AN APPLICATION OF NONNEGATIVE MATRICES TO THE SYNCHRONIZATION OF CHAOTIC OSCILLATORS

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Abstract. The following problem arose in a correspondence between the authors, concerning the exact location of the set of eigenvalues of special real matrices, arising from a problem of synchronization of chaotic oscillators.

Keywords: Cyclic matrices, eigenvalues of the Kronecker product of two matrices, theory of irreducible and reducible matrices.
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1. Introduction

The original form of this matrix problem, which arose from an application of the synchronization of chaotic oscillators, was this. Let k be a fixed positive integer and for all integers $m \geq k+1$, let $A = [a_{i,j}] \in \mathbb{R}^{m,m}$ be such that

$$(1) \qquad \begin{cases} a_{i,i} = k & (\text{all } 1 \leq i \leq m), \\ a_{i,j} = 0 \text{ or } -1 & (\text{all } i \neq j,\ 1 \leq i,j \leq m), \\ \sum_{j=1}^{m} a_{i,j} = 0 & (\text{all } 1 \leq i \leq m). \end{cases}$$

Then, where are the eigenvalues of *all* the matrices A which satisfy (1)?

For our purposes, we found it more convenient, mathematically, to express the matrix A of (1) as $A = kI_m - B$, where I_m is the $m \times m$ identity matrix and $B = [b_{i,j}] \in \mathbb{R}^{m,m}$ satisfies

$$(2) \qquad \begin{cases} b_{i,i} = 0 & (\text{all } 1 \leq i \leq m), \\ b_{i,j} = 0 \text{ or } 1 & (\text{all } i \neq j,\ 1 \leq i,j \leq m), \\ \sum_{j=1}^{m} b_{i,j} = k & (\text{all } 1 \leq i \leq m). \end{cases}$$

Thus, from (2), B is a nonnegative matrix in $\mathbb{R}^{m,m}$. Moreover, let $\rho(B)$ denote the spectral radius of B, i.e.,

$$\rho(B) := \max_{1 \leq i \leq m} \{|\lambda_i| : \lambda_i \text{ is an eigenvalue of } B\}.$$

It follows from the well-known theorem of Geršgorin's [3] that if λ is any eigenvalue of B, there is an i, with $1 \leq i \leq m$, such that $|\lambda - b_{i,i}| \leq \sum_{j=1, j\neq i}^{m} |b_{i,j}|$. But, using (2), this reduces to $|\lambda| \leq k$ for *all* eigenvalues of B, so that $\rho(B) \leq k$. On the other hand, for the vector $\mathbf{v} := [1, 1, \ldots, 1]^T$ in $\mathbb{R}^n$, we have that $B\mathbf{v} = k\mathbf{v}$, so that k is an eigenvalue of B. Thus,

$$\rho(B) = k, \text{ for all matrices } B \text{ satisfying (2).} \tag{3}$$

Next, we define the set $\mathcal{B}(k)$ as

$$\mathcal{B}(k) := \left\{B \in \mathbb{R}^{m\times m} : m \geq k+1 \text{ and } B \text{ satisfies (2)}\right\}, \tag{4}$$

and if $\sigma(B)$ denotes the eigenvalues of B, it follows that

$$\bigcup_{B\in\mathcal{B}(k)} \sigma(B) \subseteq \{z \in \mathbb{C} : |z| \leq k\}. \tag{5}$$

In addition, as the disk on the right in (5) is closed, in the usual topology of the complex plane, then the topological closure of the set of all eigenvalues of all B in $\mathcal{B}(k)$ necessarily satisfies

$$\overline{\bigcup_{B\in\mathcal{B}(k)} \sigma(B)} \subseteq \{z \in \mathbb{C} : |z| \leq k\}, \text{ for each positive integer } k. \tag{6}$$

Our goal here is to precisely determine $\overline{\bigcup_{B\in\mathcal{B}(k)}}$, for each integer $k \geq 1$. This will be done in two steps.

2. Step 1: The Case $k = 1$

Our first result is

Theorem 1. *For $k = 1$, we have*

$$\overline{\bigcup_{B\in\mathcal{B}(1)} \sigma(B)} = \{0\} \cup \{z \in \mathbb{C} : |z| = 1\}. \tag{7}$$

Remark. This states that, for any $m \times m$ matrix B in $\mathcal{B}(1)$, each eigenvalue of B is either 0, or a point on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, as shown in Fig. 1 below.

Proof. Consider any $m \times m$ matrix $B \in \mathcal{B}(1)$, where $m \geq 2$ from (4). Then, there is an $m \times m$ permutation matrix P for which PBP^T has the special form

$$PBP^T = \begin{bmatrix} R_{1,1} & R_{1,2} & \cdots & R_{1,t} \\ \mathcal{O} & R_{2,2} & \cdots & R_{2,t} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{O} & \mathcal{O} & \cdots & R_{t,t} \end{bmatrix}, \text{ with } 1 \leq t < m, \tag{8}$$

where each diagonal submatrix $R_{j,j}$ in (8) is square, and is either irreducible or a 1×1 null matrix. (The expression in (8) is called the *normal reduced form* of the matrix B; see [4, p.51].) We note that the union of the eigenvalues of the matrices $\{R_{j,j}\}_{j=1}^t$ gives all the eigenvalues of B. Because $k = 1$ in this theorem, it is easily

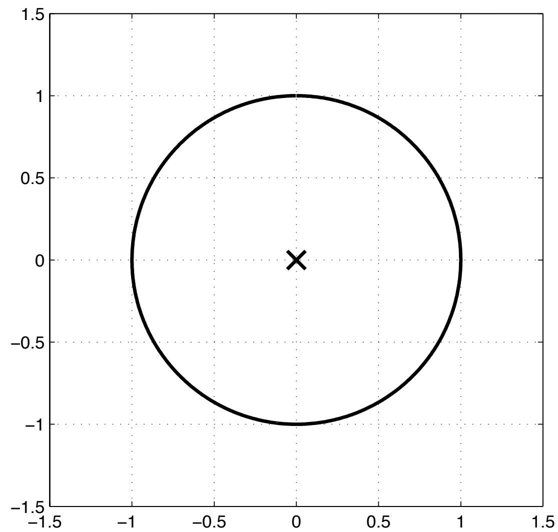

FIGURE 1. Eigenvalue location for the case $k = 1$

seen that each diagonal submatrix $R_{j,j}$ of (8) is either a 1×1 null matrix, or a full cycle square permutation matrix, such as the following $l \times l$ matrix

$$S_l = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \text{ where } l \geq 2. \tag{9}$$

The eigenvalues of S_l are given by

$$\exp\left\{ i\frac{2\pi j}{l} \right\}_{j=0}^{l-1}, \tag{10}$$

which are *uniformly spaced* (in angle) on the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

Next, consider the last square submatrix $R_{t,t}$ of (8). We further note that the final matrix $R_{t,t}$ in (8) cannot be a 1×1 null matrix, as this would imply that the last row of equation (8) consists only of zeros, which contradicts part 3 of (2). (This also shows why $t < m$ in (8).)

The previous paragraph shows that, for any $m \times m$ matrix B in $\mathcal{B}(1)$ (where $m \geq 2$), the eigenvalues of B are made up of blocks of *uniformly spaced points* on the unit circle, and possibly some zero eigenvalues, so that

$$\bigcup_{B\in\mathcal{B}(1)} \sigma(B) \subseteq \{0\} \cup \{z \in \mathbb{C} : |z| = 1\}. \tag{11}$$

Next, we consider the following $(l+1)\times(l+1)$ matrix in $\mathcal{B}(1)$, namely

$$B_{l+1} = \left[\begin{array}{c|ccccc} 0 & 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & & \\ \vdots & & & S_l & & \\ 0 & & & & & \end{array}\right],$$

where S_l is the matrix of (9) with $l \geq 2$. Then, B_{l+1} is an element of $\mathcal{B}(1)$, and moreover, all the eigenvalues of B_{l+1} are the eigenvalues of (10), plus one eigenvalue of zero. Then, on letting l tend to infinity, we easily see that the topological closure of the set of all eigenvalues of all B in $\mathcal{B}(1)$ satisfies

$$\overline{\bigcup_{B\in\mathcal{B}(1)} \sigma(B)} = \{0\} \cup \{z \in \mathbb{C} : |z| = 1\}, \tag{12}$$

the desired result of Theorem 1. This final set in eq. (12) is shown in Figure 1. ■

3. The Case $k \geq 2$

Our next result states that the spectra of matrices in $\mathcal{B}(k)$, $k \geq 2$, are vastly different from the case of $\mathcal{B}(1)$.

Theorem 2. *For any integer k with $k \geq 2$, we have*

$$\overline{\bigcup_{B\in\mathcal{B}(k)} \sigma(B)} = \{z \in \mathbb{C} : |z| \leq k\}. \tag{13}$$

Remark. This states that the collection of all eigenvalues of all B in $\mathcal{B}(k)$, for $k \geq 2$, *fills out* the disk $\{z \in \mathbb{C} : |z| \leq k\}$.

Proof. We already know from (6) that

$$\overline{\bigcup_{\text{all } B\in\mathcal{B}(k)} \sigma(B)} \subseteq \{z \in \mathbb{C} : |z| \leq k\} \ \text{(any } k \geq 2\text{)}, \tag{14}$$

so it remains only to show that equality holds in (14).

Fixing a $k \geq 2$, consider the *basic circulant permutation matrix* D_m, (cf. [1, p. 26])
(15)

$$D_m := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \text{where } D_m \in \mathbb{R}^{m,m}, \text{ where } m \geq k,$$

so that D_m satisfies (2), but with $m \geq k$. The eigenvalues of D_m are, from (9)

$$\sigma(D_m) = \left\{ \exp\left(i \frac{2\pi j}{m} \right) \right\}_{j=0}^{m-1}. \tag{16}$$

Then, we consider the associated *circulant matrix*

$$C_m(k) := I_m + D_m + D_m^2 + \cdots + D_m^{k-1} \quad (\text{any } k \geq 2), \tag{17}$$

whose eigenvalues are well-known to be given by

$$\sigma\left(C_m(k)\right) = \left\{ \sum_{j=0}^{k-1} \exp\left(i \frac{2\pi j l}{m} \right) \right\}_{l=0}^{m-1} \quad (\text{all } m \geq k). \tag{18}$$

Next, we form the *Kronecker product* $D_m \otimes C_m(k)$, given by

$$D_m \otimes C_m(k) := \begin{bmatrix} \mathcal{O} & C_m(k) & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & C_m(k) & \cdots & \mathcal{O} & \mathcal{O} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & C_m(k) \\ C_m(k) & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \end{bmatrix}, \tag{19}$$

which is an $m^2 \times m^2$ matrix, where each $\mathcal{O}$ is a null $m \times m$ matrix. It follows (cf. [2, p. 245]) that the m^2 eigenvalues of $D_m \otimes C_m(k)$ are given by

$$\sigma\left(D_m \otimes C_m(k)\right) = \left\{ \left(\sum_{l=0}^{k-1} \exp\left(i \frac{2\pi l j}{m} \right) \right) \cdot \exp\left(i \frac{2\pi s}{m} \right) \right\}_{j=0,s=0}^{m-1,m-1}. \tag{20}$$

We also have from (20) that

$$\bigcup_{m \geq k} \sigma\left(D_m \otimes C_m(k)\right) \subseteq \left\{ z \in \mathbb{C} : |z| \leq k \right\}. \tag{21}$$

To give further insights in the remainder of the proof of Theorem 2, consider the following case of $m = 8$ and $k = 2$. The associated matrix $D_8 \otimes C_8(2)$ from (19) has 64 eigenvalues, which are shown in Fig. 2, where the *small solid disks* are the

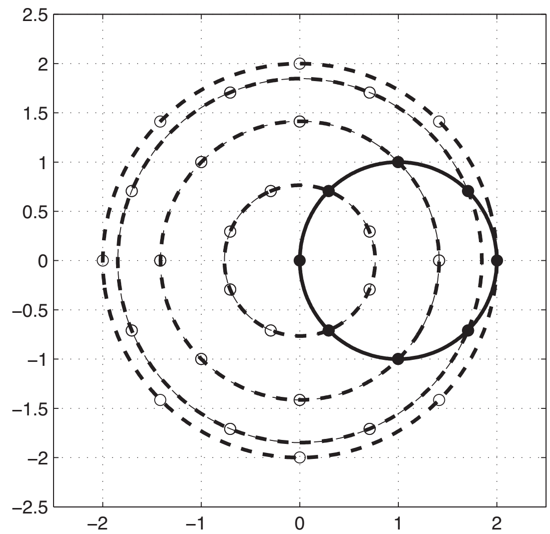

FIGURE 2. The case $k = 2$, $m = 8$.

eight eigenvalues of $C_8(2)$, which are given, from (18), as

$$\sigma\left(C_8(2)\right) = \left\{1 + \exp\left(i\frac{2\pi j}{8}\right)\right\}_{j=0}^{7}. \tag{22}$$

Then, each *small solid disk* in Fig 2 defines a specific circle, with center 0, which passes through the given solid disk, and on this circle, there are eight equally spaced zeros, where each such zero is a small *open disk* (if it is not already a solid disk). The 64 eigenvalues of $D_m \otimes C_m(k)$ are given by the case $m = 8$ and $k = 2$ in (20), where we note that some of these zeros are *multiple,* as in the case of $z = 0$.

Because of the high symmetries of the eigenvalues in Fig. 2, it is not difficult to show, in general for the case $k = 2$, that the shortest distance between *any* given eigenvalue of $D_m \otimes C_m(2)$ and its *nearest neighboring* eigenvalue is *at most* the distance between the eigenvalue $z = 2$ and its nearest eigenvalue $z' = 1 + \exp\left(i\frac{2\pi}{m}\right)$, i.e.,

$$\begin{aligned} |z - z'| &= \left|2 - \left(1 + \exp\left(i\frac{2\pi}{m}\right)\right)\right| \\ &= \left[2\left(1 - \cos\left(\frac{2\pi}{m}\right)\right)\right]^{\frac{1}{2}} \sim \frac{2\pi}{m}, \text{ as } m \to \infty. \end{aligned} \tag{23}$$

This means that, on taking the *union* of *all* eigenvalues of $D_m \otimes C_m(2)$ for *all* $m \geq 2$, these eigenvalues necessarily *fill out* the closed disk $\{z \in \mathbb{C} : |z| \leq 2\}$, which gives us that

$$\overline{\bigcup_{\text{all } m\geq 2} \sigma\left(D_m \otimes C_m(2)\right)} = \{z\in\mathbb{C} : |z|\leq 2\}. \tag{24}$$

But, as $D_m \otimes C_m(2) \in \mathcal{B}(2)$, for all $m \geq 2$, we have the result of (13), for the case $k = 2$.

To extend the above result for $k = 2$ to any positive integer $k > 2$, we define, in connection with (17), the polynomial function

$$F_k(z) := 1 + z + z^2 + \cdots + z^{k-1}, \text{ where } z = e^{i\theta}, \tag{25}$$

so that

$$F_k(z) = \frac{1-z^k}{1-z}, \text{ for any } z \neq 1. \tag{26}$$

It follows from (25) that (20) can be also expressed as

$$\sigma\left(D_m \otimes C_m(k)\right) = \left\{ F_k\left(\exp\left(i\frac{2\pi j}{m}\right)\right)\cdot \exp\left(i\frac{2\pi s}{m}\right)\right\}_{j=0,s=0}^{m-1,m-1}. \tag{27}$$

Then, $F_k(e^{i\theta})$ has the properties given in Lemma 1 below.

Lemma 1. *For any $k \geq 2$, we have*

$$\begin{cases} i) & |F_k(e^{i\theta})| \leq k \;\; (\text{all } 0\leq\theta\leq 2\pi), \text{ with } F_k(1) = k, \\ ii) & |F_k(e^{i\theta})|^2 = F_k(e^{i\theta})F_k(e^{-i\theta}) = \frac{(1-\cos(k\theta))}{1-\cos(\theta)} \;\; (\text{ all } 0<\theta<2\pi), \\ iii) & F_k(e^{i(2\pi j)/k}) = 0 \;\; (\text{all } 1\leq j\leq k-1), \\ iv) & F_k(e^{i(2\pi j)/(k-1)}) = 1 \;\; (\text{all } 1\leq j\leq k-2, \text{ for } k\geq 3), \\ v) & |F_k(e^{i\theta})|^2 \text{ is continuous in } \theta \text{ on } [0,2\pi], \text{ and is strictly} \\ & \text{decreasing to zero on the interval } \left(0,\frac{2\pi}{k}\right), \\ vi) & \text{Im } F_k(e^{i\theta}) > 0 \;\; (\text{all } 0<\theta<\frac{2\pi}{k}). \end{cases} \tag{28}$$

Proof. Items (28i) and (28ii) are immediate from (25) and (26). Then from (26), (28iii) follows. Next, from (26), we have that $F_k(z) = 1$, for $z \neq 1$, is equivalent to $z(z^{k-1} - 1) = 0$, from which (28iv) follows. Next, to obtain (28v), it can be verified from (25) and (28ii) that

$$|F_k(e^{i\theta})|^2 = k + 2\sum_{j=1}^{k-1}(k-j)\cos(j\theta), \tag{29}$$

for any θ in $[0, 2\pi]$ and for any $k \geq 2$, which give the continuity of $|F_k(e^{i\theta})|^2$ on $[0, 2\pi]$. With this expression and with standard trigonometric identities, which change with k, it can be verified that $\frac{d}{d\theta}|F_k(e^{i\theta})|^2 < 0$ on the interval $\left(0, \frac{2\pi}{k}\right)$, for *each* $k \geq 2$, which gives the desired result of (28v) that $|F_k(e^{i\theta})|^2$ is strictly decreasing to zero on $\left(0, \frac{2\pi}{k}\right)$. As an example, consider the case $k = 3$ of (29). On differentiating, one obtains from (29), with some minimal effort, that

$$\frac{d}{d\theta}|F_3(e^{i\theta})|^2 = -4\sin\theta(1 + 2\cos\theta), \tag{30}$$

which is negative on the interval $\left(0, \frac{2\pi}{3}\right)$. Finally, to establish (28*vi*), it follows from (25) that

$$\text{Im } F_k(e^{i\theta}) = \sum_{j=0}^{k-1} \sin(j\theta), \tag{31}$$

which can be verified, using trigonometric identities again, to give *vi*) of Lemma 1, for any $k \geq 2$. As an example of this, we deduce from (31) that

$$\text{Im } F_3(e^{i\theta}) = \sin\theta \cdot (1 + 2\cos\theta),$$

which is positive on the interval $\left(0, \frac{2\pi}{3}\right)$. ■

In Figures 3 and 4, we show the graphs of $F_2(e^{i\theta})$ and $F_3(e^{i\theta})$, for $0 \leq \theta \leq 2\pi$. For $k = 3$, we see, from (28*iii*) and (28*iv*), that $F_3(e^{i\theta})$, with increasing θ, first passes through zero when $\theta = \frac{2\pi}{3}$, and then through unity when $\theta = \pi$, then again through zero when $\theta = \frac{4\pi}{3}$, thereby creating a single *inner loop,* as shown in Fig. 4. In Fig. 5, we show the graph of $F_5(e^{i\theta})$, for $0 \leq \theta \leq 2\pi$, and in this case, $F_5(e^{i\theta})$, with increasing θ, similarly shows the existence of *three* inner loops. (We will come back to a discussion about these inner loops.). Note, from Figures 3–5, that $\text{Im}\, F_k(e^{i\theta})$ is positive on the interval $\left(0, \frac{2\pi}{k}\right)$, which comes from (28*vi*).

We return to the proof of Theorem 2, for a fixed $k > 2$, and a variable m, with $m \geq k$. Consider the $km + 1$ points of the curve $F_k(e^{i\theta})$, where θ takes the equally spaced values $\left\{\theta_j := \frac{2\pi j}{km}\right\}_{j=0}^{mk}$. We see from (28*vi*) that the numbers $\left\{F_k(e^{i\theta_j})\right\}_{j=0}^{m}$ are all in the closed upper-half plane, with $F_k(e^{i\theta_0}) = k$ and $F_k(e^{i\theta_m}) = 0$. Moreover, the numbers $\left\{F_k(e^{i\theta_j})\right\}$, for $0 \leq j \leq m$, have strictly decreasing moduli from (28*v*). Also, it can be verified that the successive differences of these numbers, i.e.,

$$\left|F_k\left(e^{i\frac{2\pi(j-1)}{km}}\right) - F_k\left(e^{i\frac{2\pi j}{km}}\right)\right| > 0, \text{for } j = 1, 2, \ldots, m, \tag{32}$$

are such that the *largest* of these differences is the *first one,* namely

$$\left|F_k(1) - F_k\left(e^{i\frac{2\pi}{km}}\right)\right|, \tag{33}$$

as can be seen directly in Figures 4 and 5. In particular, it can also be shown that

$$\left|F_k(1) - F_k\left(e^{i\frac{2\pi}{km}}\right)\right| \sim \frac{\pi(k-1)}{m}, \text{as } m \to \infty. \tag{34}$$

The result of (34) has two interesting consequences. First, of the points $\left\{F_k(e^{i\frac{2\pi j}{km}})\right\}_{j=0}^{m}$ in the closed upper-half plane, the first difference of these moduli in (32) is largest,

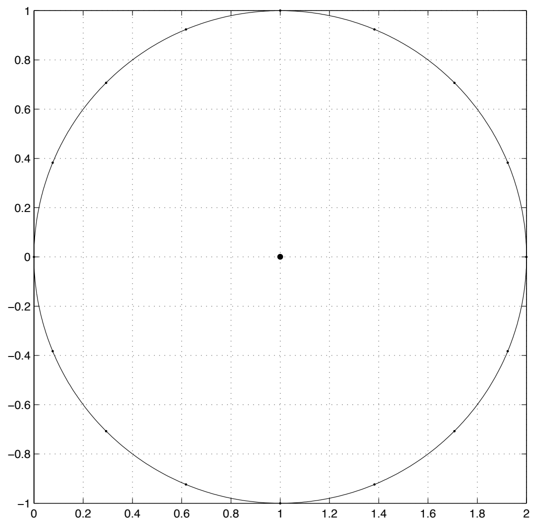

FIGURE 3. The curve $F_2(e^{i\theta})$ for $0 \leq \theta \leq 2\pi$ and the point $z = 0$.

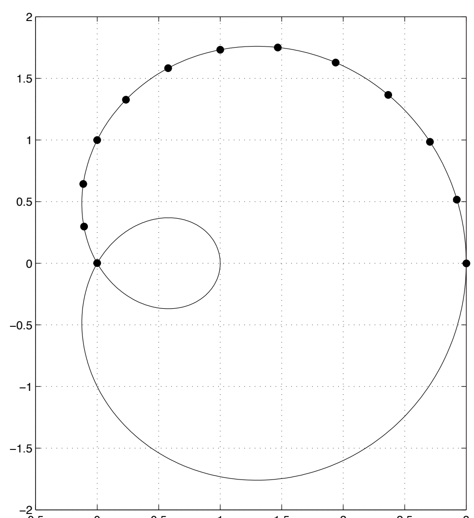

FIGURE 4. The curve $F_3(e^{i\theta})$ for $0 \leq \theta \leq 2\pi$, and the thirteen points $F_3(e^{i\theta_j})$, for $\theta_j = \frac{2\pi j}{36}$, for $j = 0, 1, \ldots, 12$, and $m = 12$.

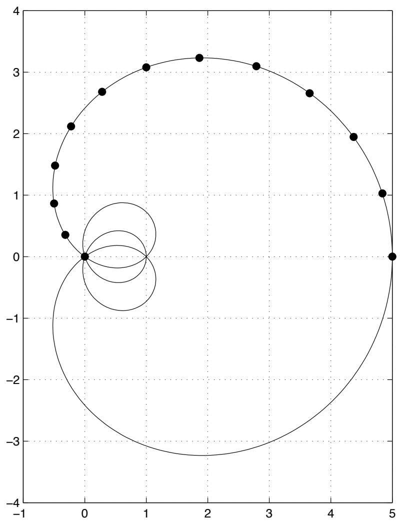

FIGURE 5. The curve $F_5(e^{i\theta})$ for $0 \le \theta \le 2\pi$, and the thirteen points $F_5(e^{i\theta_j})$, for $\theta_j = \frac{2\pi j}{60}$, for $j = 0, 1, \ldots, 12$, and $m = 12$.

which, from (34), is $\mathcal{O}\left(\frac{1}{m}\right)$, as $m \to \infty$. Then, on recalling (20), the m^2 eigenvalues of $D_m \otimes C_m(k)$ can be expressed, using (25), as

$$\sigma\left(D_m \otimes C_m(k)\right) = \left\{ \left[F_k\left(e^{i\left(\frac{2\pi l}{m}\right)}\right) \right] \cdot e^{i\frac{2\pi s}{m}} \right\}_{l=0,s=0}^{m-1,m-1}, \tag{35}$$

which creates m^2 eigenvalues, built on rotations of the points $F_k\left(e^{i\left(\frac{2\pi l}{m}\right)}\right)$ on the curve $F_k(e^{i\theta})$, by the final factor $e^{i\frac{2\pi s}{m}}$ in (35). This means that, of all m^2 eigenvalues of $D_m \otimes C_m(k)$, the difference in modulus between *any* eigenvalue of $D_m \otimes C_m(k)$ and its nearest eigenvalue, cannot exceed the difference of (34), which is $\mathcal{O}\left(\frac{1}{m}\right)$, for m large. Hence, on letting $m \to \infty$, the eigenvalues of all $D_m \otimes C_m(k)$, for all $m \ge k$, necessarily fill out the disk $\{z \in \mathbb{C} : |z| \le k\}$. Thus, as $D_m \otimes C_m(k) \in \mathcal{B}(k)$, we have the desired result of (13) of Theorem 2. ■

We return to the inner loops in Fig. 4 and 5. We note that the proof of Theorem 2, for $k > 2$, only drew upon the values of $F_k(e^{i\theta_j})$ for $\left\{\theta_j = \frac{2\pi j}{km}\right\}_{j=0}^{m-1}$, in the upper half-plane, i.e., for values of $0 \le \theta_j \le \frac{2\pi}{k}$, while the associated circles were associated with just these points. Let us now consider similarly the solid disks of Fig. 2. If we *delete* the three solid disks in the open lower half-plane, the circles generated from the five darkened points in the upper half-plane still effectively cover the whole disk $\{z \in \mathbb{C} : |z| \le 2\}$, and *this* is what is taking place here! In other words, one *only* has to work with the values of $F_k(e^{i\theta_j})$ for $\left\{\theta_j = \frac{2\pi j}{km}\right\}_{j=0}^{m-1}$ to cover the entire disk $\{z \in \mathbb{C} : |z| \le 2\}$. Similarly, for $k = 5$, the three inner

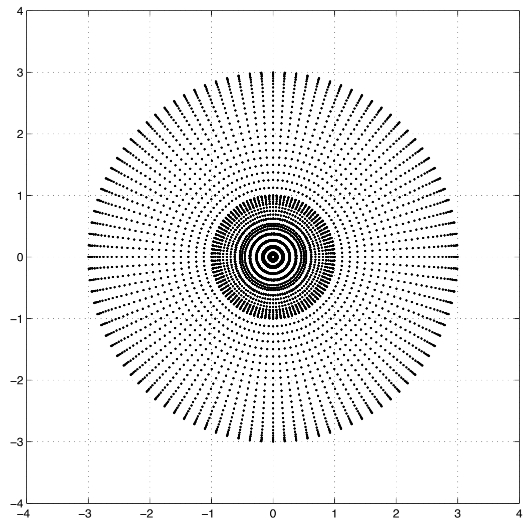

FIGURE 6. Eigenvalues of $D_{100} \otimes C_{100}(3)$.

loops of Fig. 5 can be considered to be *superfluous* in filling out the larger disk $\{z \in \mathbb{C} : |z| \leq 5\}$.

To carry this a bit further, consider the unique inner loop, in the case $k = 3$ of Fig. 4. The maximum distance, from any point of this inner loop to $z = 0$, is exactly *unity,* which means that, applying our technique of building circles and letting $m \to \infty$, only results in forcing many more eigenvalues into the *inner disk* $\{z \in \mathbb{C} : |z| \leq 1\}$ of the larger disk $\{z \in \mathbb{C} : |z| \leq 3\}$. This is precisely what causes the *darkening* of the inner unit circle in Fig. 6.

In a similar way, the maximum distance from $z = 0$ to any point of the three inner loops, for the case $k = 5$ in Fig. 5, turns out to be *exactly* 1.25, and one can directly see in Fig. 7 the associated darkening of the disk with center $z = 0$ and radius 1.25, caused by the swelling of eigenvalues in this region, because of these loops. This can be more easily seen in Fig. 8, which gives a zoomed-in view of the darkened area of Fig. 7.

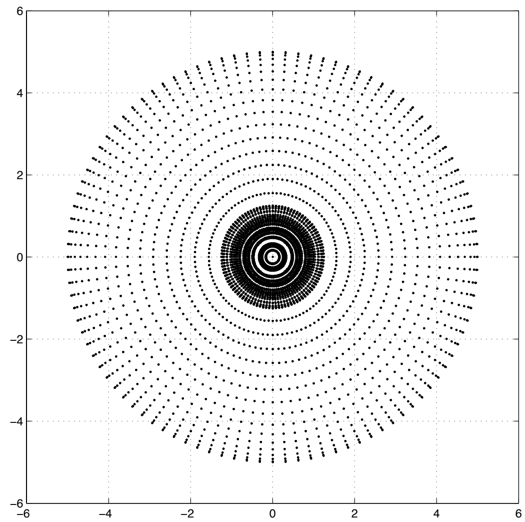

FIGURE 7. Eigenvalues of $D_{100} \otimes C_{100}(5)$.

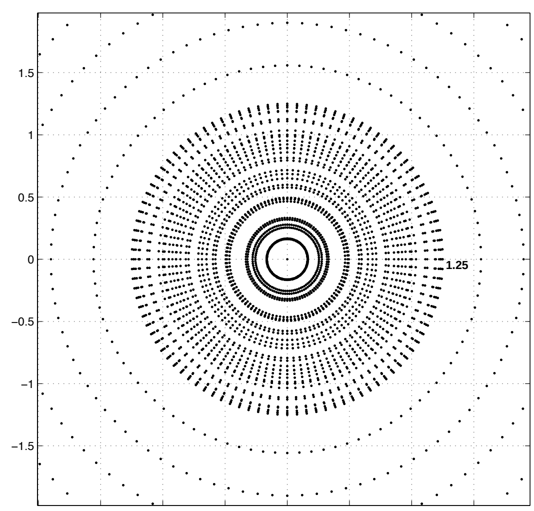

FIGURE 8. Detail of the darkened area for the Eigenvalues of $D_{100} \otimes C_{100}(5)$.

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References

[1] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1985.

[2] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis*. Cambridge University Press, Cambridge, 1991.

[3] S. Geršgorin. Über die Abgrenzung der Eigenwerte einer Matrix. *Izv. Akad. Nauk SSSR*, pages 479–754, 1931.

[4] Richard S. Varga. *Matrix Iterative Analysis*. Springer Series in Computational Mathematics. Springer, Berlin, Heidelberg, New York, second edition, 2000.

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