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Zeros of the partial sums of $\cos(z)$ and $\sin(z)$ II

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Summary. We study here in detail the location of the real and complex zeros of the partial sums of $\cos(z)$ and $\sin(z)$, which extends results of Szegö (1924) and Kappert (1996).

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1 Introduction

For any even positive integer n (written $n \in 2\mathbb{N}$), let

$$\cos_n(z) := \sum_{j=0}^{n/2} (-1)^j z^{2j}/(2j)! \quad (n \in 2\mathbb{N}) \tag{1.1}$$

denote the n-th partial sum of the function $\cos(z)$, and similarly, for any odd positive integer m (written $m \in 2\mathbb{N} - 1$), let

$$\sin_m(z) := \sum_{j=0}^{\frac{m-1}{2}} (-1)^j z^{2j+1}/(2j+1)! \quad (m \in 2\mathbb{N} - 1) \tag{1.2}$$

denote the m-th partial sum of the function $\sin(z)$. The zeros of the *normalized* partial sums $\cos_n(nz)$ and $\sin_m(mz)$ were first studied in 1924 by Szegö [11]. To describe his results, let

$$A_\infty^+ := \{z \in \mathbb{C} : |-ize^{1+iz}| = 1,\ |z| \le 1,\ \text{and } \operatorname{Im} z \ge 0\}, \tag{1.3}$$

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so that A_∞^+ is a piecewise analytic Jordan arc in the upper half-plane of the closed unit disk. (For those familiar with the work of Szegö [11], it was shown there that if $s_n(z) := \sum_{k=0}^{n} z^k/k!$ is the n-th partial sum of e^z, then all the zeros of the normalized partial sum $s_n(nz)$ tend, as $n \to \infty$, to the Jordan curve

$$D_\infty := \left\{z \in \mathbb{C} : \left|ze^{1-z}\right| = 1 \text{ and } |z| \leq 1\right\}, \tag{1.4}$$

which is commonly called the *Szegö curve*. Then, A_∞^+ of (1.3) is just the restriction to the upper half-plane of the rotation, by $\pi/2$, of the Szegö curve.) Then set

$$A_\infty := A_\infty^+ \cup \{\overline{z} \in \mathbb{C} : z \in A_\infty^+\}. \tag{1.5}$$

We remark that A_∞, the union of A_∞^+ and its reflection in the real axis, is then a piecewise-analytic Jordan curve in the closed unit disk, where $\pm i$ and $\pm 1/e$ are boundary points of A_∞. If

(1.6) $\{z_{n,j}\}_{j=1}^n$ and $\{w_{m,j}\}_{j=1}^m$ denote, respectively, the zeros of $\cos_n(nz)$ and $\sin_m(mz)$,

then Szegö showed that the set of accumulation points, of either

$$\bigcup_{n \in 2\mathbb{N}} \{z_{n,j}\}_{j=1}^n \quad \text{or} \quad \bigcup_{m \in 2\mathbb{N}-1} \{w_{m,j}\}_{j=1}^m,$$

is *precisely* the following set of points in the closed unit disk:

$$A_\infty \cup \left[-\frac{1}{e}, \frac{1}{e}\right]. \tag{1.7}$$

To illustrate this, Fig. 1.1 shows all the zeros (as dots) of $\{\cos_{2j}(2jz)\}_{j=1}^{50}$ and also the curve A_∞. One can see that the nonreal zeros of $\cos_n(nz)$ are approaching A_∞, and that the real interval $[-\frac{1}{e}, +\frac{1}{e}]$ is becoming dense with zeros of $\cos_n(nz)$. (The plot of all zeros of $\{\sin_{2j+1}((2j+1)z)\}_{j=0}^{50}$ is nearly identical with Fig. 1.1, and is given in Fig. 1.2.)

The *rate*, at which the nonreal zeros of either $\cos_n(nz)$ or $\sin_m(mz)$ approach the Jordan curve A_∞, was first studied by Kappert [7], using the techniques of Carpenter, Varga and Waldvogel [3], and his results were later shown in Varga and Carpenter [14] to be best possible. To describe these results more precisely, for any $\epsilon > 0$, let the open rectangle R_ϵ be defined by

$$R_\epsilon := \{z \in \mathbb{C} : -1 < \text{Re } z < 1 \text{ and } -\epsilon < \text{Im } z < \epsilon\}, \tag{1.8}$$

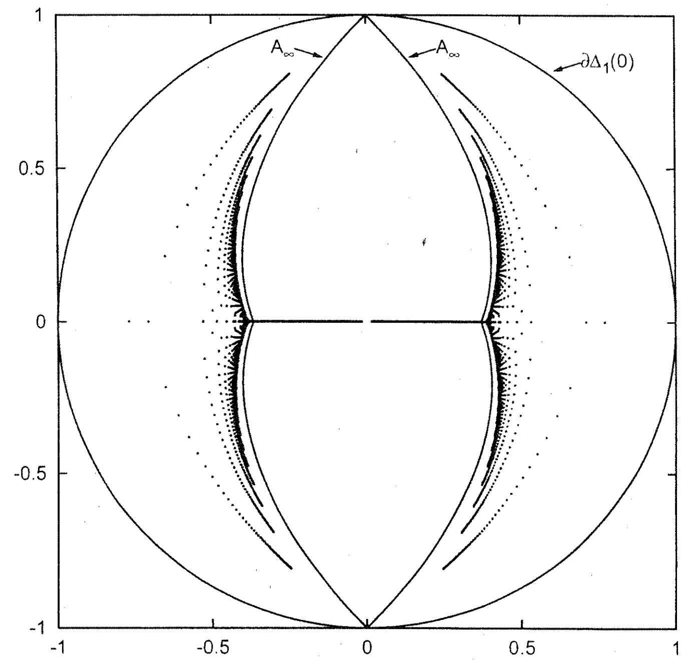

Fig. 1.1. The zeros of $\cos_n(nz)$ for $n = 2, 4, 6..., 100$

and, for any complex number α and for any $\delta > 0$, let

$$(1.9) \qquad \Delta_\delta(\alpha) := \{z \in \mathbb{C} : |z - \alpha| < \delta\}$$

denote the open disk about α with radius δ. If, as usual,

$$dist[z; T] := \inf\{|z - y| : y \in T\} \quad (z \in \mathbb{C}; T \subseteq \mathbb{C}),$$

and if, for a finite set S in $\mathbb{C}$,

$$dist[S; T] := \max\{dist[z; T] : z \in S\},$$

then, with the definitions of the zeros of (1.6), it was shown in Kappert [7] (in a slightly different form), for any fixed positive ϵ and δ with $0 < \epsilon + \delta < 1/8$, that for $n \in 2\mathbb{N}$,

$$(1.10) \quad dist[\{z_{n,j}\}_{j=1}^{n} \backslash (\Delta_\delta(i) \cup \Delta_\delta(-i) \cup R_\epsilon); A_\infty] = O\left(\frac{\log n}{n}\right),$$

as $n \to \infty$, i.e., for $n \in 2\mathbb{N}$, the maximum distance from A_∞ of a zero $z_{n,j}$ of $\cos_n(nz)$, *not* in $\Delta_\delta(i) \cup \Delta_\delta(-i) \cup R_\epsilon$, is $O(\log n/n)$, as $n \to \infty$. Moreover, the statement of (1.10) also holds for the partial sums $\sin_m(mz)$, with $m \in (2\mathbb{N} - 1)$ replacing n and $\{w_{m,j}\}_{j=1}^{m}$ replacing $\{z_{n,j}\}_{j=1}^{n}$ in (1.10). It was later shown in [14] that the result of (1.10) is *best possible*,

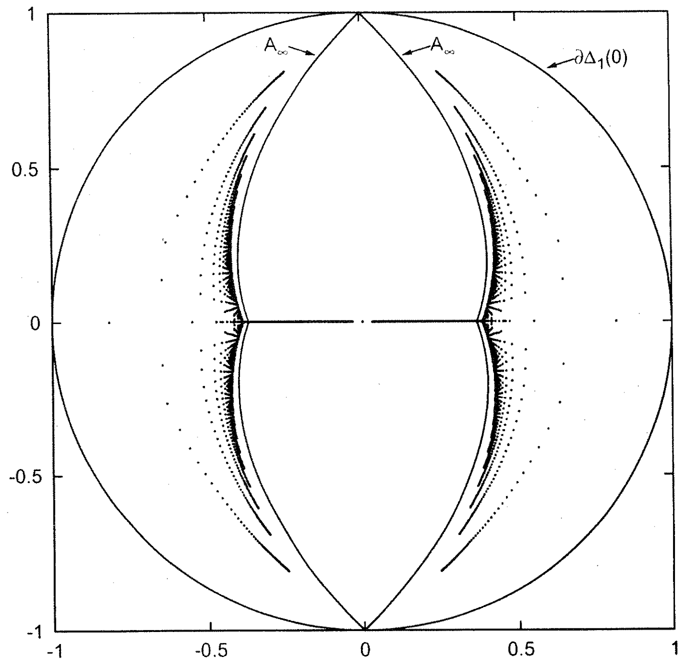

Fig. 1.2. The zeros of $\sin_n(nz)$ for $n = 1, 3, 5, ..., 101$

in the sense that there exists a positive constant $\alpha \doteq 0.25249$, such that for any fixed positive ϵ and δ with $0 < \epsilon + \delta < 1/8$,

$$(1.11)\ \varliminf_{\substack{n\to\infty \\ n\in 2\mathbb{N}}} \left\{ \frac{n}{\log n} dist \left[\{z_{n,j}\}_{j=1}^{n} \backslash (\Delta_\delta(i) \cup \Delta_\delta(-i) \cup R_\epsilon); A_\infty \right] \right\} \geq \alpha,$$

with (1.11) also holding with $m \in 2\mathbb{N} - 1$ replacing n and $\{w_{m,j}\}_{j=1}^{m}$ replacing $\{z_{n,j}\}_{j=1}^{n}$.

What remains unanswered from (1.10) and (1.11) is the behavior of the zeros of $\cos_n(nz)$, for each $n \in 2\mathbb{N}$, or of $\sin_m(mz)$, for each $m \in 2\mathbb{N} - 1$, which are near or on the real line. Figure 1.1 strongly suggests that there are *no* zeros of $\cos_n(nz)$, except for *real zeros*, in the set consisting of A_∞ and its interior, for *any* $n \in 2\mathbb{N}$, with a similar relation holding for the zeros of $\sin_m(mz)$ for any $m \in 2\mathbb{N} - 1$. That this is true is a main result, Theorem 4.3, of this paper, which will be established in Sect. 4.

It is well-known that if G is a bounded region in $\mathbb{C}$ which contains, say, μ (real) zeros of $\cos(z)$, then, from Hurwitz's Theorem (see Titchmarsh [13, p. 119]) and the fact that the sequence of polynomials $\{\cos_n(z)\}_{n\in 2\mathbb{N}}$ converges uniformly to $\cos(z)$ in any compact subset of $\mathbb{C}$, it follows that there is a positive even integer n_0 such that, for all $n \in 2\mathbb{N}$ with $n \geq n_0$, $\cos_n(z)$ possesses exactly μ real zeros in G, and these zeros tend to the zeros of $\cos(z)$, as $n \to \infty$. These μ zeros of $\cos_n(z)$, having this property, are called *Hurwitz zeros*, while the remaining zeros of $\cos_n(z)$ are called *spurious zeros*, with similar definitions applying to $\{\sin_m(z)\}_{m\in 2\mathbb{N}-1}$ and $\sin(z)$. In Sect. 4, we study the number of Hurwitz zeros of $\cos_n(nz)$, $n \in 2\mathbb{N}$, and of $\sin_m(mz)$, $m \in 2\mathbb{N}-1$, which has for us interesting number-theoretic connections.

Continuing the discussion of the real (Hurwitz or spurious) zeros of $\cos_n(nz)$ or $\sin_m(mz)$, it is known from Szegö [11] that if $z_{n,1}$ now denotes the largest real (Hurwitz or spurious) zero of $\cos_n(nz)$ for each $n \in 2\mathbb{N}$, then

$$\lim_{\substack{n\to\infty \\ n\in 2\mathbb{N}}} \left(z_{n,1} - \frac{1}{e}\right) = 0, \tag{1.12}$$

with the same result holding for the largest real zero $w_{m,1}$ of $\sin_m(mz)$, for $m \in 2\mathbb{N}-1$. In Theorem 5.2, we obtain a more precise rate of convergence of these zeros to $1/e$, as $n \in 2\mathbb{N}$ or $m \in 2\mathbb{N}-1$ tends to infinity.

2 Preliminary results

With the definition of the set A_∞^+ in (1.3), consider the piecewise analytic Jordan curve S^+, in the upper half-plane of the closed unit disk, defined by

$$S^+ := A_\infty^+ \cup \left[-\frac{1}{e}, +\frac{1}{e}\right], \tag{2.1}$$

and set

$$K^+ := S^+ \cup int(S^+). \tag{2.2}$$

Similarly, let

$$K^- := \left\{\overline{z} \in \mathbb{C} : z \in K^+\right\}, \tag{2.3}$$

so that K^- is the reflection of K^+ in the real axis, and set

$$K := K^+ \cup K^-. \tag{2.4}$$

The sets K^+ and K^- are shown in Fig. 2.1. We remark that it can be shown (see Saff and Varga [10]) that the set K^+ of (2.2) can be equivalently expressed as

$$K^+ = \left\{z \in \mathbb{C} : |-ize^{1+iz}| \leq 1, |z| \leq 1 \text{ and } Im\ z \geq 0\right\}. \tag{2.5}$$

We now assume that n is an *even* positive integer (written $n \in 2\mathbb{N}$), and, analogous to the partial sums of $\cos(nz)$ in (1.1), we consider the n-th partial sum of $\cosh(n)$, called $\cosh_n(n)$, which is similarly defined as

$$\cosh_n(n) := \sum_{k=0}^{n/2} \frac{n^{2k}}{(2k)!}, \quad (n \in 2\mathbb{N}), \tag{2.6}$$

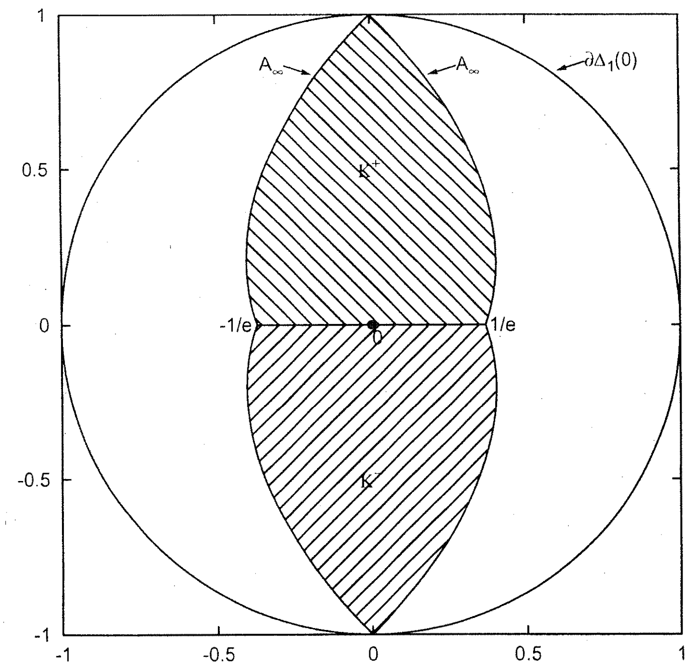

Fig. 2.1. The sets K^+ and K^-

where

$$\cosh(n) = \sum_{k=0}^{\infty} \frac{n^{2k}}{(2k)!} = \frac{e^n + e^{-n}}{2}. \tag{2.7}$$

On setting

$$Q_n := 1 - \frac{\cosh_n(n)}{\cosh(n)} \quad (n \in 2\mathbb{N}), \tag{2.8}$$

we first establish

Lemma 2.1 *For any* $n \in 2\mathbb{N}$, Q_n *of* (2.8) *satisfies*

$$0 < Q_n < \frac{1}{2}, \text{ and } \lim_{n\to\infty} Q_n = \frac{1}{2}. \tag{2.9}$$

Proof. Starting with Szegö's notation, calling $s_n(z)$ the n-th partial sum of e^z, i.e.,

$$s_n(z) := \sum_{k=0}^{n} z^k/k! \quad (z \in \mathbb{C}; n \in \mathbb{N}), \tag{2.10}$$

it is known from Szegö [11] (and is easily verified by differentiation) that

$$e^{-z}s_n(z) = 1 - \frac{1}{n!}\int_0^z \zeta^n e^{-\zeta} d\zeta \quad (z \in \mathbb{C}; n \in \mathbb{N}). \tag{2.11}$$

Next, it is known that the number τ_n, from Stirling's formula, is given by the following asymptotic series (see Henrici [6, p. 377]):

$$(2.12)\qquad \tau_n := \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} \cong 1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + \cdots \quad (n \to \infty).$$

Also, since

$$\tau_{n+1}/\tau_n = \frac{e}{\left(1+\dfrac{1}{n}\right)^{n+\frac{1}{2}}} \quad (n \in \mathbb{N}),$$

it follows (see Pólya and Szegö [8, I.168, p. 30]) that $\{\tau_n\}_{n=1}^{\infty}$ is a strictly decreasing sequence of positive numbers which tends to unity, i.e.,

$$(2.13)\qquad 1.08444 \doteq \tau_1 > \tau_2 > \tau_3 > \cdots \text{ and } \lim_{n\to\infty} \tau_n = 1.$$

Then on replacing z by nz in (2.11) and on using the definition of τ_n in (2.12), it follows that

$$(2.14)\quad e^{-nz}s_n(nz) = 1 - \frac{\sqrt{n}}{\tau_n\sqrt{2\pi}}\int_0^z (\zeta e^{1-\zeta})^n d\zeta \quad (z \in \mathbb{C}; n \in \mathbb{N}).$$

Next, we see from (2.6) and (2.10) that

$$\cosh_n(n) = [s_n(n) + s_n(-n)]/2 \quad (n \in 2\mathbb{N}),$$

and as $\cosh(n) = (e^n + e^{-n})/2$, the quantity Q_n can be equivalently expressed from (2.8) as

$$(2.15)\qquad Q_n = \left[\frac{1 - e^{-n}s_n(n)}{1 + e^{-2n}}\right] + \left[\frac{e^{-n} - s_n(-n)}{e^n + e^{-n}}\right].$$

Now, the special case of $z = 1$ in (2.14) gives

$$(2.16)\qquad 1 - e^{-n}s_n(n) = \frac{\sqrt{n}}{\tau_n\sqrt{2\pi}}\int_0^1 \left(\zeta e^{1-\zeta}\right)^n d\zeta,$$

and an application of Laplace's method (see Erdélyi [5, p. 9]), to the integral in (2.16), directly gives, with (2.12), the following asymptotic result of

$$1 - e^{-n}s_n(n) = \frac{1}{2} - \frac{\sqrt{2}}{3\sqrt{\pi}\cdot\sqrt{n}} + O\left(\frac{1}{n^{3/2}}\right), \quad \text{as } n \to \infty.$$

(Curiously, the above equation can be derived from a statement (no proof) in 1913 of Ramanujan [9]. Proofs for his statement were later obtained independently in 1928 by Szegö [12] and Watson [15].) The above equation indicates that, for all n sufficiently large, the terms of the sequence $\{1 - \mathrm{e}^{-n} s_n(n)\}_{n=1}^{\infty}$ are positive and strictly increasing to $\frac{1}{2}$, as $n \to \infty$, and, in fact, these consequences hold for all $n \in \mathbb{N}$:

$$(2.17) \qquad 0 < 1 - \mathrm{e}^{-n} s_n(n) < \frac{1}{2} \text{ and } \lim_{n\to\infty} (1 - \mathrm{e}^{-n} s_n(n)) = \frac{1}{2}.$$

Thus, the first bracketed term of Q_n in (2.15) satisfies (for all $n \in \mathbb{N}$)

$$(2.18) \quad 0 < \frac{1 - \mathrm{e}^{-n} s_n(n)}{1 + \mathrm{e}^{-2n}} < \frac{1}{2} \text{ and } \lim_{n\to\infty} \left(\frac{1 - \mathrm{e}^{-n} s_n(n)}{1 + \mathrm{e}^{-2n}} \right) = \frac{1}{2}.$$

We also have, from the definition of $\cosh_n(n)$ in (2.6), that

$$0 < \cosh_n(n)/\cosh(n) < 1, \quad (n \in 2\mathbb{N}),$$

which implies from (2.8) that

$$(2.19) \qquad 0 < Q_n \quad (n \in 2\mathbb{N}).$$

To treat the second bracketed term for Q_n in (2.15), we note that

$$(2.20) \qquad \mathrm{e}^{-n} - s_n(-n) = \sum_{k=n+1}^{\infty} \frac{(-n)^k}{k!} \quad (n \in 2\mathbb{N}),$$

where the above sum is an alternating series with strictly decreasing (in absolute value) terms. But as $n \in 2\mathbb{N}$, the first term in the sum in (2.20) is negative, so that $\mathrm{e}^{-n} - s_n(-n) < 0$. Hence, the second bracketed term for Q_n in (2.15) is negative, and, with (2.19) and (2.20), this gives

$$0 < Q_n < \frac{1 - \mathrm{e}^{-n} s_n(n)}{1 + \mathrm{e}^{-2n}} < \frac{1}{2} \quad (n \in 2\mathbb{N}),$$

the desired first result of (2.9) of Lemma 2.1. Again, as $\mathrm{e}^{-n} - s_n(-n)$ is an alternating series with strictly decreasing terms, then taking its first term in (2.20) gives, with (2.12),

$$\left| \frac{\mathrm{e}^{-n} - s_n(-n)}{\mathrm{e}^{n} + \mathrm{e}^{-n}} \right| < \frac{n^{n+1}/(n+1)!}{\mathrm{e}^{n} + \mathrm{e}^{-n}}$$

$$= \frac{n^{n+1}}{(n+1)n!\mathrm{e}^{n}(1 + \mathrm{e}^{-2n})} = \frac{n}{(n+1)\tau_n\sqrt{2\pi n}(1 + \mathrm{e}^{-2n})}.$$

Hence with (2.13), the above display gives that

$$(2.21) \qquad \lim_{n\to\infty} \left(\frac{\mathrm{e}^{-n} - s_n(-n)}{\mathrm{e}^{n} + \mathrm{e}^{-n}} \right) = 0.$$

Thus, on applying (2.21) and the last part of (2.18) to the sum for Q_n in (2.15), we obtain the final result of (2.9) of Lemma 2.1. □

Though not needed above, we remark that the following more precise form of (2.21), which can be derived from results of Copson [4] and Buckholtz [1], is

$$\left|\frac{e^{-n} - s_n(-n)}{e^n + e^{-n}}\right| = \frac{1}{\sqrt{2\pi n}}\left\{\frac{1}{2} - \frac{1}{6n} + O\left(\frac{1}{n^2}\right)\right\}, \quad \text{as } n \to \infty.$$

Continuing, fix any $n \in 2\mathbb{N}$ and consider the meromorphic function

$$R_n(z) := \frac{\cos(nz) - \cos_n(nz)}{\cos(nz)}, \tag{2.22}$$

which has simple poles in the zeros of $\cos(nz)$, i.e., in the points

$$J_n := \left\{\left(j - \frac{1}{2}\right)\frac{\pi}{n} : j \text{ is an integer}\right\}. \tag{2.23}$$

For any z with $Im\ z \geq 0$ and with $z \notin J_n$, a short calculation using (1.1) gives

$$R_n(z) = \frac{2e^{-n}\sum\limits_{k=\frac{n}{2}+1}^{\infty} n^{2k}[-ize^{1+iz}]^n(-iz)^{2k-n}/(2k)!}{(1 + e^{2inz})},$$

and, on taking absolute values in the above sum and using the definition of K^+ in (2.5), we have

$$|R_n(z)| \leq \frac{2e^{-n}\sum\limits_{k=\frac{n}{2}+1}^{\infty} n^{2k}/(2k)!}{|1 + e^{2inz}|} \quad (z \in K^+ \backslash J_n),$$

which we can also write, using the definitions of (2.6) and (2.8), as

$$|R_n(z)| \leq \frac{2e^{-n}[\cosh(n) - \cosh_n(n)]}{|1 + e^{2inz}|} = \frac{2e^{-n} \cdot \cosh(n) \cdot Q_n}{|1 + e^{2inz}|}.$$

As $Q_n < \frac{1}{2}$ for all $n \in 2\mathbb{N}$ from (2.9) of Lemma 2.1, the above display reduces to

$$|R_n(z)| < \frac{e^{-n}\cosh(n)}{|1 + e^{2inz}|} = \frac{1 + e^{-2n}}{2|1 + e^{2inz}|} \quad (z \in K^+ \backslash J_n).$$

But on writing $z = x + iy$ where $y \geq 0$, the above inequality can also be expressed as

$$|R_n(z)| < \frac{e^{ny}(1 + e^{-2n})}{4|\cos(nz)|} \quad (z \in K^+ \backslash J_n). \tag{2.24}$$

We are interested in those points of K^+ where the upper bound in (2.24) is at most unity. To this end, let Θ_n^+ be the subset in the upper half-plane, *without restriction to* K^+, for which the upper bound in (2.24) *exceeds* unity, i.e.,

$$(2.25)\ \Theta_n^+ := \left\{ z = x + iy \in \mathbb{C} \text{ with } y \geq 0 : \frac{e^{ny}(1 + e^{-2n})}{4|\cos(nz)|} > 1 \right\}.$$

It is evident from (2.25) that any sufficiently small neighborhood, of a real point in J_n, is in Θ_n^+, so that Θ_n^+ is a nonempty open subset of the upper half-plane. In particular, for any integer j, consider the j-th element of J_n, i.e., from (2.23),

$$(2.26) \qquad x_j := \left(j - \frac{1}{2} \right) \frac{\pi}{n}, \text{ so that } \cos(nx_j) = 0,$$

and consider any

$$(2.27) \qquad z = x_j + x + iy \text{ with } y \geq 0.$$

(This sets the stage for finding that portion of Θ_n^+, called $\Theta_{n,j}^+$, which is centered about x_j.) With (2.27), a short calculation shows that

$$|\cos(nz)|^2 = \frac{1}{2} \left\{ \cosh(2ny) - \cos(2nx) \right\},$$

so that the final inequality in (2.25) is equivalent to

$$(2.28) \qquad \left[\frac{1}{2} - \frac{1}{8}(1 + e^{-2n})^2 \right] e^{2ny} + \frac{1}{2} e^{-2ny} < \cos(2nx),$$

i.e.,

$$\Theta_{n,j}^+ := \left\{ z = x_j + x + iy \text{ with } y \geq 0 : \left[\frac{1}{2} - \frac{1}{8}(1 + e^{-2n})^2 \right] e^{2ny} \right.$$

$$(2.29) \qquad \left. + \frac{1}{2} e^{-2ny} < \cos(2nx) \right\}.$$

With $2ny =: v$, the left side of the inequality in (2.28) serves to define the function

$$(2.30)\ g_n(v) := \left[\frac{1}{2} - \frac{1}{8}(1 + e^{-2n})^2 \right] e^v + \frac{1}{2} e^{-v}, \quad (v \geq 0; n \in 2\mathbb{N}).$$

This function has a unique minimum, on the interval $[0, +\infty)$, at the point

$$\tilde{v}_n := \frac{1}{2} \log \left\{ \frac{4}{4 - (1 + e^{-2n})^2} \right\}, \text{ with } \tilde{v}_n \downarrow \tilde{v}_\infty = \frac{\log(\frac{4}{3})}{2} \doteq 0.14384,$$

as $n \to \infty$, with g_n strictly decreasing on $[0, \tilde{v}_n]$, and strictly increasing to $+\infty$ on $[\tilde{v}_n, +\infty)$. The minimum of $g_n(v)$ on $[0, +\infty)$ is given by

$$g_n(\tilde{v}_n) = \frac{1}{2}\left[4 - \left(1 + e^{-2n}\right)^2\right]^{\frac{1}{2}} < 1.$$

But as g_n is strictly increasing to $+\infty$ on $[\tilde{v}_n, +\infty)$, there is a unique $\hat{v}_n > \tilde{v}_n$ for which $g_n(\hat{v}_n) = 1$, where

$$\hat{v}_n \downarrow \hat{v}_\infty = \log 2 \doteq 0.69314, \text{ as } n \to \infty, \quad (n \in 2\mathbb{N}).$$

We remark that the factor $(1 + e^{-2n})^2$ in (2.30) is a *nuisance factor*, and can effectively be replaced by unity, since $\tilde{v}_n$ and $\tilde{v}_\infty$ agree to five significant decimal digits for *all* $n \geq 6$, the same being true for $\hat{v}_n$ and $\hat{v}_\infty$. (The choice of 6 in $n \geq 6$ comes from the fact (see [7]) that $\cos_n(nz)$ has some nonreal zeros if and only if $n \geq 6$.)

Because we are interested in the boundary points of $\Theta^+_{n,j}$, denoted by $\partial\Theta^+_{n,j}$, we consider the case when the final inequality in (2.29) is replaced by equality, i.e., with $u := 2nx$ and $v := 2ny$,

$$g_n(v) = \cos(u). \tag{2.31}$$

Now, (2.31), with $n = \infty$, produces the boundary of a *generic upper half-oval* in $Im\ z \geq 0$, where $z = u + iv$, which is shown as the shaded portion in Fig. 2.2. We note from Fig. 2.2 that this upper half-oval is slightly *pinched inward* near the real axis; this is a consequence of g_∞ being strictly decreasing on the interval $[0, \tilde{v}_\infty)$. It is also evident from (2.31) that this generic upper half-oval is symmetric in u, and that its maximum half-width in u is $\pi/6$, and its maximum half-width in v is $\log 2$.

From the above definitions, the nonreal boundary of the upper half-oval $\Theta^+_{n,j}$ is then given by

$$\partial\Theta^+_{n,j} = \{z = x_j + x + iy \text{ with } y > 0 : g_n(2ny) = \cos(2nx)\}, \tag{2.32}$$

and the above construction shows (see (2.25)) that

$$\Theta^+_n = \bigcup_{j \text{ an integer}} \Theta^+_{n,j}, \tag{2.33}$$

i.e., Θ^+_n is an infinite set of identical equally-spaced half-ovals in $Im\ z \geq 0$, where the upper half-oval $\Theta^+_{n,j}$ is centered about the real point x_j of (2.26). Because of the normalizations in (2.31) to the variables $u = 2nx$ and $v = 2ny$, it follows that the maximum half-width in x of $\Theta^+_{n,j}$ is essentially $\pi/(12n)$, and its maximum half-width in y is essentially $\log 2/(2n)$ (where "essentially" means up to the nuisance factor $(1 + e^{-2n})^2$). Note that since

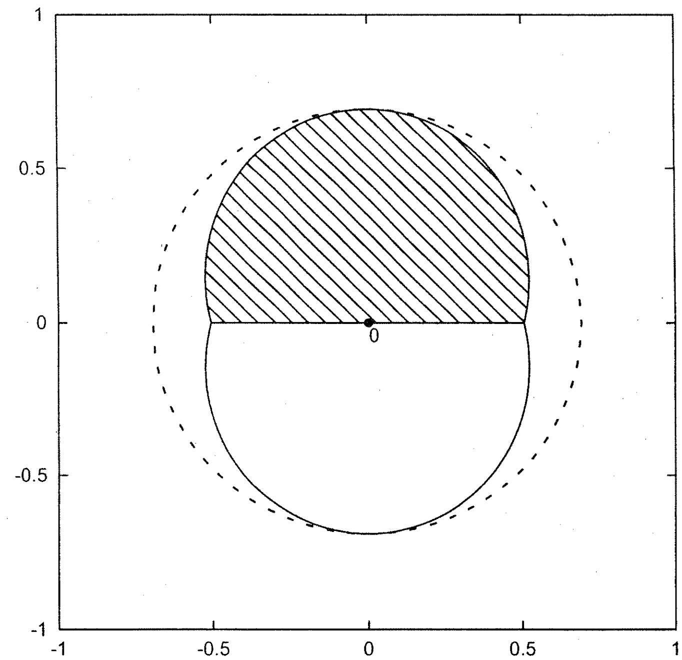

Fig. 2.2. Generic upper-half and lower-half ovals

the distance between the centers of adjacent half-ovals is π/n from (2.23), and since the maximum half-width of these ovals is essentially $\pi/(12n)$, these ovals are necessarily *non-intersecting*.

Next, it follows from the definition of Θ_n^+ in (2.25) and the inequality in (2.24) that

$$|R_n(z)| < 1 \quad (z \in K^+ \backslash \Theta_n^+),$$

i.e., from (2.22),

$$\left|1 - \frac{\cos_n(nz)}{\cos(nz)}\right| < 1 \quad (z \in K^+ \backslash \Theta_n^+). \tag{2.34}$$

This immediately tells us that $K^+ \backslash \Theta_n^+$ contains *no* zeros of $\cos_n(nz)$, for any $n \in 2\mathbb{N}$. But a similar argument, applied now to $Im\ z \leq 0$, gives that

$$\left|1 - \frac{\cos_n(nz)}{\cos(nz)}\right| < 1 \quad (z \in K^- \backslash \Theta_n^-), \tag{2.35}$$

where K^- and Θ_n^- are defined to be the reflections, in the real axis, of K^+ and Θ_n^+. On combining the results of (2.34) and (2.35) and on defining

$$\Theta_{n,j} := \Theta_{n,j}^+ \cup \Theta_{n,j}^-, \tag{2.36}$$

so that

$$\Theta_n = \bigcup_{j \text{ an integer}} \Theta_{n,j} \quad (n \in 2\mathbb{N}), \tag{2.37}$$

we have

$$\left|1 - \frac{\cos_n(nz)}{\cos(nz)}\right| < 1 \quad (z \in K \backslash \Theta_n). \tag{2.38}$$

We remark that Θ_n is now an infinite set of open identical equally-spaced non-intersecting (full) ovals $\Theta_{n,j}$, where $\Theta_{n,j}$ is centered about the real point x_j of (2.26). As an immediate consequence of (2.38), we have the result of

Lemma 2.2 *With the definitions of K and Θ_n in* (2.4) *and* (2.37), $\cos_n(nz)$ *has no zeros in $K \backslash \Theta_n$, for any $n \in 2\mathbb{N}$.*

To extend the result of Lemma 2.2, it is evident from (2.25) that $\cos(nz)$ cannot vanish on the boundary of any $\Theta_{n,j}$. Suppose, for some integers n and j, that $\Theta_{n,j} \subset K$, so that $\partial\Theta_{n,j} \subseteq K$. From (2.38), we then obtain

$$|\cos(nz) - \cos_n(nz)| < |\cos(nz)| \quad (z \in \partial\Theta_{n,j}). \tag{2.39}$$

Now, an application of Rouché's Theorem gives us that $\cos_n(nz)$ has as many zeros in $\Theta_{n,j}$ as does $\cos(nz)$. But as $\cos(nz)$ has exactly one simple real zero in $\Theta_{n,j}$ (in the point $x_j = (j - 1/2)\frac{\pi}{n}$), then $\cos_n(nz)$ also has one simple zero in $\Theta_{n,j}$. As $\cos_n(nz)$ is a real polynomial, this zero of $\cos_n(nz)$ is also necessarily real. This gives us the result of

Lemma 2.3 *For any $n \in 2\mathbb{N}$ and for any integer j such that $\Theta_{n,j} \subset K$, then $\Theta_{n,j}$ contains exactly one simple (real) zero of* $\cos(nz)$ *and exactly one simple (real) zero of* $\cos_n(nz)$.

For any $n \in 2\mathbb{N}$, assume that $K \cap \Theta_n$ consists *only* of ovals which are fully contained in K, as is the case in Fig. 2.3 for $n = 14$. Then the results of Lemmas 2.2 and 2.3 give that $\cos_n(nz)$ has no nonreal zeros in K, and that $\cos_n(nz)$ and $\cos(nz)$ have the same number of simple real zeros in K. What is not covered is the case when *some* oval $\Theta_{n,j}$ is *not* fully contained in K. This will be explored more carefully in Sect. 4.

It is interesting to examine Fig. 2.3 again. First, we claim that the four real zeros of $\cos_{14}(14z)$ in K and the four real zeros $\left\{(j - 1/2)\frac{\pi}{14}\right\}_{j=-1}^{2}$ of $\cos(14z)$, (which are the centers of the ovals $\{\Theta_{14,j}\}_{j=-1}^{2}$), *coincide* to plotting accuracy! As we shall see, these four real zeros of $\cos_{14}(14z)$ in K will be designated as *Hurwitz zeros*. Note also that $\cos_{14}(14z)$ has two real zeros outside of K which can be seen, from their spacings, to be much less accurate approximations to the next two real zeros of $\cos(14z)$, which lie outside of K. These two real zeros, along with all its nonreal zeros, will be similarly designated as *spurious zeros* of $\cos_{14}(14z)$.

With the preceding notations and results for $\cos_n(nz)$ of this section, it is easy to obtain the corresponding results for the normalized partial sums

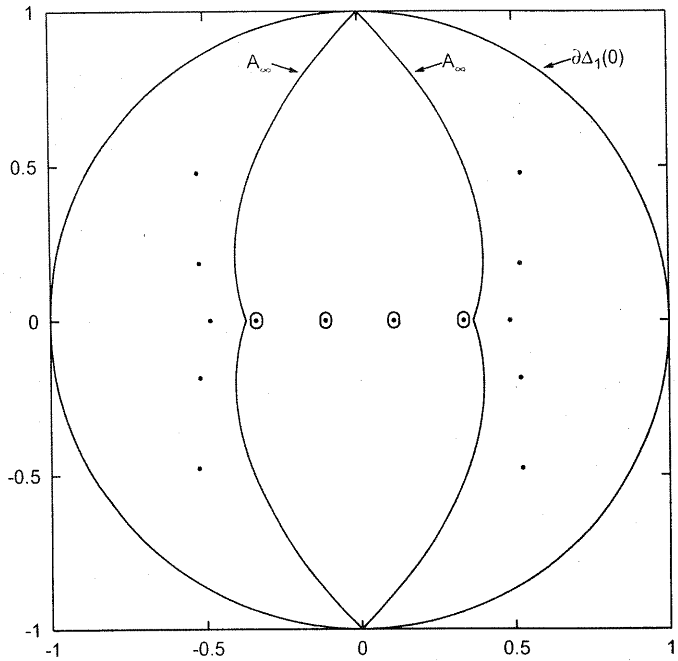

Fig. 2.3. Zeros of $\cos_{14}(14z)$ and the ovals of Θ_{14} inside K

$\sin_m(mz)$ of (1.2), where $m \in 2\mathbb{N}-1$. In this case, we now analogously define

$$\sinh_m(m) := \sum_{j=0}^{\frac{m-1}{2}} \frac{m^{2j+1}}{(2j+1)!} \quad (m \in 2\mathbb{N}-1), \tag{2.40}$$

where

$$\sinh(m) = \sum_{j=0}^{\infty} \frac{m^{2j+1}}{(2j+1)!} = \frac{e^m - e^{-m}}{2}. \tag{2.41}$$

Then, if

$$Q_m := 1 - \frac{\sinh_m(m)}{\sinh(m)} \quad (m \in 2\mathbb{N}-1), \tag{2.42}$$

the proof of Lemma 2.2 can be easily modified to give the result of

Lemma 2.4 *For any* $m \in 2\mathbb{N}-1$, Q_m *of* (2.42) *satisfies*

$$0 < Q_m < \frac{1}{2}, \text{ and } \lim_{m\to\infty} Q_m = \frac{1}{2}. \tag{2.43}$$

Continuing, fix $m \in 2\mathbb{N}-1$, and consider now the meromorphic function

$$R_m(z) := \frac{\sin(mz) - \sin_m(mz)}{\sin(mz)}, \tag{2.44}$$

which has simple poles in the zeros of $\sin(mz)$, i.e., in the points

$$J_m := \left\{ \frac{j\pi}{m} : j \text{ is an integer} \right\}. \tag{2.45}$$

It can then be verified, in analogy with (2.24), that

$$|R_m(z)| < \frac{e^{my}(1 - e^{-2m})}{4|\sin(mz)|} \quad (z \in K^+ \backslash J_m). \tag{2.46}$$

The corresponding ovals in this case, called $\Theta_{m,j}$, can be verified to be ovals centered about the points $x_j := \frac{j\pi}{m}$ of J_m, which are now defined, in the upper half-plane by

$$\begin{aligned} \Theta^+_{m,j} := \Bigg\{ z = x_j + x + iy \text{ with } y \geq 0 : \left[\frac{1}{2} - \frac{1}{8}\left(1 - e^{-2m}\right)^2 \right] e^{2my} \\ + \frac{1}{2} e^{2my} < \sin(2mx) \Bigg\}, \end{aligned} \tag{2.47}$$

where again, with reflections in the real axis, one obtains the full oval $\Theta_{m,j}$. With

$$\Theta_m := \bigcup_{j \text{ an integer}} \Theta_{m,j} \quad (m \in 2\mathbb{N} - 1), \tag{2.48}$$

we again have an infinite set of open identical equally-spaced non-intersecting ovals, where now each $\Theta_{m,j}$ is centered about x_j of J_m. Moreover, we note that the definition of $\Theta^+_{m,j}$ in (2.47) essentially differs from that of $\Theta^+_{n,j}$ in (2.29) only by a new "nuisance factor," i.e., $(1 - e^{-2m})^2$ in (2.47) rather than $(1 + e^{-2n})^2$ in (2.29). The analogs of Lemmas 2.2 and 2.3 then directly follow.

Lemma 2.5 *With the definitions of K and Θ_m, $\sin_m(mz)$ has no zeros in $K \backslash \Theta_m$, for any $m \in 2\mathbb{N} - 1$.*

Lemma 2.6 *For any $m \in 2\mathbb{N} - 1$ and for any integer j such that $\Theta_{m,j} \subseteq K$, then $\Theta_{m,j}$ contains exactly one simple (real) zero of $\sin(mz)$ and one simple (real) zero of $\sin m(mz)$.*

3 Some elementary number-theoretic connections

To begin, the simple (real) zeros of $\cos(nz)$ are given by the set J_n of (2.23), and if k_n denotes the exact number of zeros of $\cos(nz)$ in the interval $\left[0, \frac{1}{e}\right]$

for each $n \in 2\mathbb{N}$, then, with $\lfloor x \rfloor$ denoting the integer part of any real number x, it is easily verified that

$$k_n := \left\lfloor \frac{n}{e\pi} + \frac{1}{2} \right\rfloor. \tag{3.1}$$

As $\cos(nz)$ is an even function of z, the total number of real zeros of $\cos(nz)$ in the interval $[-\frac{1}{e}, +\frac{1}{e}]$ is thus $2k_n$, and these zeros are given by $\{x_j := (j - 1/2)\frac{\pi}{n}\}_{j=-k_n+1}^{k_n}$.

We can of course associate an oval $\Theta_{n,j}$ (defined in Sect. 2) with each of the zeros $\{x_j\}_{j=-k_n+1}^{k_n}$, and we now ask if these ovals, $\{\Theta_{n,j}\}_{j=-k_n+1}^{k_n}$, are all fully contained in K. Geometrically, this turns out to be equivalent to asking if Θ_{n,k_n} is fully contained in K.

Note that the boundary of K *bows outward* appreciably at the points $z = \pm\frac{1}{e}$, as can be seen in Fig. 2.1, while the right real boundary of $\Theta_{n,j}$ is more nearly vertical, as can be seen in Fig. 2.2. From this, it can be rigorously established that the oval Θ_{n,k_n} is fully contained in K if and only if

$$\text{the largest real point of } \partial\Theta_{n,k_n} \text{ is at most } \frac{1}{e}, \tag{3.2}$$

where $\partial\Theta_{n,k_n}$ denotes the boundary of Θ_{n,k_n}. To make this more tractable, the real points of $\partial\Theta_{n,j}$ satisfy, from (2.25),

$$\frac{1 + e^{-2n}}{4} = |\cos(nx)|. \tag{3.3}$$

Writing $x := (j - 1/2)\frac{\pi}{n} + \frac{\xi_n}{n}$, then $|\cos(nx)| = |\sin(\xi_n)|$, and on defining

$$\xi_n^+ := \sin^{-1}\left(\frac{1 + e^{-2n}}{4}\right) > 0 \quad (n \in 2\mathbb{N}), \tag{3.4}$$

the real points of $\partial\Theta_{n,j}$ are then $\frac{(2j-1)\pi}{2n} \pm \frac{\xi_n^+}{n}$. Hence, the condition of (3.2) becomes equivalently

$$\left(k_n - \frac{1}{2}\right) + \frac{\xi_n^+}{\pi} \le \frac{n}{e\pi}. \tag{3.5}$$

On the other hand, from the definition of k_n in (3.1), we can write

$$\frac{n}{e\pi} + \frac{1}{2} = k_n + t_n, \text{ where } 0 < t_n < 1, \tag{3.6}$$

so that t_n is the *fractional part* of $\frac{n}{e\pi} + \frac{1}{2}$. The condition of (3.2) then reduces to

$$\frac{\xi_n^+}{\pi} \le t_n \quad (n \in 2\mathbb{N}). \tag{3.7}$$

The condition of (3.7) is thus a *necessary and sufficient condition* that all $2k_n$ ovals $\{\Theta_{n,j}\}_{j=-k_n+1}^{k_n}$ are fully contained in K. It can also be seen from (3.4) that the numbers $\{\xi_n^+\}_{n\in 2\mathbb{N}}$ are positive numbers which are strictly decreasing with n, where, to five decimal digits,

$$\frac{\xi_n^+}{\pi} \doteq 0.08043 \quad (\text{for } \textit{all } n \in 2\mathbb{N} \text{ with } n \geq 6). \tag{3.8}$$

On checking with several number theorists, it appears *not* to be known if $1/e\pi$ is *irrational*. If it were irrational, then the fractional parts $\{t_n\}_{n\in 2\mathbb{N}}$ of (3.6) would be *uniformly distributed* in $[0, 1]$.

As a consequence of Lemma 2.3 and the above considerations, we have

Lemma 3.1 *For any $n \in 2\mathbb{N}$, determine k_n and t_n from* (3.1) *and* (3.6), *and ξ_n^+ from* (3.4). *If* (3.7) *is valid, then there are exactly $2k_n$ ovals $\Theta_{n,j}$ which are fully contained in K. Thus, the number of (simple) real zeros of* $\cos(nz)$ *and the number of (simple) real zeros of* $\cos_n(nz)$ *in the interval* $[-\frac{1}{e}, +\frac{1}{e}]$ *are both exactly $2k_n$.*

Given any $n \in 2\mathbb{N}$, it is very easy to determine, from (3.4) and (3.6), if the associated constants ξ_n^+ and t_n satisfy the inequality of (3.7). For the first 50 even positive integers, $n = 2, 4, \cdots, 100$, this inequality *fails* only for the four numbers $n = 22,\ 30,\ 56$, and 90. For example, $t_{30} \doteq 0.01299$, while $\xi_{30}^+/\pi \doteq 0.08043$. However, on determining (with high precision) the zeros of $\cos_n(nz)$ for these four cases, it turns out, in *each* of these four cases, that $\cos(nz)$ and $\cos_n(nz)$ have the *same* number of real zeros (i.e., $2k_n$) in the interval $[-\frac{1}{e}, +\frac{1}{e}]$. In other words, the conditions of Lemma 3.1 are sufficient, but not necessary, for $\cos(nz)$ and $\cos_n(nz)$ to have the same number of zeros, namely, $2k_n$, in the interval $[-\frac{1}{e}, +\frac{1}{e}]$. This leads us to consider in Sect. 4, for those cases where t_n does *not* satisfy (3.7), other sufficient conditions which guarantee that $\cos(nz)$ and $\cos_n(nz)$ have the same number, namely $2k_n$, of simple real zeros in a slightly larger interval $(-\rho_n, +\rho_n)$, where ρ_n is defined in (4.3).

To obtain analogous results for the normalized partial sums $\sin_m(mz)$ of (1.2), where $m \in 2\mathbb{N} - 1$, we recall that the simple (real) zeros of $\sin(mz)$ are given by the set J_m of (2.45), and if k_m now denotes the exact number of zeros of $\sin(mz)$ in the interval $(0, 1/e]$ for each $m \in 2\mathbb{N} - 1$, we then have that

$$k_m = \left\lfloor \frac{m}{e\pi} \right\rfloor. \tag{3.9}$$

As $\sin(mz)$ is an odd function with a simple zero at $z = 0$, the total number of real zeros of $\sin(mz)$ in $[-1/e, +1/e]$ is exactly $2k_m + 1$, and these zeros are given by $\{x_j := \frac{j\pi}{m}\}_{j=-k_m}^{+k_m}$. Then, with the definition of the ovals $\Theta_{m,j}$

of (2.47), we similarly have that the oval Θ_{m,k_m} is fully contained in K if and only if

$$\text{(3.10)} \qquad \text{the largest real point of } \partial\Theta_{m,k_m} \text{ is at most } \frac{1}{e}.$$

On calling $\frac{k_m \pi}{m} + \frac{\xi_m^+}{m}$ the largest real point of Θ_{m,k_m}, we see from (2.46) that

$$\frac{1-\mathrm{e}^{-2m}}{4} = \left|\sin\left(m\left[\frac{k_m\pi}{m} + \frac{\xi_m^+}{m}\right]\right)\right| = \sin \xi_m^+,$$

so that (see (3.4))

$$\text{(3.11)} \qquad \xi_m^+ = \sin^{-1}\left(\frac{1-\mathrm{e}^{-2m}}{4}\right) > 0 \quad (m \in 2\mathbb{N}-1),$$

where again (see (3.8)) we have that

$$\text{(3.12)} \qquad \frac{\xi_m^+}{\pi} \doteq 0.08043 \quad (\text{for } \textit{all } m \in 2\mathbb{N}-1 \text{ with } m \geq 5).$$

If t_m is the fractional part of $\frac{m}{e\pi}$, then with (3.9), we have

$$\text{(3.13)} \qquad \frac{m}{e\pi} = k_m + t_m, \text{where } 0 < t_m < 1 \quad (m \in 2\mathbb{N}-1),$$

and it can be verified that the oval Θ_{m,k_m} is fully contained in K if and only if

$$\text{(3.14)} \qquad \frac{\xi_m^+}{\pi} \leq t_m \quad (m \in 2\mathbb{N}-1).$$

which of course is basically the same as (3.7). This establishes

Lemma 3.2 *For any $m \in 2\mathbb{N}-1$, determine k_m and t_m from (3.9) and (3.13), and ξ_m^+ from (3.11). If (3.14) is valid, then there are exactly $2k_m+1$ ovals $\Theta_{m,j}$ which are fully contained in K. Thus, the number of (simple) real zeros of $\sin(mz)$ and the number of (simple) real zeros of $\sin_m(mz)$ in the interval $[-\frac{1}{e}, +\frac{1}{e}]$ are both exactly $2k_m+1$.*

As in the case of $n \in 2\mathbb{N}$, it similarly turns out that for the first 50 odd positive integers $m = 1, 3, \cdots, 99$, the inequality of (3.14) *fails* only for the four numbers $m = 9, 43, 69$, and 77. (We note that these four cases, out of fifty, give a ratio $\frac{4}{50} = 0.08000$, which approximates the numbers in (3.8) and (3.12). This would be expected if $1/e\pi$ were irrational!) But on determining (with high precision) the zeros of $\sin_m(mz)$ in these four cases, it again turns out in *each* of these four cases, that $\sin(mz)$ and $\sin_m(mz)$ have the *same* number of real zeros (i.e., $2k_m+1$) in the interval $[-\frac{1}{e}, +\frac{1}{e}]$.

4 Extensions of the results of Sect. 3

As mentioned in Sect. 3, there are cases ($n = 22, 30, 56$, and 90 for $n \in 2\mathbb{N}$ with $n \leq 100$, and $m = 9, 43, 69$, and 77 for $m \in 2\mathbb{N} - 1$ with $m \leq 99$) for which (3.7) and (3.14) are not valid, i.e., there are cases where the ovals Θ_{n,k_n} and Θ_{m,k_m}, from Sect. 2, are not fully contained in the set K of (2.4). Thus, not all the hypotheses of Lemmas 3.1 and 3.2 are valid in these cases. The point of this section is to show, by a modified argument, that, when (3.7) is not valid, the conclusions of Lemma 3.1 are valid for a slightly larger interval $(-\rho_n, +\rho_n)$, (where ρ_n will be defined in (4.3)), for all $n \in 2\mathbb{N}$, with a similar extension holding for Lemma 3.2.

Consider any $n \in 2\mathbb{N}$ for which (3.7) is *not* valid, i.e. (see (3.7) and (3.8)),

$$0 < t_n < \frac{\xi_n^+}{\pi} \doteq 0.08043, \tag{4.1}$$

so that the oval Θ_{n,k_n} is not fully contained in K. The center point of Θ_{n,k_n} is given by $x_{k_n} := (k_n - 1/2)\pi/n$, where from (3.6), we have that

$$x_{k_n} = \frac{1}{e} - \frac{\pi t_n}{n}. \tag{4.2}$$

Next, since the half-width in x of Θ_{n,k_n} is essentially $\pi/(12n)$ and its half-width in y is essentially $\log 2/2n$, it can be shown that the disk, centered at x_{k_n}, and having radius $\log 2/(2n)$, essentially contains Θ_{n,k_n}, i.e.,

$$\Theta_{n,k_n} \subseteq \{z \in \mathbb{C} : |z - x_{k_n}| \leq \log 2/(2n)\},$$

up to nuisance factors $(1+\mathrm{e}^{-2n})$. (The associated dotted circle for the above disk is shown also in Fig. 2.2.) Thus from (4.2),

$$x_{k_n} + \frac{\log 2}{2n} = \frac{1}{e} - \frac{\pi t_n}{n} + \frac{\log 2}{2n} = \frac{1}{e}\left\{1 + \frac{1}{n}\left[-e\pi t_n + \frac{e\log 2}{2}\right]\right\}.$$

But as $t_n > 0$ from (4.1) and as $(e\log 2)/2 \doteq 0.94208$, it follows that

$$x_{k_n} + \frac{\log 2}{2n} < \frac{1}{e}\left\{1 + \frac{1}{n}\left[\frac{e\log 2}{2}\right]\right\} < \frac{1}{e}\left\{1 + \frac{1}{n}\right\} =: \rho_n. \tag{4.3}$$

From the definition of ρ_n in (4.3), we see that $\{\rho_n\}_{n\in 2\mathbb{N}}$ is a sequence of positive numbers, less than unity, which strictly decrease to $1/e$, i.e.,

$$0.55182 \doteq \rho_2 > \rho_4 > \cdots, \text{ with } \lim_{2n\to\infty} \rho_{2n} = 1/e. \tag{4.4}$$

Moreover, using the upper bound of (4.1) for t_n, then with (4.2) and (4.3), we obtain

$$\rho_n - x_{k_n} = \frac{1}{n}\left[\frac{1}{e} + \pi t_n\right] < \frac{0.62056}{n} < \frac{\pi}{n}. \tag{4.5}$$

As the spacing between successive zeros of $\cos(nz)$ is π/n, the final inequality in (4.5) guarantees that the zero, x_{k_n}, is necessarily the *largest* zero of $\cos(nz)$ in the disk $\{z \in \mathbb{C} : |z| \le \rho_n\}$, so that $\cos(nz)$ has exactly $2k_n$ zeros in the interval $(-\rho_n, +\rho_n)$. This will be used below.

Continuing, from the definition of $R_n(z)$ in (2.22) we have

$$R_n(z) = \left(\sum_{j=\frac{n+2}{2}}^{\infty} (-1)^j \frac{(nz)^{2j}}{(2j)!} \right) / [\cos(nz)] \quad (z \notin J_n), \tag{4.6}$$

so that for any z with $|z| \le \rho_n$ and $z \notin J_n$,

$$|R_n(z)| \le \left(\sum_{j=\frac{n+2}{2}}^{\infty} \frac{(n\rho_n)^{2j}}{(2j)!} \right) / |\cos(nz)|$$

$$= \frac{(n\rho_n)^{n+2}}{(n+2)!|\cos(nz)|} \left\{ 1 + \frac{n^2\rho_n^2}{(n+3)(n+4)} + \frac{n^4\rho_n^4}{\prod_{j=3}^{6}(n+j)} + \cdots \right\}$$

$$< \frac{(n\rho_n)^{n+2}}{(n+2)!|\cos(nz)|} \{1 + \rho_n^2 + \rho_n^4 + \cdots\}$$

$$= \frac{(n\rho_n)^{n+2}}{(n+2)!|\cos(nz)|(1-\rho_n^2)},$$

where, from (4.4), the positive numbers ρ_n^2, for $n \in 2\mathbb{N}$, are all less then unity. Then, with Stirling's formula (2.12) (where $\tau_n > 1$ for all n from (2.13)), a short calculation shows that

$$|R_n(z)| < \frac{\gamma_n}{|\cos(nz)|} \quad (|z| \le \rho_n,\ z \notin J_n), \tag{4.7}$$

where

$$\gamma_n := \frac{(e\rho_n)^{n+2}}{\sqrt{2\pi n}(e^2 - e^2\rho_n^2)} \quad (n \in 2\mathbb{N}), \tag{4.8}$$

(so that no nuisance factors now arise here). But, as $e\rho_n = 1 + \frac{1}{n}$ from (4.3), then for all $n \in 2\mathbb{N}$,

$$\gamma_n = \frac{(1+\frac{1}{n})^{n+2}}{\sqrt{2\pi n}\,\{e^2 - (1+\frac{1}{n})^2\}} = \frac{e}{\sqrt{2\pi n}(e^2-1)} \left\{ 1 + O\left(\frac{1}{n}\right) \right\}, \tag{4.9}$$

as $n \to \infty$. It can be verified (see again [8, I. 168, p. 30]) that $\{\gamma_n\}_{n\in 2\mathbb{N}}$ is a sequence of positive numbers which strictly decrease to zero, i.e.,

$$0.27789 \doteq \gamma_2 > \gamma_4 > \gamma_6 \cdots, \text{ with } \lim_{2n\to\infty} \gamma_{2n} = 0. \tag{4.10}$$

As in Sect. 2, we can now find analogous new ovals $\{\hat{\Theta}_{n,j}\}_{j=-k_n+1}^{k_n}$, where $\hat{\Theta}_{n,j}$ is centered about x_j of (2.26), which are defined, for $-k_n+1 \le j \le k_n$, in the upper half-plane by

$$\hat{\Theta}_{n,j}^{+} := \left\{ z = x_j + x + iy \text{ with } y \ge 0 : \frac{\gamma_n}{|\cos(nz)|} > 1 \right\}, \tag{4.11}$$

and by a reflection of $\hat{\Theta}_{n,j}^{+}$ in the real axis. It can also be verified, as in Sect. 2, that the half-widths in x of the ovals $\{\hat{\Theta}_{n,j}\}_{j=-k_n+1}^{k_n}$ are now given by

$$\hat{\xi}_n/n, \text{ where } \hat{\xi}_n := \sin^{-1}(\gamma_n) > 0, \tag{4.12}$$

and the half-widths in y of $\{\hat{\Theta}_{n,j}\}_{j=-k_n+1}^{k_n}$ are similarly given by

$$\hat{\zeta}_n/n, \text{ where } \hat{\zeta}_n := \sinh^{-1}(\gamma_n) > 0. \tag{4.13}$$

(We remark that $\{\hat{\xi}_n\}_{n\in 2\mathbb{N}}$ is a sequence of positive numbers which strictly decreases to zero.) These new ovals $\{\hat{\Theta}_{n,j}\}_{j=-k_n+1}^{k_n}$ are considerably smaller, and more nearly circular, than the original ovals $\{\Theta_{n,j}\}_{j=-k_n+1}^{k_n}$. For example, if $n = 22$ the maximum half-width in x and y of the original ovals $\{\Theta_{22,j}\}_{j=-21}^{22}$ are, respectively,

$$0.011\,485\,466 \text{ and } 0.015\,753\,345,$$

while those of the new ovals $\{\hat{\Theta}_{22,j}\}_{j=-21}^{22}$ are, respectively,

$$0.001\,785\,012 \text{ and } 0.001\,784\,095.$$

We note that the right real boundary point of $\partial\hat{\Theta}_{n,k_n}$, i.e., $x_{k_n} + \hat{\xi}_n/n$, satisfies (see (4.2) and (4.12))

$$x_{k_n} + \frac{\hat{\xi}_n}{n} = \frac{1}{e} - \frac{\pi t_n}{n} + \frac{\sin^{-1}(\gamma_n)}{n} < \frac{1}{e} + \frac{\sin^{-1}(\gamma_n)}{n},$$

since $t_n > 0$ from (3.6). We claim that the above upper bound for the right real boundary point of $\partial\hat{\Theta}_{n,k_n}$ is always less than ρ_n of (4.3), i.e.,

$$x_{k_n} + \frac{\hat{\xi}_n}{n} < \frac{1}{e} + \frac{\sin^{-1}(\gamma_n)}{n} < \rho_n = \frac{1}{e} + \frac{1}{en} \quad (\text{all } n \in 2\mathbb{N}), \tag{4.14}$$

since the final inequality above is equivalent to

$$\gamma_n < \sin\left(\frac{1}{e}\right) \doteq 0.35964 \quad (n \in 2\mathbb{N}),$$

which follows from (4.10). Hence, all the real points of $\{\hat{\Theta}_{n,j}\}_{j=-k_n+1}^{k_n}$ lie in $(-\rho_n, +\rho_n)$, for all $n \in 2\mathbb{N}$.

Next, we have from (4.7) and the definition of $\hat{\Theta}_{n,j}$ in (4.11) that, for $-k_n + 1 \leq j \leq k_n$,

$$\left|1 - \frac{\cos_n(nz)}{\cos(nz)}\right| < 1 \quad (z \in \partial\hat{\Theta}_{n,j}),$$

so that as $\cos(nz) \neq 0$ on $\partial\hat{\Theta}_{n,j}$, we have

$$|\cos(nz) - \cos_n(nz)| < |\cos(nz)| \quad \left(z \in \partial\hat{\Theta}_{n,j}\right). \tag{4.15}$$

Applying Rouché's Theorem again, as in the proof of Lemma 3.1, we obtain

Lemma 4.1 *For any $n \in 2\mathbb{N}$ for which the associated constant t_n of* (3.6) *satisfies* (4.1)*, each oval of* $\{\hat{\Theta}_{n,j}\}_{j=-k_n+1}^{k_n}$ *(see (4.11)) contains exactly one simple (real) zero of* $\cos(nz)$ *and one simple (real) zero of* $\cos_n(nz)$*. Thus, the number of (simple) real zeros of* $\cos(nz)$ *and the number of (simple) real zeros of* $\cos_n(nz)$ *in the interval* $(-\rho_n, +\rho_n)$ *are exactly* $2k_n$.

In a completely similar fashion, using an obvious notation, we also obtain

Lemma 4.2 *For any $m \in 2\mathbb{N} - 1$ for which the associated constant t_m of* (3.13) *satisfies* (4.1) *(with m replacing n), each oval of* $\{\hat{\Theta}_{m,j}\}_{j=-k_m}^{k_m}$ *contains exactly one simple (real) zero of* $\sin(mz)$ *and one simple (real) zero of* $\sin_m(mz)$*. Thus, the number of (simple) real zeros of* $\sin(mz)$ *and the number of (simple) real zeros of* $\sin_m(mz)$ *in the interval* $(-\rho_m, +\rho_m)$ *is exactly* $2k_m + 1$.

Since the assumptions of (3.7) and (4.1) cover all possibilities for t_n for any $n \in 2\mathbb{N}$ (with analogous results for any $m \in 2\mathbb{N} - 1$), then putting the above results together with the results of Sect. 3, we have now established

Theorem 4.3 *For any $n \in 2\mathbb{N}$, $\cos_n(nz)$ has no nonreal zeros in the set K of* (2.4)*. Moreover, with the definition of k_n of* (3.1)*, if t_n satisfies (3.7), then* $\cos(nz)$ *has exactly* $2k_n$ *real (and simple) zeros in the interval* $[-1/e, +1/e]$*, and* $\cos_n(nz)$ *also has exactly* $2k_n$ *real and simple zeros in this interval. If, on the other hand, t_n satisfies (4.1), then* $\cos(nz)$ *and* $\cos_n(nz)$ *each have* *exactly* $2k_n$ *real and simple zeros in the interval (see (4.3))* $(-\rho_n, +\rho_n)$. *Similarly, for any* $m \in 2\mathbb{N}-1$, $\sin_m(mz)$ *has no nonreal zeros in* K. *Moreover, with the definition of* k_m *in* (3.9), *if* t_m *satisfies (3.14), then* $\sin(mz)$ *has exactly* $2k_m + 1$ *real (and simple) zeros in the interval* $[-1/e, +1/e]$, *and* $\sin_m(mz)$ *also has exactly* $2k_m + 1$ *real (and simple) zeros in this interval. If, on the other hand,* t_m *does not satisfy (3.14), then* $\sin(mz)$ *and* $\sin_m(mz)$ *each has exactly* $2k_m + 1$ *zeros in the interval (see (4.3) with* m *replacing* n*)* $(-\rho_m, +\rho_m)$.

We remark that earlier Buckholtz [2] had shown, with a very short proof, that no zero of any normalized partial sum $s_n(nz)$ of e^z can lie on or within the Szegö curve D_∞ of (1.4), where e^z has, of course, no zeros. The proof of the analog of his result in Theorem 4.3, for the set K and for the zeros of $\cos_n(nz)$, any $n \in 2\mathbb{N}$, or $\sin_m(mz)$, any $m \in 2\mathbb{N}-1$, has been complicated by the existence of real zeros of $\cos(nz)$ and $\sin(mz)$ which lie in K.

For our purposes, for any $n \in 2\mathbb{N}$, we *define*, as *Hurwitz zeros* of $\cos_n(nz)$, all those real zeros of $\cos_n(nz)$ in the interval $[-1/e, +1/e]$ if t_n satisfies (3.7), and if t_n does not satisfy (3.7), all those real zeros of $\cos_n(nz)$ in the interval (see (4.3)) $[-\rho_n, +\rho_n]$, with all remaining zeros being defined as *spurious zeros*. (Analogous definitions hold for Hurwitz and spurious zeros of $\sin_m(mz)$, for any $m \in 2\mathbb{N}-1$). Thus, spurious zeros of $\cos_n(nz)$ include all nonreal zeros and real zeros not in the associated intervals mentioned above. Consider, for example, $n = 14$, for which t_{14} does satisfy (3.7). As we see in Fig. 2.3, there are two real zeros $\pm z_{14,1}$ of $\cos_{14}(14z)$ which both lie outside the interval $[-1/e, +1/e]$, so that these zeros are, by definition, spurious zeros of $\cos_{14}(14z)$. As another example, consider $n = 30$, for which t_{30} does not satisfy (3.7). In this case, there are two real zeros $\pm z_{30,1}$ of $\cos_{30}(30z)$ which do not lie in $[-\rho_{30}, +\rho_{30}]$, i.e.,

$$|z_{30,1}| = 0.42772 > \rho_{30} = 0.38014,$$

so that $\cos_{30}(30z)$ possesses two real spurious zeros.

The reason, for designating certain real zeros of $\cos_n(nz)$ as *Hurwitz zeros*, can be explained as follows. Given any $n \in 2\mathbb{N}$, determine the associated numbers k_n and t_n from (3.1) and (3.6), and fix the open rectangle defined by

$$T_{n,\epsilon} := \left\{ z = x + iy : |x| < \frac{n}{e}(1+\epsilon) \text{ and } |y| < \epsilon \right\}, \tag{4.16}$$

where ϵ is fixed with $0 < \epsilon < 1/2$. If (3.7) is valid, we know from Lemma 3.1 that the real points of the (original) ovals $\{\Theta_{n,j}\}_{j=-k_n+1}^{k_n}$ all lie in the interval $[-1/e,\ +1/e]$, and hence, there are exactly $2k_n$ real (and distinct) zeros of $\cos(nz)$ and $\cos_n(nz)$ in the interval, $[-1/e, +1/e]$. In particular, the right real boundary point of Θ_{n,k_n}, given by

$$\frac{\left(k_n - \frac{1}{2}\right)\pi}{n} + \frac{\xi_n^+}{n}, \tag{4.17}$$

must satisfy

$$\frac{\left(k_n - \frac{1}{2}\right)\pi}{n} + \frac{\xi_n^+}{n} \leq \frac{1}{e}. \tag{4.18}$$

On passing to the next positive even integer, $n+2$, in $2\mathbb{N}$, the right real boundary point of the oval Θ_{n+2,k_n} then satisfies, from (4.17),

$$\frac{\left(k_n - \frac{1}{2}\right)\pi}{n+2} + \frac{\xi_{n+2}^+}{n+2} < \frac{\left(k_n - \frac{1}{2}\right)\pi}{n+2} + \frac{\xi_n^+}{n+2} \leq \frac{n}{(n+2)e}, \tag{4.19}$$

where the first inequality above follows since the terms of $\{\xi_n^+\}_{n\in 2\mathbb{N}}$ are strictly decreasing, and the second inequality above follows from (4.18). The final inequality in (4.19) gives that the real points of the oval $\Theta_{n+2,\,k_n}$ lie in $(-1/e, +1/e)$, so that $\cos((n+2)z)$ and $\cos_{n+2}((n+2)z)$ continue to have exactly $2k_n$ real (and distinct) zeros in $(-1/e, +1/e)$. The same then is true for *all* larger integers in $2\mathbb{N}$. But, on considering the *unnormalized* functions $\cos(z)$ and $\cos_n(z)$, we first see, on multiplying by n in (4.18), that $\cos(z)$ and $\cos_n(z)$ each have exactly $2k_n$ zeros in the rectangle $T_{n,\epsilon}$ of (4.16), and, moreover, on multiplying by $(n+2)$ in (4.19), we similarly deduce that $\cos(z)$ and $\cos_{n+2}(z)$ each have exactly $2k_n$ zeros in $T_{n,\epsilon}$, as is thus the case for all $s \in 2\mathbb{N}$ with $s \geq n$. (That these $2k_n$ zeros of $\cos_s(z)$ actually *tend* to zeros of $\cos(z)$ in $T_{n,\epsilon}$, as $s \to \infty$, can be achieved by the results of (4.7)-(4.8), with an appropriate sequence of ρ_n's which tends to zero.) Thus, when t_n satisfies (3.7), this justifies our definition of Hurwitz zeros here!

Again, fixing any $n \in 2\mathbb{N}$, the case when t_n does not satisfy (3.7) is similar. In this case, we have from (4.14) that

$$\frac{(k_n - \frac{1}{2})\pi}{n} + \frac{\hat{\xi}_n}{n} < \rho_n = \frac{n+1}{en}, \tag{4.20}$$

so that, from Theorem 4.3, $\cos(nz)$ and $\cos_n(nz)$ each have exactly $2k_n$ real and simple zeros in $(-\rho_n, +\rho_n)$. On passing to the next positive even integer, $n+2$, in $2\mathbb{N}$, it follows from (4.20) that

$$\frac{(k_n - \frac{1}{2})\pi}{n+2} + \frac{\hat{\xi}_{n+2}}{n+2} < \frac{(k_n - \frac{1}{2})\pi}{n+2} + \frac{\hat{\xi}_n}{n+2} < \frac{n+1}{e(n+2)} < \frac{1}{e}, \tag{4.21}$$

where the first inequality of (4.21) again follows since the terms of $\{\hat{\xi}_n\}_{n\in 2\mathbb{N}}$ are strictly decreasing, and the second inequality follows from (4.20). The final inequality in (4.21), on multiplication by $(n+2)$, then gives us that $\cos(z)$ and $\cos_s(z)$, for all s in $2\mathbb{N}$ with $s \geq n+2$, each have exactly $2k_n$ zeros in $T_{n,\epsilon}$ of (4.16), again justifying our definition of Hurwitz zeros here. As can be imagined, analogous results hold for $\sin(z)$ and $\sin_m(z), m \in 2\mathbb{N}-1$.

5 Improvement of Szegö's result of (1.12)

Our next goal is to improve Szegö's result of (1.12). To begin, it follows from [3, eqs. (2.9) and (2.13)] that (2.14) can be expressed also as

$$e^{-nz}s_n(nz) = 1 - \frac{(ze^{1-z})^n}{\tau_n\sqrt{2\pi n}}\left(\frac{z}{1-z}\right)\left\{1 - \frac{1}{(n+1)(1-z)^2} + O\left(\frac{1}{n^2}\right)\right\}, \tag{5.1}$$

uniformly on any compact subset of $\overline{\Delta_1}(0)\backslash\{1\}$, as $n \to \infty$. Then for $n \in 2\mathbb{N}$, the identities

$$\begin{aligned} 2\cos(nz) &= e^{inz} + e^{-inz}, \text{ and} \\ 2\cos_n(nz) &= s_n(inz) + s_n(-inz), \end{aligned} \tag{5.2}$$

for all $z \in \mathbb{C}$, when coupled with (5.1) where z is replaced by $\pm iz$, give, for $z = x$ real, the result of

$$\begin{aligned} \cos_n(nx) &= \cos(nx) \\ &- \frac{(-1)^{\frac{n+2}{2}}(ex)^n}{\tau_n\sqrt{2\pi n}}\left(\frac{x^2}{1+x^2}\right)\left\{1 - \frac{(3-x^2)}{(n+1)(1+x^2)^2} + O\left(\frac{1}{n^2}\right)\right\}, \end{aligned} \tag{5.3}$$

which holds uniformly on the real interval $[-1,+1]$, as $n \to \infty$. (We remark that (5.3) is a more precise form of Szegö [12, eq. (15)].) On setting

$$h_n(x) := \frac{(ex)^n}{\tau_n\sqrt{2\pi n}}\left(\frac{x^2}{1+x^2}\right)\left\{1 - \frac{(3-x^2)}{(n+1)(1+x^2)^2} + O\left(\frac{1}{n^2}\right)\right\}, \tag{5.4}$$

then (5.3) can be expressed as

$$\cos_n(nx) = \cos(nx) - (-1)^{\frac{n+2}{2}}h_n(x) \tag{5.5}$$

for $-1 \leq x \leq 1$ and $n \in 2\mathbb{N}$. We note, from the Maclaurin expansions of $\cos_n(nx)$ and $\cos(nx)$, that $h_n(x)$ can also be expressed as

$$h_n(x) = \frac{(nx)^{n+2}}{(n+2)!} - \frac{(nx)^{n+4}}{(n+4)!} + \frac{(nx)^{n+6}}{(n+6)!} - \cdots \quad (-1 \leq x \leq 1; n \in 2\mathbb{N}), \tag{5.6}$$

which similarly appears, for $z = x$, in the numerator of (4.6). The above series is an alternating series with strictly decreasing (in absolute value) terms for $x \in (0, 1]$. As the same is true for the series obtained by term-by-term differentiation of $h_n(x)$, with respect to x, this gives, from (5.4) and (5.6), in a straight-forward way, the result of

Lemma 5.1 *For each $n \in 2\mathbb{N}$, $h_n(x)$ of (5.4) is an even function which is a strictly increasing positive function on $(0, 1]$, with $h_n(0) = 0$ and $h_n(1) > 1$. Moreover, since $h_n(1/e) < 1$ from (5.4), there is a unique ω_n, with $1/e < \omega_n < 1$, such that*

$$h_n(\omega_n) = 1 \quad (n \in 2\mathbb{N}). \tag{5.7}$$

The reason for introducing the function $h_n(x)$ of (5.4) is this. From (5.5), the positive real zeros of $\cos_n(nx)$ are exactly those abscissas x in $(0, 1]$ where the curves, $\cos(nx)$ and $(-1)^{\frac{n+2}{2}} h_n(x)$, *cross*. For $n = 20$ (so that $(-1)^{\frac{n+2}{2}} = -1$ in this case), we show in Fig. 5.1 the curves $\cos(20x)$ and $-h_{20}(x)$, for $x \in [0, 1/2]$, with small solid disks indicating the crossings of these curves. We see in Fig. 5.1 that there are two such crossings in $[0, 1/e]$ which are close to zeros of $\cos(20x)$, and that there are two further crossings in $[1/e, 1/2]$, giving (from evenness considerations) a total of eight real zeros of $\cos_{20}(20x)$. In general, it is clear from (5.5) that we can write

$$(-1)^{\frac{n+2}{2}} h_n\left(\frac{u}{n}\right) = \cos(u) - \cos_n(u) \quad (u \in [-n, +n]). \tag{5.8}$$

Thus, from power series expansions, the smallest positive zero of $\cos_n(u)$ converges *geometrically* to the smallest positive zero, $\pi/2$, of $\cos(u)$, as $n \to \infty$. In fact, we observe, from the positivity of $h_n(x)$, that the smallest positive zero of $\cos_n(x)$ is to the

$$\text{left (right) of } \pi/2 \text{ if } \frac{n+2}{2} \text{ is even (odd)} \quad (n \in 2\mathbb{N}),$$

with similar statements holding for successively larger positive zeros of $\cos(u)$, for $u \in [0, n/e)$. In addition, the rate of geometric convergence, of real zeros of $\cos_n(u)$ to real zeros of $\cos(u)$, *decreases* with each subsequent larger positive zeros of $\cos(u)$. (We remark that these deductions cannot be deduced from the equal-sized ovals $\Theta_{n,j}$ of Sect. 2.) We also remark that this geometric convergence of real zeros of $\cos_n(nz)$ to real zeros of $\cos(nz)$ in K is why, in Fig. 2.3, these zeros coincided to plotting accuracy!

The situation concerning $\sin_m(mz)$ and $\sin(mz)$, $m \in 2\mathbb{N}-1$, is nearly the same. In this case, the analog of (5.2) is now

$$\begin{aligned} 2i\sin(mz) &= e^{imz} - e^{-imz}, \text{ and} \\ 2i\sin_m(mz) &= s_m(imz) - s_m(-imz), \end{aligned} \tag{5.9}$$

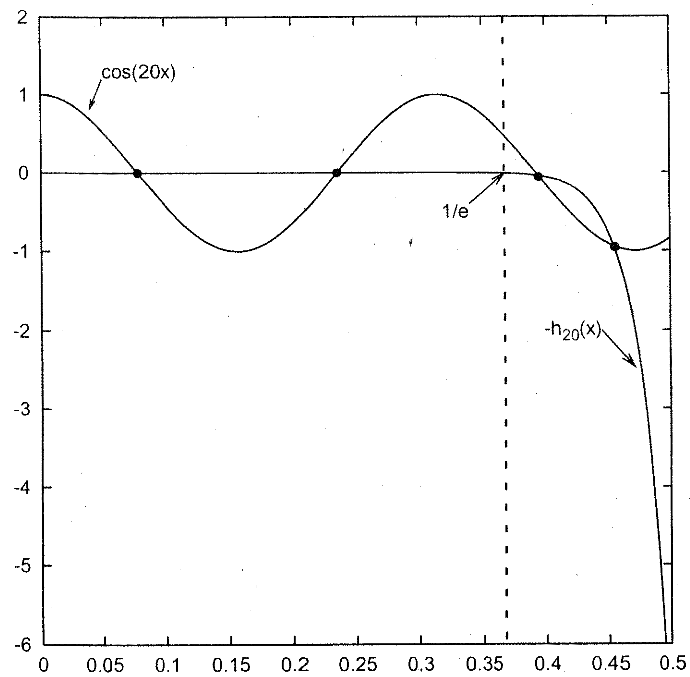

Fig. 5.1. The curves of $\cos_n(20x)$ and $-h_{20}(x)$

for all $z \in \mathbb{C}$, and on using (5.9) with (5.1), where z is replaced by $\pm iz$, we similarly obtain, for x real,

$$\text{(5.10)} \qquad \sin_m(mx) = \sin(mx) - (-1)^{\frac{m+1}{2}} h_m(x),$$

where $h_m(x)$ is given in both (5.4) and (5.6), but with n replaced by $m \in 2\mathbb{N}-1$. Again, the zeros of $\sin_m(mz)$ are the abscissas, in $[-1,+1]$, where the curves, $\sin(mx)$ and $(-1)^{\frac{m+1}{2}} h_m(x)$, *cross*. Noting that $h_m(x)$ is an odd function, the number of real zeros of $\sin_m(mx)$, in $[-1,+1]$, is necessarily odd. Also, we see that Lemma 5.1 is also valid for all $m \in 2\mathbb{N}-1$, in that for any $m \in 2\mathbb{N}-1$, there is a unique ω_m, with $1/e < \omega_m < 1$ such that

$$\text{(5.11)} \qquad h_m(\omega_m) = 1 \quad (m \in 2\mathbb{N}-1).$$

Similarly from (5.10), we can write

$$\text{(5.12)} \quad (-1)^{\frac{m+1}{2}} h_m\left(\frac{u}{m}\right) = \sin(u) - \sin_m(u) \quad (u \in [-m,+m]),$$

so from power series expansions again, the smallest positive zero of $\sin_m(u)$ converges geometrically to the smallest positive zero, π, of $\sin(u)$, as $m \to \infty$, where the convergence for this zero is from the

$$\text{left (right) if } \frac{m+1}{2} \text{ is even (odd),}$$

with similar statements holding for successively larger positive zeros of $\sin(u)$, for $u \in [0, m/e]$, where the rate of geometric convergence decreases with each subsequent larger zero.

We next estimate ω_n of (5.7), for $n \in 2\mathbb{N}$, as $n \to \infty$. Using the representation of (5.4), it can be verified, again in a straight-forward way, that

$$\omega_n = \frac{1}{e}\left(1 + \frac{\log n}{2n} + O\left(\frac{\log n}{n^2}\right)\right), \text{ as } n \to \infty. \tag{5.13}$$

Recalling the fact that $h_n(x)$ is strictly increasing on $(0, 1]$ and that $|\cos(nx)| \leq 1$, it follows, as $h_n(\omega_n) := 1$, that there are *no* crossings of the curves $\cos(nx)$ and $(-1)^{\frac{n+2}{2}} h_n(x)$ for any $x > \omega_n$, where $\omega_n > \frac{1}{e}$. Thus, if $z_{n,1}$ is the largest real (Hurwitz or spurious) zero of $\cos_n(nz)$, then

$$z_{n,1} \leq \omega_n. \tag{5.14}$$

If $z_{n,1}$ is a Hurwitz zero (written $z^H_{n,1}$), then $z^H_{n,1}$ is contained in both $\hat{\Theta}_{n,k_n}$ and $[0, \rho_n]$, so that, from (4.12) and (4.3);

$$x_{k_n} - \frac{\hat{\xi}_n}{n} \leq z^H_{n,1} \leq \rho_n = \frac{1}{e} + \frac{1}{en}.$$

With (4.2) and the fact that $0 < t_n < 1$ from (3.6), the above inequalities become

$$\frac{1}{e} - \frac{1}{n} - \frac{\hat{\xi}_n}{n} < z^H_{n,1} \leq \frac{1}{e} + \frac{1}{en},$$

which implies from (4.12) that

$$dist\left[z^H_{n,1}; \frac{1}{e}\right] = O\left(\frac{1}{n}\right), \text{ as } n \to \infty. \tag{5.15}$$

On the other hand, if $z_{n,1}$ is a spurious zero (written $z^S_{n,1}$), then, by definition,

$$\left\{\begin{array}{ll} \frac{1}{e} < z^S_{n,1} \leq \omega_n, & \text{if } t_n \text{ satisfies (3.7), or} \\ \rho_n < z^S_{n,1} \leq \omega_n, & \text{if } t_n \text{ does not satisfy (3.7).} \end{array}\right\}$$

Then, from (4.3) and (5.13), it follows, in either case, that

$$dist\left[z^S_{n,1}; \frac{1}{e}\right] \leq \frac{\log n}{2en} + O\left(\frac{\log n}{n^2}\right), \text{ as } n \to \infty. \tag{5.16}$$

Combining (5.13) and (5.14) (with a similar calculation holding for $\sin_m(mz)$), we have the result of

Theorem 5.2 *Let* $z_{n,1}$ *denote the largest real zero of* $\cos_n(nz)$, *where* $n \in 2\mathbb{N}$. *Then,*

$$\overline{\lim_{\substack{n\to\infty \\ n\in 2\mathbb{N}}}} \left(\frac{n}{\log n} dist\,[z_{n,1}; 1/e] \right) \le \frac{1}{2e} \doteq 0.18394. \tag{5.17}$$

Similarly, if $w_{m,1}$ *denotes the largest real zero of* $\sin_m(mz)$, *where* $m \in 2\mathbb{N} - 1$, *then*

$$\overline{\lim_{\substack{m\to\infty \\ m\in 2\mathbb{N}-1}}} \left(\frac{m}{\log m} dist\,[w_{m,1}; 1/e] \right) \le \frac{1}{2e} \doteq 0.18394. \tag{5.18}$$

The results of (5.17) and (5.18) of Theorem 5.2 are of course similar to the result of (1.10), which is known to be best possible. It is likely that (5.17) and (5.18) give the *correct* dependence on $n \in 2\mathbb{N}$ and $m \in 2\mathbb{N} - 1$, but an open question is if equality holds in either (5.17) or (5.18).

6 An open problem

The precise behavior of the zeros of $\cos_n(nz)$, $n \in 2\mathbb{N}$, or of $\sin_m(mz)$, $m \in 2\mathbb{N} - 1$ near the points $\pm i$ of K has not been treated here or in the literature, and this is also an intriguing open problem. We plan to investigate this in the future.

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