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On Smallest Isolated Gerschgorin Disks for Eigenvalues. II

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§ 1. Introduction

If a given irreducible $n \times n$ complex matrix A admits an isolated Gerschgorin disk, then it is well known that this disk contains exactly one eigenvalue of A. For improved bounds for this isolated eigenvalue, it is natural to consider positive diagonal similarity transformations applied to A to reduce the radius of this isolated disk, and in fact, it was shown in [3] and [4] that algorithms exist which yield the *smallest* such isolated Gerschgorin disk under positive diagonal similarity transformations which contains this isolated eigenvalue of A.

The main purpose of this note is to show that the basic iteration of the first (linear) algorithm of [4] *majorizes* a similar iteration which can be applied directly to the original $n \times n$ matrix A, and this latter iteration actually *converges* to the isolated eigenvalue of the matrix A. In other words, the first algorithm of [4] can be used to *directly* estimate the isolated eigenvalue of A, rather than its best bound via Gerschgorin-type arguments.

§ 2. Main Result

To begin, we assume as in [4] that the given irreducible $n \times n$ complex matrix $A = (a_{i,j})$ admits an *isolated first Gerschgorin disk*, i.e., there exists a vector $\boldsymbol{x} > \mathbf{0}$ with $\boldsymbol{x}^T = (x_1, x_2, \ldots, x_n)$ such that

$$|a_{1,1} - a_{j,j}| - \Lambda_j(\boldsymbol{x}) - \Lambda_1(\boldsymbol{x}) \geqq 0 \quad \text{for all} \quad 2 \leqq j \leqq n, \tag{1}$$

where

$$\Lambda_j(\boldsymbol{x}) = \frac{1}{x_j} \sum_{\substack{k=1 \\ k \neq j}}^{n} |a_{j,k}| \, x_k, \qquad 1 \leqq j \leqq n. \tag{2}$$

Thus, P_1, defined as the set of all vectors $\boldsymbol{x} > \mathbf{0}$ for which (1) is valid, is nonempty. We partition A as follows:

$$A = \left[\begin{array}{c|c} a_{1,1} & \hat{\boldsymbol{\beta}}^T \\ \hline \hat{\boldsymbol{\gamma}} & A_{2,2} \end{array} \right] \tag{3}$$

where $A_{2,2}$ is an $(n-1) \times (n-1)$ matrix, and $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\gamma}}$ are vectors with $(n-1)$ components. With $B = A - a_{1,1} I_n$ and $\tilde{B} = A_{2,2} - a_{1,1} I_{n-1}$, it follows from (3) that

$$B = \left[\begin{array}{c|c} 0 & \hat{\boldsymbol{\beta}}^T \\ \hline \hat{\boldsymbol{\gamma}} & \tilde{B} \end{array} \right]. \tag{4}$$

Now, we define an $n \times n$ matrix Q, used in [4], which will majorize the matrix B. Specifically, let[1]

$$(5) \qquad Q = \left[\begin{array}{c|c} 0 & |\hat{\boldsymbol{\beta}}|^T \\ \hline -|\hat{\boldsymbol{\gamma}}| & \tilde{Q} \end{array}\right],$$

where the $(n-1)\times(n-1)$ matrix $\tilde{Q}=(\tilde{q}_{i,j})$, $1 \leqq i, j \leqq n-1$, is defined from the $(n-1)\times(n-1)$ matrix $\tilde{B}=(\tilde{b}_{i,j})$, $1\leqq i,j\leqq n-1$, by

$$(6) \qquad \tilde{q}_{i,i}=|\tilde{b}_{i,i}|; \quad \tilde{q}_{i,j}=-|\tilde{b}_{i,j}|, \quad i\neq j, \quad 1\leqq i,j\leqq n-1.$$

As in [4], it is convenient to normalize all vectors $\boldsymbol{x}\in P_1$, by setting $x_1=1$. Denoting the remaining column vector with $n-1$ components by $\hat{\boldsymbol{x}}$, let

$$(7) \qquad \sup_{\boldsymbol{x}\in P_1} |\hat{\beta}|^T \hat{\boldsymbol{x}} \equiv \sigma; \qquad \inf_{\boldsymbol{x}\in P_1} |\hat{\beta}|^T \hat{\boldsymbol{x}} \equiv \mu.$$

Since A is irreducible and P_1 is nonempty, it follows that both μ and σ are positive real numbers. We shall also assume for simplicity that $0<\mu<\sigma$, the case $\mu=\sigma$ being essentially trivial.

Lemma. For any complex number z with $|z|\leqq\sigma$, $\tilde{B}-zI_{n-1}$ is an H-matrix, and

$$(8) \qquad |(\tilde{B}-zI_{n-1})^{-1}| \leqq (\tilde{Q}-|z|\,I_{n-1})^{-1}.$$

Proof. If $|z|\leqq\sigma$, then $(\tilde{Q}-|z|\,I_{n-1})$ is an M-matrix from Lemma 2 of [4]. Moreover, from

$$(9) \qquad \begin{aligned} |\tilde{b}_{i,i}-z| &\geqq |\tilde{b}_{i,i}|-|z| = \tilde{q}_{i,i}-|z|, && 1\leqq i\leqq n-1, \\ |\tilde{b}_{i,j}| &= -\tilde{q}_{i,j}, \quad i\neq j, && 1\leqq i,j\leqq n-1, \end{aligned}$$

it follows that $|\tilde{B}-zI_{n-1}| \geqq \tilde{Q}-|z|\,I_{n-1}$, which proves that $\tilde{B}-zI_{n-1}$ is an H-matrix as defined originally by OSTROWSKI [2]. The inequality of (8) is then a well known consequence of [2], completing the proof.

The previous lemma shows us that $(\tilde{B}-zI_{n-1})^{-1}$ is defined for any z with $|z|\leqq\sigma$. With this, we now define the following mapping $T(z)$ for any complex z with $|z|\leqq\sigma$:

$$(10) \qquad T(z) = -\hat{\boldsymbol{\beta}}^T (\tilde{B}-zI_{n-1})^{-1} \tilde{\boldsymbol{\gamma}},$$

and we consider the *method of successive substitution*

$$(11) \qquad \lambda_{k+1} = T(\lambda_k)$$

applied to any initial λ_0 with $|\lambda_0|\leqq\sigma$. In the notation of [4], we can write that

$$(12) \qquad g(s) = |\hat{\boldsymbol{\beta}}|^T (\tilde{Q}-sI_{n-1})^{-1} |\hat{\boldsymbol{\gamma}}|$$

for *any* real number $s\leqq\sigma$. From (8), (10), and (11), it follows that

$$(13) \qquad |\lambda_{k+1}| \leqq g(|\lambda_k|),$$

[1] Here, we are using the notation that if $C=(c_{i,j})$ is an $m\times n$ matrix, then $|C| = (|c_{i,j}|)$ is the associated $m \times n$ matrix with nonnegative elements.

if $|\lambda_k| \leqq \sigma$. Thus, defining $t_0 \equiv |\lambda_0| \leqq \sigma$, and $t_{k+1} \equiv g(t_k)$, $k \geqq 0$, we have from Theorem 1 of [4] that the sequence $\{t_k\}_{k=0}^{\infty}$ is *monotone decreasing* with $\lim_{t\to\infty} t_k = \mu > 0$. Moreover, since $g(t)$ is strictly increasing for any $t \leqq \sigma$ (Lemma 3 of [4]), it follows inductively from (13) that

$$|\lambda_k| \leqq t_k, \quad k \geqq 0. \tag{14}$$

This gives us that the sequence $\{\lambda_k\}_{k=0}^{\infty}$ is at least bounded. We now show that the sequence $\{\lambda_k\}_{k=0}^{\infty}$ is convergent. For any λ with $|\lambda| < \sigma$ and any ε sufficiently small, consider the function $T(\lambda+\varepsilon)$. From (10), we can write $T(\lambda+\varepsilon)$ as

$$T(\lambda+\varepsilon) = -\hat{\boldsymbol{\beta}}^T \{I_{n-1} - \varepsilon S(\lambda)\}^{-1} S(\lambda)\, \hat{\boldsymbol{\gamma}}, \tag{15}$$

where

$$S(\lambda) \equiv (\tilde{B} - \lambda I_{n-1})^{-1}. \tag{15'}$$

For ε sufficiently small, expanding the matrix $(I - \varepsilon S(\lambda))^{-1}$ in a power series in ε yields

$$T(\lambda+\varepsilon) - T(\lambda) = -\hat{\boldsymbol{\beta}}^T \{\varepsilon S(\lambda) + \varepsilon^2 S^2(\lambda) + \cdots\} S(\lambda)\, \hat{\boldsymbol{\gamma}}, \tag{16}$$

which shows that $T(\lambda)$ is analytic. Since $|S(\lambda)| \leqq (\tilde{Q} - |\lambda|\, I_{n-1})^{-1}$ from the lemma, then

$$T'(\lambda) = -\hat{\boldsymbol{\beta}}^T S^2(\lambda)\, \hat{\boldsymbol{\gamma}}, \tag{17}$$

so that

$$|T'(\lambda)| \leqq |\hat{\boldsymbol{\beta}}|^T (\tilde{Q} - |\lambda|\, I_{n-1})^{-2} |\hat{\boldsymbol{\gamma}}| = g'(|\lambda|), \tag{18}$$

the last equality following in a similar way from (12). Since $g(s)$ is monotone increasing for any real s with $s \leqq \sigma$, then $g'(s) > 0$ for all $s \leqq \sigma$. Moreover, it can be verified from the results of [4] that $g'(\mu) < 1$ and $g'(\sigma) > 1$ (cf. Fig. 1 of [4]). Thus, there exists a ζ with $\mu < \zeta < \sigma$ such that $g'(s) < 1$ for all $s < \zeta$, and it therefore follows from (18) that $|T'(\lambda)| < 1$ for any complex number with $|\lambda| < \zeta$. This brings us to

Theorem 1. Let A be an irreducible $n \times n$ matrix which admits a first isolated Gerschgorin disk, and assume that the quantities μ and σ of (7) satisfy $\mu < \sigma$. For any $|\lambda_0| < \sigma$, the iterative method $\lambda_{i+1} = T(\lambda_i)$ is convergent, i.e., $\lim_{i\to\infty} \lambda_i = \lambda$, and λ is the unique eigenvalue of the matrix B in the disk $|z| \leqq \mu$.

Proof. For any λ_0 with $|\lambda_0| < \sigma$, we have from (14) that $|\lambda_k| \leqq t_k$ for all $k \geqq 0$, and from [4], we have that the t_k decrease monotonically to μ. Thus for all k sufficiently large, it follows that $|\lambda_k| < \zeta$, and thus $|T'(\lambda_k)| < 1$ for all k sufficiently large. But this is a well known sufficient condition for the convergence of the method of successive approximations. Hence,

$$\lim_{k\to\infty} \lambda_k = \lambda, \quad \text{and} \quad T(\lambda) = \lambda. \tag{19}$$

From (10), this means that

$$\lambda = -\hat{\boldsymbol{\beta}}^T (\tilde{B} - \lambda I_{n-1})^{-1} \hat{\boldsymbol{\gamma}}. \tag{20}$$

As in [1], this implies that λ is an eigenvalue of B. More precisely, the vector $\boldsymbol{w}$ with first component unity and the remaining $n-1$ components given by $\boldsymbol{w} \equiv -(\tilde{B} - \lambda I_{n-1})^{-1}\, \hat{\boldsymbol{\gamma}}$ is then easily seen to be an eigenvector of B, corresponding to the eigenvalue λ. That $|\lambda| \leqq \mu$ is obvious from (14), and that λ is the *unique* eigenvalue of B in the disk $|z| \leqq \mu$ is a simple consequence of the fact that the matrix A admits an isolated first Gerschgorin disk, which completes the proof.

In applying this procedure, we remark that it is sufficient to start with any $\boldsymbol{x} \in P_1$ for which strict inequality is valid for at least one j, $2 \leqq j \leqq n$, in (1). With this vector $\boldsymbol{x}$, one can define $\lambda_0 \equiv \hat{\boldsymbol{\beta}}^T \cdot \hat{\boldsymbol{x}}$; the strict inequality for at least one component in (1) then yields both that $\mu < \sigma$ and that $|\lambda_0| < \sigma$, and convergence is then guaranteed by Theorem 1.

§ 3. An Example

To illustrate the preceding results, consider the following matrix

$$A = \begin{bmatrix} 1 & i/2 & i/2 \\ 1/2 & 4 & i/2 \\ 1/2 & 1/2 & 6 \end{bmatrix}, \tag{21}$$

which was also considered in [4]. As mentioned in [4], each of the eigenvalues of A can be isolated by positive diagonal similarity transformations, and in fact the vector $\xi = (1, 1, 1)^T$ is simultaneously in the associated sets P_1, P_2, and P_3. The following table gives the first four iterates of (11) for each of the eigenvalues of A, together with the actual eigenvalues of A.

Table

k	$1+\lambda_k$	$4+\lambda_k$	$6+\lambda_k$
0	$1+i$	$4.5+0.5i$	7
1	$1.0254-0.1189i$	$4.0822-0.1024i$	$5.9912+0.1318i$
2	$0.9897-0.1255i$	$4.0081-0.0708i$	$5.9935+0.1890i$
3	$0.9896-0.1243i$	$4.0115-0.0638i$	$5.9983+0.1889i$
Actual	$0.9897-0.1243i$	$4.0121-0.0642i$	$5.9982+0.1885i$

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