

MINIMAL GERSCHGORIN SETS II

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The Gerschgorin Circle Theorem, which yields n disks whose union contains all the eigenvalues of a given $n \times n$ matrix $A = (a_{i,j})$, applies equally well to any matrix $B = (b_{i,j})$ of the set Ω_A of $n \times n$ matrices with $b_{i,i} = a_{i,i}$ and $|b_{i,j}| = |a_{i,j}|$, $1 \leq i, j \leq n$. This union of n disks thus bounds the entire spectrum $S(\Omega_A)$ of the matrices in Ω_A . The main result of this paper is a precise characterization of $S(\Omega_A)$, which can be determined by extensions of the Gerschgorin Circle Theorem based only on the use of positive diagonal similarity transformations, permutation matrices, and their intersections.

Given any $n \times n$ complex matrix $A = (a_{i,j})$, it is well known that the simplest of Gerschgorin arguments, which depends upon row sums of the moduli of off-diagonal entries of the matrix $X^{-1}AX$, X a positive diagonal matrix, yields the union of n disks which contains all the eigenvalues of A . It is clear that this union of n disks necessarily contains all the eigenvalues of any $n \times n$ matrix in the set Ω_A defined as follows: $B = (b_{i,j}) \in \Omega_A$ if $b_{i,i} = a_{i,i}$, $1 \leq i \leq n$, and $|b_{i,j}| = |a_{i,j}|$ for all $1 \leq i, j \leq n$, $i \neq j$. Hence, this union of n Gerschgorin disks can be viewed as giving bounds for the entire spectrum $S(\Omega_A) = \{z \mid \det(zI - B) = 0 \text{ for some } B \in \Omega_A\}$ of the set Ω_A .

It is logical to ask to what extent the spectrum $S(\Omega_A)$ can be more precisely determined by extensions of Gerschgorin's original argument [3]. In the previous paper [6], it was shown that

$$(1.1) \quad \partial G(\Omega_A) \subset S(\Omega_A) \subset G(\Omega_A),$$

where $G(\Omega_A)$ is the *minimal Gerschgorin set* deduced from A and $\partial G(\Omega_A)$ is its boundary. The first inclusion of (1.1) states that every point of the boundary $\partial G(\Omega_A)$ of the minimal Gerschgorin set is then an eigenvalue of some $B \in \Omega_A$. We now extend the results of [6] by making use of results of Schneider [4], and Camion and Hoffman [1]. In so doing, we shall *precisely* determine $S(\Omega_A)$.

To begin, let $P_\phi = (\delta_{i,\phi(j)})$ be an $n \times n$ permutation matrix, where ϕ is a permutation of the integers $1 \leq i \leq n$ and $\delta_{i,j}$ is the Kronecker delta function, and let $X = \text{diag}(x_1, x_2, \dots, x_n)$, where $\mathbf{x} > \mathbf{0}$. Given $B \in \Omega_A$, we define the $n \times n$ matrix $M^\phi(\mathbf{x})$ by

$$(1.2) \quad M^\phi(\mathbf{x}) = (X^{-1}BX - \lambda I)P_\phi = (m_{i,j}),$$

so that

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$$(1.3) \quad m_{i,j} = b_{i,\phi(j)} x_{\phi(j)} / x_i - \lambda \delta_{i,\phi(j)}, \quad 1 \leq i, j \leq n.$$

Following Schneider [4], if λ is an eigenvalue of B , then $M^\phi(\mathbf{x})$ is surely singular and thus not strictly diagonally dominant. Hence,

$$(1.4) \quad |m_{i,i}| \leq \sum_{j \neq i} |m_{i,j}|$$

must be true for at least one i , $1 \leq i \leq n$. Defining first

$$(1.5) \quad A_i(\mathbf{x}) \equiv \left(\sum_{j \neq i} |a_{i,j}| x_j \right) / x_i, \quad 1 \leq i \leq n,$$

then (1.4) implies that either

$$(1.6) \quad |\lambda - a_{i,i}| \leq A_i(\mathbf{x}) \quad \text{if} \quad \phi(i) = i,$$

or

$$(1.6') \quad 2a_{\phi(i)} |a_{i,\phi(i)}| / x_i \leq |\lambda - a_{i,i}| + A_i(\mathbf{x}) \quad \text{if} \quad \phi(i) \neq i.$$

For any complex number σ , we consequently define

$$(1.7) \quad r_i^\phi(\sigma; \mathbf{x}) \equiv A_i(\mathbf{x}) - |\sigma - a_{i,i}| \quad \text{if} \quad \phi(i) = i,$$

and let

$$(1.7') \quad r_i^\phi(\sigma; \mathbf{x}) \equiv |\sigma - a_{i,i}| + A_i(\mathbf{x}) - 2|a_{i,\phi(i)}| x_{\phi(i)} / x_i \quad \text{if} \quad \phi(i) \neq i.$$

With this, we next define the set $G_i^\phi(\mathbf{x})$ as

$$(1.8) \quad G_i^\phi(\mathbf{x}) \equiv \{\sigma \mid r_i^\phi(\sigma; \mathbf{x}) \geq 0\}, \quad 1 \leq i \leq n.$$

If $\phi(i) = i$, then $G_i^\phi(\mathbf{x})$ reduces to the familiar Gerschgorin disk $|z - a_{i,i}| \leq A_i(\mathbf{x})$. If $\phi(i) \neq i$, we observe from (1.7') that $G_i^\phi(\mathbf{x})$ is the closed exterior of a disk, and is thus an *unbounded* set.

Defining $G^\phi(\mathbf{x})$ to be the union of the sets $G_i^\phi(\mathbf{x})$:

$$(1.9) \quad G^\phi(\mathbf{x}) \equiv \bigcup_{i=1}^n G_i^\phi(\mathbf{x}),$$

the inequalities of (1.6) and (1.6') show that if $\lambda \in S(\Omega_A)$, then $\lambda \in G_i^\phi(\mathbf{x})$ for some i , and hence $\lambda \in G^\phi(\mathbf{x})$. Thus, $S(\Omega_A) \subset G^\phi(\mathbf{x})$ for every $\mathbf{x} > \mathbf{0}$, and we then have that

$$(1.10) \quad G^\phi(\Omega_A) \equiv \bigcap_{\mathbf{x} > \mathbf{0}} G^\phi(\mathbf{x}),$$

called the *minimal Gerschgorin set relative to the permutation ϕ* , is such that

$$(1.11) \quad S(\Omega_A) \subset G^\phi(\Omega_A)$$

for every permutation ϕ . It is clear that $G^\phi(\Omega_A)$ is a *closed* set for

any permutation ϕ . Since $G_i^\phi(x)$ is a bounded set only when $\phi(i) = i$, it follows that $G^\phi(\Omega_A)$ is a bounded set only when ϕ is the identity permutation. We remark that the results of [6] are for the special case when ϕ is the identity permutation.

Since (1.11) is valid for any permutation ϕ , it then follows that

$$(1.12) \quad S(\Omega_A) \subset H(\Omega_A),$$

where

$$(1.13) \quad H(\Omega_A) \equiv \bigcap_{\phi} G^\phi(\Omega_A).$$

In § 2, we first characterize (Theorem 1) the minimal Gerschgorin sets $G^\phi(\Omega_A)$, and then show (Theorem 2) that their boundaries $\partial G^\phi(\Omega_A)$ are subsets of $S(\Omega_A)$. Finally, using a result of Camion and Hoffman [1], we prove (Theorem 3) in § 3 our main result that

$$(1.14) \quad S(\Omega_A) = H(\Omega_A).$$

Summarizing, the now elementary Gerschgorin Circle Theorem [3], applied to a particular matrix A , actually gives eigenvalue bounds for a set Ω_A of related matrices. Our main result is that the *exact* spectrum $S(\Omega_A)$ of Ω_A can be determined from extensions of the Gerschgorin Circle Theorem based only on positive diagonal similarity transformations, permutation matrices, and intersections.

In § 4, we include an extension of a result of [6] concerning the number of eigenvalues of any $B \in \Omega_A$ in a bounded component of $G^\phi(\Omega_A)$. Finally, in § 5 we include several examples to show how $S(\Omega_A)$ can be determined.

2. The Function $\nu_\phi(\sigma)$. In order to determine $G^\phi(\Omega_A)$, let σ be any complex number, and consider the real $n \times n$ matrix $Q^\phi(\sigma) = (q_{i,j})$ whose entries are defined by

$$(2.1) \quad q_{i,j} = (-1)^{\delta_{i,j}} |a_{i,\phi(j)} - \sigma \delta_{i,\phi(j)}|, \quad 1 \leq i, j \leq n.$$

Since the off-diagonal entries of $Q^\phi(\sigma)$ are nonnegative, then $Q^\phi(\sigma)$ is *essentially nonnegative* [2; 5, p. 260], and hence we can associate with the matrix $Q^\phi(\sigma)$ the real number $\nu_\phi(\sigma)$, where $\nu_\phi(\sigma)$ is the (possibly multiple) eigenvalue of $Q^\phi(\sigma)$ with largest real part. From the Perron-Frobenius theory of nonnegative matrices [5, pp. 46-47], $\nu_\phi(\sigma)$ corresponds to a nonnegative eigenvector $\mathbf{y} \geq \mathbf{0}$, i.e., $Q^\phi(\sigma)\mathbf{y} = \nu_\phi(\sigma)\mathbf{y}$, and it is further known that

$$(2.2) \quad \nu_\phi(\sigma) = \inf_{u > 0} \max_{1 \leq i \leq n} \left\{ \frac{(Q^\phi(\sigma)\mathbf{u})_i}{u_i} \right\}.$$

We remark that $\nu_\phi(\sigma)$ is a continuous function of σ .

THEOREM 1. *Let $A = (a_{i,j})$ be an $n \times n$ complex matrix, let ϕ be any permutation, and let σ be a complex number. Then, $\sigma \in G^\phi(\Omega_A)$ if and only if $\nu_\phi(\sigma) \geq 0$.*

Proof. From the definitions of $Q^\phi(\sigma)$ in (2.1) and $r_i^\phi(\sigma; \mathbf{x})$ in (1.7)-(1.7'), it follows that

$$(2.3) \quad r_i^\phi(\sigma; \mathbf{x}) = \left(\frac{x_{\phi(i)}}{x_i} \right) \left[\frac{(Q^\phi(\sigma)\mathbf{z})_i}{z_i} \right], \text{ where } z_i \equiv x_{\phi(i)}.$$

Now, if $\sigma \in G^\phi(\Omega_A)$, then $\sigma \in G^\phi(\mathbf{x})$ for every $\mathbf{x} > \mathbf{0}$. But for every $\mathbf{x} > \mathbf{0}$, there is an i such that $\sigma \in G_i^\phi(\mathbf{x})$, so that $r_i^\phi(\sigma; \mathbf{x}) \geq 0$. Since $\mathbf{x} > \mathbf{0}$, then $(x_{\phi(i)}/x_i)$ is positive for all $1 \leq i \leq n$, and it therefore follows from (2.2) that

$$\max_{1 \leq i \leq n} [(Q^\phi(\sigma)\mathbf{z})_i/z_i] \geq 0 \text{ for every } \mathbf{x} > \mathbf{0}.$$

Clearly, as $\mathbf{x} > \mathbf{0}$ runs over all positive vectors, so does the corresponding vector $\mathbf{z} > \mathbf{0}$. Hence, $\nu_\phi(\sigma) \geq 0$ from (2.2). Conversely, assume that $\nu_\phi(\sigma) \geq 0$. From (2.2) and (2.3), it follows that $r_i^\phi(\sigma; \mathbf{x}) \geq 0$ for some i for every $\mathbf{x} > \mathbf{0}$. Hence, $\sigma \in G^\phi(\mathbf{x})$ for every $\mathbf{x} > \mathbf{0}$, and thus $\sigma \in G^\phi(\Omega_A)$, which completes the proof.

Our interest turns now to the boundary $\partial G^\phi(\Omega_A)$ of the minimal Gerschgorin set $G^\phi(\Omega_A)$. As usual, it is defined by

$$(2.4) \quad \partial G^\phi(\Omega_A) = \overline{G^\phi(\Omega_A)} \cap \overline{G^\phi(\Omega_A)'},$$

where $\overline{G^\phi(\Omega_A)'}$ is the closure of the complement $G^\phi(\Omega_A)'$ of $G^\phi(\Omega_A)$. It follows from Theorem 1 that $G^\phi(\Omega_A)'$ is the set of all σ which satisfy $\nu_\phi(\sigma) < 0$. Similarly, the boundary $\partial G^\phi(\Omega_A)$ of the minimal Gerschgorin set is the set of all σ for which $\nu_\phi(\sigma) = 0$, and to which there exists a sequence of complex numbers $\{z_j\}_{j=1}^\infty$ with $\lim_{j \rightarrow \infty} z_j = \sigma$ such that $\nu_\phi(z_j) < 0$.

As in [6], we now show that every point of the boundary $\partial G^\phi(\Omega_A)$ is an eigenvalue of some matrix $B \in \Omega_A$.

THEOREM 2. *Let $A = (a_{i,j})$ be an $n \times n$ complex matrix, and let ϕ be any permutation. If $\nu_\phi(\sigma) = 0$, then σ is an eigenvalue of some matrix $B \in \Omega_A$, and thus $\sigma \in S(\Omega_A)$.*

Proof. If $\nu_\phi(\sigma) = 0$, then there exists a vector $\mathbf{y} \geq \mathbf{0}$ with $\mathbf{y} \neq \mathbf{0}$ such that $Q^\phi(\sigma)\mathbf{y} = \mathbf{0}$. Writing $(\sigma - a_{k,k}) = |\sigma - a_{k,k}| \exp(i\psi_k)$, $1 \leq k \leq n$, let the $n \times n$ matrix $B = (b_{k,j})$ be defined by

$$(2.5) \quad b_{k,k} = a_{k,k}; b_{k,j} = |a_{k,j}| \exp i \{ \psi_k + \pi[-1 + \delta_{k,\phi(k)} + \delta_{j,\phi(k)}] \}, k \neq j.$$

It is evident that $B \in \Omega_A$, and if $y_j = z_{\phi(j)}$, it can be verified (upon considering separately the cases when $\phi(i) = i$ and $\phi(i) \neq i$) that $Q^\phi(\sigma)y = 0$ is equivalent to

$$(2.6) \quad \sum_{j=1}^n b_{k,j} z_j = \sigma z_k, \quad 1 \leq k \leq n.$$

Since $y \neq 0$, then $z \neq 0$, and we conclude from (2.6) that σ is an eigenvalue of B , which completes the proof.

In order to prove a somewhat stronger result, let $\sigma \in \partial G^\phi(\Omega_A)$. Then, $\nu_\phi(\sigma) = 0$ and $\sigma \in S(\Omega_A)$. But as $S(\Omega_A) \subset G^\phi(\Omega_A)$ from (1.11), we have the

COROLLARY 1. *Let A be an $n \times n$ complex matrix. Then, for any permutation ϕ ,*

$$(2.7) \quad \partial G^\phi(\Omega_A) \subset \partial S(\Omega_A).$$

In [6], an interesting geometrical property of the boundary $\partial G^\phi(\Omega_A)$ was given when ϕ was the identity permutation, and A was assumed to be irreducible. In that case, each boundary point of $G^\phi(\Omega_A)$ was shown to be the intersection of n Gerschgorin circles. An analogous result is true for an arbitrary permutation ϕ , under slightly stronger hypotheses.

COROLLARY 2. *Let A be an $n \times n$ complex matrix, let ϕ be any permutation, and let $\sigma \in \partial G^\phi(\Omega_A)$. If $Q^\phi(\sigma)$ is irreducible, then there exists a vector $x > 0$ such that $\sigma \in \partial G_i^\phi(x)$ for all $1 \leq i \leq n$.*

Proof. If $Q^\phi(\sigma)$ is irreducible, then $Q^\phi(\sigma)$ is essentially positive [5, p. 257]. Thus, there exists a vector $z > 0$ such that $Q^\phi(\sigma)z = \nu_\phi(\sigma)z$. But, if $\sigma \in \partial G^\phi(\Omega_A)$, then $\nu_\phi(\sigma) = 0$, and $Q^\phi(\sigma)z = 0$. Letting $x > 0$ be defined component-wise by $z_i = x_{\phi(i)}$, it then follows from (2.3) that $r_i^\phi(\sigma; x) = 0$ for all $1 \leq i \leq n$. Now, $r_i^\phi(\sigma; x)$ is obviously a continuous function of σ from (1.7)–(1.7'), and from (1.8) we deduce that $\partial G_i^\phi(x) = \{ \mu \mid r_i^\phi(\mu; x) = 0 \}$. Hence, $\sigma \in \partial G_i^\phi(x)$ for all $1 \leq i \leq n$, which completes the proof.

We remark that if ϕ is the identity permutation, then $Q^\phi(\sigma)$ is irreducible for any σ if and only if A is irreducible. For general ϕ , it is not difficult to show that A irreducible implies that $Q^\phi(\sigma)$ is irreducible when $\sigma \neq a_{i,i}$ for any i .

3. Main Result. We shall now show that $S(\Omega_A) = H(\Omega_A) \equiv \bigcap_\phi G^\phi(\Omega_A)$. Since $S(\Omega_A) \subset H(\Omega_A)$ by (1.12), it suffices to prove that

$S(\Omega_A)' \subset H(\Omega_A)'$, where $S(\Omega_A)'$ denotes the complement of $S(\Omega_A)$. This last inclusion will follow quite easily from the following theorem of Camion and Hoffman [1]:

Given an arbitrary $n \times n$ complex matrix $B = (b_{i,j})$, let $\mathring{\Omega}_B$ be the set of all matrices $C = (c_{i,j})$ with $|c_{i,j}| = |b_{i,j}|$ for all $1 \leq i, j \leq n$. Then, if all matrices $C \in \mathring{\Omega}_B$ are nonsingular, there exists a positive diagonal matrix $X = \text{diag}(x_1, \dots, x_n)$, $x_i > 0$, and a permutation matrix $P_\phi = (\delta_{i,\phi(j)})$ such that the matrix $M \equiv BXP_\phi = (m_{i,j})$ is strictly diagonally dominant, i.e.,

$$(3.1) \quad |m_{i,i}| > \sum_{j \neq i} |m_{i,j}| \quad \text{for all } 1 \leq i \leq n.$$

We first prove

LEMMA 1. $\sigma \in S(\Omega_A)'$ if and only if each $R \in \mathring{\Omega}_{A-\sigma I}$ is nonsingular.

Proof. It is clear that each $R \in \mathring{\Omega}_{A-\sigma I}$ can be uniquely expressed as $R = D(B - \sigma I)$, where $D = \text{diag}(e^{i\psi_1}, \dots, e^{i\psi_n})$, ψ_j is real, and $B \in \Omega_A$. Then, $\sigma \in S(\Omega_A)'$ implies that $\det(B - \sigma I) \neq 0$ for any $B \in \Omega_A$. But as $|\det D| = 1$, then $\det R = \det D \cdot \det(B - \sigma I) \neq 0$ for any $R \in \mathring{\Omega}_A$. The converse follows similarly.

Now, suppose $\sigma \in S(\Omega_A)'$. From Lemma 1 and the result of Camion and Hoffman applied to $B = A - \sigma I$, there exists a positive diagonal matrix $X = \text{diag}(x_1, \dots, x_n)$ and a permutation matrix $P_\phi = (\delta_{i,\phi(j)})$ such that the matrix $M \equiv (A - \sigma I)XP_\phi \equiv (m_{i,j})$ is strictly diagonally dominant, where

$$(3.2) \quad m_{i,j} = (a_{i,\phi(j)} - \sigma \delta_{i,\phi(j)})x_{\phi(j)}.$$

Comparing (3.2) with the definition of $Q^\phi(\sigma)$ in (2.1) and setting $z_j \equiv x_{\phi(j)}$, $1 \leq j \leq n$, (3.1) can be equivalently expressed as

$$(3.3) \quad 0 > \sum_{j \neq i} |m_{i,j}| - |m_{i,i}| = (Q^\phi(\sigma)z)_i, \quad 1 \leq i \leq n.$$

Since $z > 0$, it follows from (2.2) that $\nu_\phi(\sigma) < 0$, and hence from Theorem 1 we deduce that $\sigma \notin G^\phi(\Omega_A)$. Consequently, $\sigma \notin S(\Omega_A)$ implies that $\sigma \notin G^\phi(\Omega_A)$, which in turn implies that $\sigma \notin H(\Omega_A)$, or

$$(3.4) \quad S(\Omega_A)' \subset H(\Omega_A)'.$$

This, coupled with the result that $S(\Omega_A) \subset H(\Omega_A)$, gives us

THEOREM 3. *Let $A = (a_{i,j})$ be any $n \times n$ complex matrix. Then*

$$S(\Omega_A) = H(\Omega_A).$$

4. *Disconnected minimal gerschgorin sets.* A familiar result of Gerschgorin [3] states that if k disks of the Gerschgorin set $G^I(x)$ (where I is the identity permutation) are disjoint from the remaining $n - k$ disks, then these k disks contain exactly k eigenvalues of any matrix $B \in \Omega_A$. In this section, we give a generalization of this result (cf. Theorem 5 of [6]). For a given $n \times n$ matrix $A = (a_{i,j})$ and an arbitrary permutation ϕ , let $G_j^\phi(\Omega_A)$ denote the nonempty disjoint closed connected components of the minimal Gerschgorin set $G^\phi(\Omega_A)$:

$$(4.1) \quad G^\phi(\Omega_A) = \bigcup_{j=1}^m G_j^\phi(\Omega_A), \quad 1 \leq m \leq n.$$

For each *bounded* component $G_j^\phi(\Omega_A)$, let the *order* s_j^ϕ be defined as the number of diagonal elements $a_{i,i}$ of A contained in $G_j^\phi(\Omega_A)$ for which $\phi(i) = i$. We shall show that each matrix $B \in \Omega_A$ contains exactly s_j^ϕ eigenvalues in each bounded component $G_j^\phi(\Omega_A)$ of the minimal Gerschgorin set $G^\phi(\Omega_A)$.

To begin, we enlarge the set Ω_A . An $n \times n$ matrix $B = (b_{i,j})$ is defined to be an element of the extended set Ω_A^ϕ if

$$(4.2) \quad \begin{cases} |b_{i,i} - a_{i,i}| \leq |a_{i,\phi(i)}|, & \phi(i) \neq i, \\ |b_{i,j}| \leq |a_{i,j}|, & 1 \leq i, j \leq n, \text{ for which } j \neq i \text{ and } j \neq \phi(i). \end{cases}$$

Clearly, $\Omega_A \subset \Omega_A^\phi$.

LEMMA 2. *Given $B \in \Omega_A^\phi$, then $G^\phi(\Omega_B) \subset G^\phi(\Omega_A)$.*

Proof. For any vector $u > 0$ and any complex number σ , consider the vector $Q_B^\phi(\sigma)u$, where we are using an obvious subscript notation. With $B \in \Omega_A^\phi$, one verifies from (4.2) and (2.1) that $Q_B^\phi(\sigma)u \leq Q_A^\phi(\sigma)u$ for any $u > 0$ and any σ , from which it follows that

$$(4.3) \quad \max_{1 \leq i \leq n} \left\{ \frac{(Q_B^\phi(\sigma)u)_i}{u_i} \right\} \leq \max_{1 \leq i \leq n} \left\{ \frac{(Q_A^\phi(\sigma)u)_i}{u_i} \right\}.$$

Thus, from (2.2), $\nu_{\phi,B}(\sigma) \leq \nu_{\phi,A}(\sigma)$. Hence, by Theorem 1, $\sigma \in G^\phi(\Omega_B)$ implies that $\sigma \in G^\phi(\Omega_A)$, which completes the proof.

For this extended set Ω_A^ϕ , we remark that it can be further shown that $S(\Omega_A^\phi) = G^\phi(\Omega_A)$ for any permutation ϕ . This generalizes another result (Theorem 6) of [6].

In the spirit of Gerschgorin's original continuity argument [3], we prove

THEOREM 4. Let $A = (a_{i,j})$ be any $n \times n$ complex matrix, and let ϕ be any permutation. If $G^\phi(\Omega_A)$ has a bounded component $G_j^\phi(\Omega_A)$ of order s_j^ϕ , then, for any matrix $B \in \Omega_A$, B contains exactly s_j^ϕ eigenvalues in $G_j^\phi(\Omega_A)$.

Proof. For any $B = (b_{i,j}) \in \Omega_A$, consider the family of matrices $B_m(\alpha) = (b_{i,j}(\alpha))$ defined by

$$(4.4) \quad \begin{cases} b_{i,i}(\alpha) = b_{i,i}, & 1 \leq i \leq n; \\ b_{i,\phi(i)}(\alpha) = b_{i,\phi(i)}[m(1-\alpha) + \alpha] & \text{when } \phi(i) \neq i; \\ b_{i,j}(\alpha) = \alpha b_{i,j} & \text{for any } 1 \leq i, j \leq n \text{ for which } j \neq i \text{ and } j \neq \phi(i). \end{cases}$$

By definition, $B_m(\alpha) \in \Omega_A^\phi$ for all $0 \leq \alpha \leq 1$ and all $m \geq 1$, and $B_m(1) = B$. Moreover, $B_m(\alpha) \in \Omega_{B_m(\alpha')}^\phi$ for all $0 \leq \alpha \leq \alpha' \leq 1$. Thus, from Lemma 2, $G^\phi(\Omega_{B_m(\alpha)}) \subset G^\phi(\Omega_A)$ for all $0 \leq \alpha \leq 1$ and all $m \geq 1$, and it is clear that the set $G^\phi(\Omega_{B_m(\alpha)})$ increases monotonically with α . We shall show that $B_m(0)$ has exactly s_j^ϕ eigenvalues in the bounded component $G_j^\phi(\Omega_A)$, and the theorem will follow by continuously increasing α from zero to unity.

From (4.4), the only possibly nonzero entries of the matrix $B_m(0)$ are $b_{i,i}(0)$ and $b_{i,\phi(i)}(0)$ where $\phi(i) \neq i$. Hence, by considering the disjoint cycles of the permutation ϕ , we can find an $n \times n$ permutation matrix P such that

$$(4.5) \quad PB_m(0)P^T = \begin{bmatrix} B_{1,1} & & 0 \\ & B_{2,2} & \\ 0 & & \ddots \\ & & & B_{N,N} \end{bmatrix}, \quad 1 \leq N < n.$$

Here, $B_{1,1}$ is a diagonal matrix corresponding to all disjoint cycles with $\phi(i) = i$. The other matrices $B_{j,j}$ have the cyclic form

$$(4.6) \quad B_{j,j} = \begin{bmatrix} b_{1,1}^{(j)} & b_{1,2}^{(j)} & 0 \\ & \ddots & \vdots \\ & 0 & b_{r_{j-1},r_j}^{(j)} \\ b_{r_j,1}^{(j)} & & b_{r_j,r_j}^{(j)} \end{bmatrix}, \quad 2 \leq j \leq N,$$

where the off-diagonal entries of $B_{j,j}$ are, from (4.4), given by $mb_{i,\phi(i)}$, $\phi(i) \neq i$. Obviously, the eigenvalues of all the $B_{j,j}$ are the eigenvalues of $B_m(0)$.

The spectrum of matrices of the form (4.6) is discussed in Example 1 of the next section, and in § 6 of [6]. We now assert that

$$(4.7) \quad |b_{1,2}^{(j)} b_{2,3}^{(j)} \cdots b_{r_j,1}^{(j)}| \neq 0 \quad \text{for any } 2 \leq j \leq N.$$

Otherwise, $b_{k,\phi(k)} = 0$ for some integer k , where $\phi(k) \neq k$, and, as shown

in the next section, this implies that $G^\phi(\Omega_A)$ is the entire complex plane. This contradicts the hypothesis that $G^\phi(\Omega_A)$ has a bounded component. From (4.4), we can write the product in (4.7) as $m^{r_j} \cdot K_j$, where K_j is independent of m and α . Then, it is readily verified that the eigenvalues λ of $B_{j,j}$ satisfy

$$(4.8) \quad \prod_{k=1}^{r_j} |b_{k,k}^{(j)} - \lambda| = m^{r_j} \cdot K_j, \quad 2 \leq j \leq N,$$

for any $B_m(0)$ derived from $B \in \Omega_A$. Since $B_m(0) \in \Omega_A^\phi$ for all $m \geq 1$, we may choose m to be arbitrarily large, and it is clear from (4.8) that the eigenvalues of $B_{j,j}$ must lie in an unbounded component of $G^\phi(\Omega_A)$ for any $2 \leq j \leq N$. Hence, the number of eigenvalues of $B_m(0)$ which lie in the bounded component $G_j^\phi(\Omega_A)$ is just the number of diagonal entries of $B_{1,1}$ in $G_j^\phi(\Omega_A)$, which by definition is precisely s_j^ϕ . Now, increasing α continuously from zero to unity, it follows that B has exactly s_j^ϕ eigenvalues in $G_j^\phi(\Omega_A)$, which completes the proof.

We remark that the order s_j^ϕ of a bounded component $G_j^\phi(\Omega_A)$ is a positive integer. For, if s_j^ϕ were zero, no $B \in \Omega_A$ would have an eigenvalue in $G_j^\phi(\Omega_A)$, so that $S(\Omega_A) \cap G_j^\phi(\Omega_A)$ would be empty, which is a contradiction.

5. Some examples. We now give three examples to illustrate our results concerning the sets $S(\Omega_A)$, $G^\phi(\Omega_A)$, and $H(\Omega_A)$.

EXAMPLE 1. It was previously shown [6] for the matrix

$$(5.1) \quad A = \begin{bmatrix} a_{1,1} & a_{1,2} & & & 0 \\ & 0 & a_{2,2} & a_{2,3} & \\ & & & & a_{n-1,n} \\ & 0 & & & \\ a_{n,1} & & & & a_{n,n} \end{bmatrix},$$

where

$$(5.2) \quad |a_{1,2} a_{2,3} \cdots a_{n,1}| = 1,$$

that $\partial G^I(\Omega_A) = S(\Omega_A)$, I being the identity permutation. Let ψ be the permutation¹ $(1\ 2\ 3 \cdots n)$. If ϕ is any permutation other than ψ or I , there is a positive integer k , $1 \leq k \leq n$, such that $\phi(k) \neq k$, and $\phi(k) \neq \psi(k)$, so that $a_{k,\phi(k)} = 0$. Thus, from (1.7'),

$$(5.2) \quad r_k^\phi(\sigma; \mathbf{x}) = |\sigma - a_{k,k}| + |a_{k,\psi(k)}| x_{\psi(k)}/x_k > 0$$

for all $\mathbf{x} > 0$, and for all complex numbers σ . Hence, we deduce from

¹ That is, in this section we are describing a permutation by its disjoint cycles.

(2.2), (2.3), and Theorem 1 that $G^\phi(\Omega_A)$ is the entire complex plane. This argument shows more generally for an arbitrary matrix A that any permutation ϕ which places a zero on the diagonal of $Q^\phi(\sigma)$ yields a minimal Gerschgorin set $G^\phi(\Omega_A)$ which is the entire complex plane.

For $\phi = I$, it was shown [6] for the matrix of (5.1) that

$$(5.3) \quad G^I(\Omega_A) = \left\{ \sigma \mid \prod_{i=1}^n |\sigma - a_{i,i}| \leq 1 \right\}$$

and in an identical fashion, we can show that

$$(5.4) \quad G^\psi(\Omega_A) = \left\{ \sigma \mid \prod_{i=1}^n |\sigma - a_{i,i}| \geq 1 \right\}.$$

Hence, it follows that

$$(5.5) \quad S(\Omega_A) = H(\Omega_A) = G^I(\Omega_A) \cap G^\psi(\Omega_A) = \partial G^I(\Omega_A).$$

EXAMPLE 2. Consider the matrix

$$(5.6) \quad A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

In this case, there are only three permutations, corresponding to $\phi = I$, $\phi = (13)$, and $\phi = (23)$, for which $G^\phi(\Omega_A)$ is not the entire complex plane, and it is readily verified that

$$(5.7) \quad \begin{cases} G^I(\Omega_A) = \{ \sigma \mid |2 - \sigma|^2 \cdot |1 - \sigma| \leq |1 - \sigma| + |2 - \sigma| \}, \\ G^{(13)}(\Omega_A) = \{ \sigma \mid |2 - \sigma|^2 \cdot |1 - \sigma| \geq |1 - \sigma| - |2 - \sigma| \}, \\ G^{(23)}(\Omega_A) = \{ \sigma \mid |2 - \sigma|^2 \cdot |1 - \sigma| \geq -|1 - \sigma| + |2 - \sigma| \}. \end{cases}$$

The boundaries $\partial G^\phi(\Omega_A)$ are obviously determined by choosing the equality signs in (5.7). The spectrum $S(\Omega_A)$ in this case is a multiply connected region and is illustrated in Figure 1.

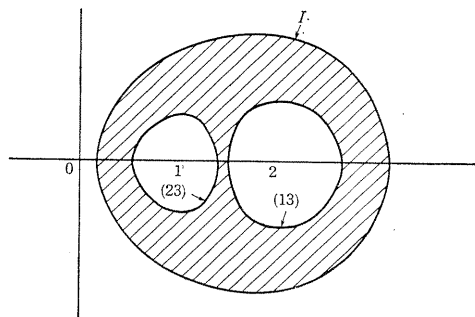


Fig. 1

EXAMPLE 3. Consider the matrix

$$(5.8) \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -5 & -1 & -1 \end{bmatrix},$$

which is the companion matrix of the polynomial

$$p_4(z) = z^4 + z^3 + z^2 + 5z + 1.$$

As previously shown, any permutation ϕ which places a zero on the diagonal of $Q^\phi(\sigma)$ yields a minimal Gerschgorin set $G^\phi(\Omega_A)$ which is the entire complex plane. Consequently, we need consider only the permutations I , (1234) , (234) , and (34) . The associated minimal Gerschgorin sets are given by

$$(5.9) \quad \begin{cases} G^I(\Omega_A) = \{\sigma \mid |\sigma|^3 \cdot |1 + \sigma| \leq 1 + 5|\sigma| + |\sigma|^2\}, \\ G^{(1234)}(\Omega_A) = \{\sigma \mid |\sigma|^3 \cdot |1 + \sigma| \geq 1 - 5|\sigma| - |\sigma|^2\}, \\ G^{(234)}(\Omega_A) = \{\sigma \mid |\sigma|^3 \cdot |1 + \sigma| \geq -1 + 5|\sigma| - |\sigma|^2\}, \\ G^{(34)}(\Omega_A) = \{\sigma \mid |\sigma|^3 \cdot |1 + \sigma| \geq -1 - 5|\sigma| + |\sigma|^2\}. \end{cases}$$

The last minimal Gerschgorin set $G^{(34)}(\Omega_A)$ is the entire complex plane, and thus yields no boundary components of $S(\Omega_A)$. The set $G^{(234)}(\Omega_A)$ yields, however, two separate boundaries, and $G^{(1234)}(\Omega_A)$ has a bounded component. Applying Theorem 4, we can assert that each matrix of the set Ω_A has exactly one eigenvalue in this component, and hence each matrix of Ω_A has exactly one eigenvalue in the inner annular region of Figure 2.

These examples have interesting common features. In each ex-

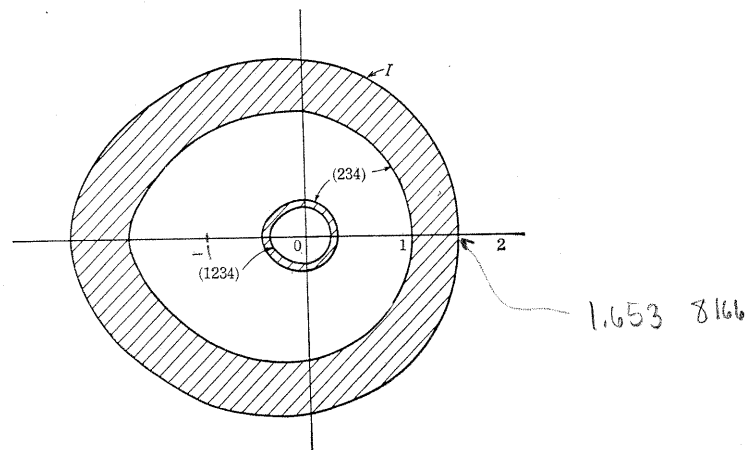


Fig. 2

ample, the minimum number of permutations necessary to define all the boundary components of $S(\Omega_A)$ does not exceed the order n of the matrix A . Similarly, the total number of boundary components of $S(\Omega_A)$ does not exceed $2n$. We conjecture this to be true in general. We do point out that examples can be constructed where these upper bounds are attained.

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