbb{N} - 1$, denotes the zeros of $\sin_m(mz)$ of (1.16), then (3.21) holds with m replacing n and $\{w_{m,k}\}_{k=1}^{m}$ replacing $\{\hat{z}_{n,k}\}_{k=1}^{n}$, and this result is best possible in the sense of the previous theorems.

In a subsequent paper, we will carefully study the location and density behavior of all real and complex zeros of $\cos_n(nz)$, $n \in 2\mathbb{N}$, and of $\sin_m(mz)$, $m \in 2\mathbb{N} - 1$.

References

[1] J.D. Buckholtz, A characterization of the exponential series, Part II, Amer. Math. Monthly 73 (1966) 121–123.
[2] A.J. Carpenter, R.S. Varga and J. Waldvogel, Asymptotics for the zeros of the partial sums of e^z. I, Rocky Mountain J. Math. 21 (1991) 99–120.
[3] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 2 (Wiley, New York, 1977).
[4] M. Kappert, On the zeros of the partial sums of cos(z) and sin(z), Numer. Math. 74 (1996) 397–417.
[5] G. Szegő, Über eine Eigenschaft der Exponentialreihe, Sitzungsber. Math. Ges. 23 (1924) 50–64.
[6] E.C. Titchmarsh, *The Theory of Functions*, 2nd ed. (Oxford Univ. Press, London, 1939).