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Numerical Methods of High-Order Accuracy for Nonlinear Boundary Value Problems

II. Nonlinear Boundary Conditions*

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§ 1. Introduction

In [3, 4], we considered the numerical approximation of the solution of the following real nonlinear two-point boundary value problem

$$\mathscr{L}[u(x)] = f(x, u(x)), \quad 0 < x < 1, \tag{1.1}$$

with Dirichlet boundary conditions

$$D^k u(0) = D^k u(1) = 0, \quad D = \frac{d}{dx}, \quad 0 \leqq k \leqq n-1, \tag{1.2}$$

where

$$\mathscr{L}[u(x)] = \sum_{j=0}^{n} (-1)^{j+1} D^j [p_j(x) D^j u(x)], \quad n \geqq 1. \tag{1.3}$$

Basically, the Rayleigh-Ritz-Galerkin method for (1.1)—(1.2) was applied in [4] to a variety of finite dimensional subspaces, such as polynomial and spline subspaces, these subspaces having been selected in part with an eye toward efficient digital computation.

Our aim here is simply to extend the results of [4] to *nonlinear* boundary conditions. Although such extensions will be explored more fully in [5], we restrict ourselves here, for ease of exposition, to the case $n=1$ of (1.1), i.e.,

$$\mathscr{L}[u(x)] = D\{p_1(x) D u(x)\} - p_0(x) u(x) = f(x, u(x)), \quad 0 < x < 1, \tag{1.4}$$

with boundary conditions

$$D u(0) = \psi_0(u(0)); \quad D u(1) = -\psi_1(u(1)). \tag{1.5}$$

We assume that $p_1(x) \in C^1[0, 1]$, $p_0(x) \in C^0[0, 1]$, and that there exists a constant ω such that

$$p_1(x) \geqq \omega > 0 \quad \text{for all} \quad x \in [0, 1]. \tag{1.6}$$

The given real functions $\psi_0(t)$ and $\psi_1(t)$ are assumed to be continuously differentiable for all real t, and to satisfy

$$\psi_0(0) = 0, \quad \psi_1(0) = 0, \tag{1.7}$$

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and we assume further that there exist two constants a and b such that

$$D\psi_0(t) \geqq a, \quad D\psi_1(t) \geqq b \quad \text{for all real } t. \tag{1.8}$$

We remark that the reduction to the case of (1.7) can be made, without essential loss of generality, by a suitable change of the dependent variable. More explicitly, if the conditions of (1.7) are not satisfied, then define

$$\begin{aligned} u(x) &\equiv v(x) + \eta(x), \\ \eta(x) &\equiv x(1-x)\{x\psi_1(0) + (1-x)\psi_0(0)\}, \\ \tilde{f}(x, v) &\equiv f(x, v + \eta(x)) - \mathscr{L}[\eta(x)], \\ \tilde{\psi}_0(t) &\equiv \psi_0(t) - D\eta(0), \\ \tilde{\psi}_1(t) &\equiv \psi_1(t) + D\eta(1). \end{aligned}$$

The boundary value problem of (1.4)—(1.5) then becomes

$$\mathscr{L}[v(x)] = \tilde{f}(x, v(x)), \quad 0 < x < 1, \tag{1.4'}$$

and

$$Dv(0) = \tilde{\psi}_0(v(0)); \quad Dv(1) = \tilde{\psi}_1(v(1)), \tag{1.5'}$$

where $\tilde{\psi}_0(t)$ and $\tilde{\psi}_1(t)$ now satisfy (1.7). Thus, $u(x)$ is a solution of (1.1)—(1.2) if and only if $v(x) \equiv u(x) - \eta(x)$ is a solution of (1.4')—(1.5'), and the two problems are equivalent.

To give a concrete example of boundary conditions which arise in practice and satisfy (1.7) and (1.8), consider the *linear* boundary conditions (cf. [2]) of

$$\sigma_0 u(0) - \sigma_0' D u(0) = 0; \quad \sigma_1 u(1) + \sigma_1' D u(1) = 0, \quad \sigma_0' \neq 0, \quad \sigma_1' \neq 0, \tag{1.5''}$$

for which $\psi_0(t) = \frac{\sigma_0}{\sigma_0'} t$ and $\psi_1(t) = -\frac{\sigma_1}{\sigma_1'} t$. Problems with boundary conditions (1.5'') arise in linear diffusion theory. Later, in §5, we shall show how radiative-type boundary conditions can also be considered by our formulation.

To begin our discussion, we define S to be the linear space of all real-valued absolutely continuous functions $w(x)$ defined on $[0, 1]$ such that $Dw(x) \in L^2[0, 1]$. As a consequence of the Sobolev Imbedding Theorem [15, p. 174], we remark that S is in reality the well-known Sobolev space $W^{1,2}[0, 1]$. We assume as in [4, Eq. (1.4)] that, given the two real constants a and b of (1.8), there exist a positive constant K and a real constant β such that

$$\begin{aligned} \|w\|_{L^\infty} &\equiv \sup_{x \in [0,1]} |w(x)| \\ &\leqq K \left\{ \int_0^1 [p_1(x)(Dw(x))^2 + (p_0(x) + \beta)(w(x))^2]\, dx \right. \\ &\quad \left. + a p_1(0)(w(0))^2 + b p_1(1)(w(1))^2 \right\}^{\frac{1}{2}} \end{aligned} \tag{1.9}$$

for all $w \in S$. When a and b of (1.8) are nonnegative, we can deduce from the inequality of (1.6) that such constants K and β do exist (see Lemma 1), i.e., the inequality of (1.9) is *not* an added assumption in this case.

Next, we introduce the finite quantity (see Lemma 2)

$$(1.10)\quad \Lambda = \inf_{\substack{w\in S\\ w\neq 0}} \frac{\int_0^1 [p_1(x)\,(Dw(x))^2 + p_0(x)\,(w(x))^2]\,dx + a\,p_1(0)\,(w(0))^2 + b\,p_1(1)\,(w(1))^2}{\int_0^1 (w(x))^2\,dx},$$

and we assume that the functions $f(x, u)$ and $\frac{\partial f(x,u)}{\partial u}$ are real and continuous in both variables, and that there exists a constant γ such that

$$(1.11)\quad \frac{\partial f(x,u)}{\partial u} \equiv f_u(x,u) \geqq \gamma > -\Lambda \quad \text{for all} \quad x\in[0,1], \quad \text{and all real } u,$$

where Λ is defined in (1.10). This latter assumption is similar to that in [4].

With these assumptions, the basic results of [4] carry over rather easily, and for this reason, few proofs will be given here in detail. To outline the subsequent material, §2 briefly lists the basic results patterned after [4]. In §§3 and 4, these basic results are applied to the particular subspaces $P^{(N)}$ of polynomials, and to the subspaces $Sp(L, \pi, z)$ of L-splines, the later including both the Hermite subspaces $H^{(m)}(\pi)$ and the natural spline subspaces $Sp^{(m)}(\pi)$ as special cases. In §5, extensions will be considered, and in §6, numerical results for a representative problem will be described.

§ 2. Variational Formulation

We begin with

Lemma 1. If the constants a and b of (1.8) are nonnegative, then there exist a positive constant K and a real constant β such that (1.9) is valid for all $w\in S$.

Proof. By assumption (1.6), $p_1(x) \geqq \omega > 0$ in $[0, 1]$, and it follows with

$$(2.1)\quad K_1 \equiv \frac{1}{\sqrt{\omega}} \quad \text{and} \quad \beta \equiv \omega + \max_{x\in[0,1]} (-p_0(x))$$

that

$$(2.2)\quad \|w\|_{W^{1,2}} \equiv \left\{\int_0^1 [(Dw(x))^2 + (w(x))^2]\,dx\right\}^{\frac12} \leqq K_1 \left\{\int_0^1 [p_1(x)\,(Dw(x))^2 + (p_0(x)+\beta)\,(w(x))^2]\,dx\right\}^{\frac12}$$

for all $w\in S$, where $\|w\|_{W^{1,2}}$ is the *Sobolev norm* of w. On the other hand, Sobolev's inequality [15, p. 174] in one dimension gives us that there exists a positive constant K_2 such that

$$(2.3)\quad \|w\|_{L^\infty} \leqq K_2 \|w\|_{W^{1,2}} \quad \text{for all} \quad w\in S,$$

and thus combining (2.2) and (2.3) yields

$$(2.4)\quad \|w\|_{L^\infty} \leqq K_3 \left\{\int_0^1 [p_1(x)\,(Dw(x))^2 + (p_0(x)+\beta)\,(w(x))^2]\,dx\right\}^{\frac12}, \quad K_3 \equiv K_1 K_2,$$

for all $w\in S$. The inequality of (1.9) is now evident since, with a and b nonnegative by hypothesis, the terms $a\,p_1(0)\,(w(0))^2$ and $b\,p_1(1)\,(w(1))^2$ in (1.9) are nonnegative, and the right-hand side of (1.9) can be bounded below in terms of the right-hand side of (2.4). Q.E.D.

It is also interesting to remark that the proof of Lemma 1 shows that constants K and β can similarly be determined for the case in which a and b in (1.8) are *slightly negative*. More precisely, with the constant $K_3 = K_1 K_2$ determined in (2.4) from the Sobolev inequality, the assumption that $a \leqq 0$ and $b \leqq 0$ with

$$K_3^2\{|a|\, p_1(0) + |b|\, p_1(1)\} < 1 \tag{2.5}$$

again allows one to find constants K and β satisfying (1.9).

Lemma 2. *With the assumption of* (1.9), *the quantity* Λ *defined in* (1.10), *satisfies* $\Lambda \geqq \frac{1}{K^2} - \beta$, *and* Λ *is thus a finite number.*

Proof. Since $\|w\|_{L^\infty} \geqq \|w\|_{L^2}$, the desired inequality follows immediately from (1.9). Q.E.D.

As in [4], we make the essential hypothesis that (1.4)—(1.5) admits a classical solution. This assumption will be discussed in detail in [5]. Then, as in [4], we have

Theorem 1. *With the assumptions of* (1.9) *and* (1.11), *let* $\varphi(x)$ *be a classical solution of* (1.4)—(1.5). *Then,* $\varphi(x)$ *strictly minimizes the following functional*

$$\begin{aligned} F[w] &= \int_0^1 \left\{ \tfrac{1}{2} p_1(x) \left(D w(x)\right)^2 + \tfrac{1}{2} p_0(x) \left(w(x)\right)^2 + \int_0^{w(x)} f(x, \eta)\, d\eta \right\} dx \\ &\quad + p_1(0) \int_0^{w(0)} \psi_0(\lambda)\, d\lambda + p_1(1) \int_0^{w(1)} \psi_1(\lambda)\, d\lambda, \qquad w \in S, \end{aligned} \tag{2.6}$$

over the space S, *and* $\varphi(x)$ *is the unique solution of* (1.4)—(1.5).

Proof. It is readily verified with the above assumptions that

$$F[w] \geqq F[\varphi] + \frac{(\Lambda + \gamma)}{2} \int_0^1 [w(x) - \varphi(x)]^2\, dx \quad \text{for all} \quad w \in S, \tag{2.7}$$

from which the result follows. Q.E.D.

We now proceed to describe the approximation scheme. Let S_M be any *finite* dimensional subspace of dimension M of S, and let $\{w_i(x)\}_{i=1}^M$ be M linearly independent functions in S_M. Upon considering $F\left[\sum_{j=1}^M u_j w_j(x)\right]$, the inequality of (2.7) allows us to prove, exactly as in Theorem 2 of [4], that the minimization of $F[w]$ over S_M determines a *unique* function in S_M. We state this as

Theorem 2. *With the assumptions of* (1.9) *and* (1.11), *there exists a unique function* $\hat{w}_M(x) = \sum_{j=1}^M \hat{u}_j w_j(x)$ *in* S_M *which minimizes the functional* $F[w]$ *over* S_M.

If $h(x)$ is a continuous function on [0, 1], and a_1 and b_1 are two real constants, define

$$\begin{aligned} \|w\|_{h a_1 b_1} = \Big\{ &\int_0^1 \left[p_1(x) \left(D w(x)\right)^2 + \left(p_0(x) + h(x)\right) \left(w(x)\right)^2\right] dx \\ &+ a_1 p_1(0) \left(w(0)\right)^2 + b_1 p_1(1) \left(w(1)\right)^2 \Big\}^{\frac{1}{2}} \end{aligned} \tag{2.8}$$

for all $w \in S$. If $h(x) \equiv \mu$, we write simply $\|w\|_{\mu a_1 b_1}$ for $\|w\|_{h a_1 b_1}$.

Lemma 3. If $\Gamma \geqq h(x) \geqq \gamma' > -\Lambda$ for all $x \in [0, 1]$, and $a_1 \geqq a$ and $b_1 \geqq b$, then $\|w\|_{h a_1 b_1}$ and $\|w\|_{\gamma a b}$ (where γ is the constant of (1.11)) are both norms on S, and they are moreover *equivalent*.

Proof. From the definition of Λ in (1.10) and the hypotheses, it follows that

$$(\|w\|_{h a_1 b_1})^2 \geqq (\Lambda + \gamma')(\|w\|_{L^2})^2 \quad \text{and} \quad (\|w\|_{\gamma a b})^2 \geqq (\Lambda + \gamma)(\|w\|_{L^2})^2$$

for all $w \in S$. Hence, these quantities are norms on S. Finally, to establish the equivalence of these norms, it can be verified from the hypotheses that

$$c_1 (\|w\|_{h a_1 b_1})^2 \leqq (\|w\|_{\gamma a b})^2 \leqq c_2 (\|w\|_{h a_1 b_1})^2, \tag{2.9}$$

for all $w \in S$, where possible choices for the positive constants c_1 and c_2 are given by

$$c_1 = \left\{1 + \frac{\max(\Gamma - \gamma; 0)}{(\Lambda + \gamma)} + K^2[(a_1 - a) p_1(0) + (b_1 - b) p_1(1)] \cdot \left[1 + \frac{\max(\beta - \gamma; 0)}{(\Lambda + \gamma)}\right]\right\}^{-1}, \tag{2.10}$$

and

$$c_2 = \left(1 + \frac{\max(\gamma - \gamma'; 0)}{(\Lambda + \gamma')}\right), \tag{2.10'}$$

where K, β, Λ, and γ are the constants of (1.9)—(1.11). Q.E.D.

The following consequence of Lemma 3 is proved exactly as in [4].

Corollary. If assumption (1.9) is satisfied for some real β, then it is also satisfied for every γ' with $\gamma' > -\Lambda$.

With this corollary, it is now evident that, for the constant γ of (1.11), the inequality of (1.9) is valid for $\beta = \gamma$, and we write this now as

$$\|w\|_{L^\infty} \leqq K \|w\|_{\gamma a b}, \quad \text{for all} \quad w \in S. \tag{2.11}$$

In the spirit of Lemma 3, we similarly establish

Lemma 4. If the constants a and b of (1.8) are nonnegative, then the Sobolev norm $\|w\|_{W^{1,2}}$ of (2.1) and the norm $\|w\|_{\gamma a b}$ are equivalent on S.

Next, the following a priori bounds for both the solution $\varphi(x)$ and any best approximation $\hat{w}_M(x)$ can be established. The proof, which makes use of the basic assumptions of (1.9)—(1.11), is similar to that of Lemma 4 of [4], and is omitted. We remark that the assumptions of (1.7) are explicitly used at this point.

Lemma 5. Let $w(x)$ be any function in S such that

$$F[w] \leqq 0 = F[0]. \tag{2.12}$$

Then, the following a priori bound is valid:

$$\|w\|_{\gamma a b} \leqq L, \tag{2.13}$$

where L is a known function (cf. [4, Lemma 4]) of Λ, γ, and $\mathscr{M}$ where $\mathscr{M} \equiv \max_{0 \leqq x \leqq 1} |f(x, 0)|$. Consequently, from (2.11),

$$\|w\|_{L^\infty} \leqq L', \tag{2.14}$$

where L' is a known function of $K, \beta, \Lambda, \gamma$, and $\mathscr{M}$.

Having selected some finite-dimensional subspace S_M of S of dimension M, spanned by $\{w_i(x)\}_{i=1}^M$, the unique element $\hat{w}_M(x)$ in S_M which minimized $F[w]$ over S_M can be characterized by the conditions $\frac{\partial F\left[\sum_{i=1}^M \hat{u}_i w_i(x)\right]}{\partial u_j} = 0$, $1 \leq j \leq M$, which gives us the equations

$$\text{(2.15)} \quad \int_0^1 \{p_1(x) D\hat{w}_M(x) D w_j(x) + p_0(x)\hat{w}_M(x) w_j(x) + f(x, \hat{w}_M(x)) w_j(x)\} dx + p_1(0)\psi_0(\hat{w}_M(0)) w_j(0) + p_1(1)\psi_1(\hat{w}_M(1)) w_j(1) = 0, \quad 1 \leq j \leq M.$$

Similarly, if $\varphi(x)$ is a classical solution of (1.4)–(1.5), an integration by parts gives us

$$\text{(2.16)} \quad \int_0^1 \{p_1(x) D\varphi(x) D w_j(x) + p_0(x)\varphi(x) w_j(x) + f(x, \varphi(x)) w_j(x)\} dx + p_1(0)\psi_0(\varphi(0)) w_j(0) + p_1(1)\psi_1(\varphi(1)) w_j(1) = 0, \quad 1 \leq j \leq M.$$

If we define an inner product on S by

$$\text{(2.17)} \quad \langle u, v\rangle_{h_M a_M b_M} \equiv \int_0^1 \{p_1(x) D u(x) D v(x) + p_0(x) u(x) v(x) + h_M(x) u(x) v(x)\} dx + a_M p_1(0) u(0) v(0) + b_M p_1(1) u(1) v(1), \quad u, v \in S,$$

then subtracting (2.16) from (2.15) gives an equation which can be simply expressed as

$$\text{(2.18)} \quad \langle \hat{w}_M - \varphi, w_j\rangle_{h_M a_M b_M} = 0, \quad 1 \leq j \leq M,$$

where

$$\text{(2.19)} \quad \begin{aligned} h_M(x) &= f_u(x, \Theta_1\varphi(x) + (1-\Theta_1)\hat{w}_M(x)), \quad 0 \leq x \leq 1, \\ a_M &= D\psi_0(\Theta_2\varphi(0) + (1-\Theta_2)\hat{w}_M(0)), \\ b_M &= D\psi_1(\Theta_3\varphi(1) + (1-\Theta_3)\hat{w}_M(0)), \quad 0 < \Theta_i < 1, \quad i = 1, 2, 3. \end{aligned}$$

Because the expression of (2.18) shows $\hat{w}_M$ to be the projection of φ on S_M with respect to the inner product of (2.18), it follows that

$$\text{(2.20)} \quad \|\hat{w}_M - \varphi\|_{h_M a_M b_M} = \inf_{w \in S_M} \|w - \varphi\|_{h_M a_M b_M}.$$

On the other hand, the basic assumptions of (1.8) and (1.11) and the result of Lemma 5 show us that the quantities h_M, a_M, and b_M can be bounded above and below by

$$\text{(2.21)} \quad \begin{aligned} \gamma &\leq h_M(x) \leq \Gamma \equiv \sup_{\substack{0 \leq x \leq 1 \\ |u| \leq L'}} f_u(x, u) \\ a &\leq a_M \leq A \equiv \sup_{|t| \leq L'} D\psi_0(t), \\ b &\leq b_M \leq B \equiv \sup_{|t| \leq L'} D\psi_1(t). \end{aligned}$$

Hence, with the inequalities of (2.9), (2.11), (2.20), and (2.21), we obtain the following chain of inequalities

$$\text{(2.22)} \quad \begin{aligned} \|\hat{w}_M - \varphi\|_{L^\infty} &\leq K\|\hat{w}_M - \varphi\|_{\gamma a b} \leq C_1 \|\hat{w}_M - \varphi\|_{h_M a_M b_M} = C_1 \inf_{w \in S_M} \|w - \varphi\|_{h_M a_M b_M} \\ &\leq C_2 \inf_{w \in S_M} \|w - \varphi\|_{\Gamma A B}, \end{aligned}$$

where it is important to observe that the constants C_1 and C_2 are *independent* of the subspace S_M. We now state this as our main result.

Theorem 3. Let $\varphi(x)$ be the solution of (1.4)—(1.5), subject to the assumptions of (1.8), (1.9), and (1.11), let S_M be any finite dimensional subspace of S, and let $\hat{w}_M(x)$ be the unique function which minimizes $F[w]$ over S_M. Then, there exist a constant M_1 which can be explicitly determined a priori and is independent of S_M, such that the following error bound is valid:

$$\|\hat{w}_M - \varphi\|_{L^\infty} \leqq K\|\hat{w}_M - \varphi\|_{\gamma a b} \leqq M_1 \inf_{w \in S_M} \|w - \varphi\|_{\Gamma A B}. \tag{2.23}$$

If, in addition, the constants a and b of (1.8) are nonnegative, then there similarly exist constants M_2 and M_3 which can also be determined a priori and are independent of S_M, such that

$$\|\hat{w}_M - \varphi\|_{L^\infty} \leqq M_2 \|\hat{w}_M - \varphi\|_{W^{1,2}} \leqq M_3 \inf_{w \in S_M} \|w - \varphi\|_{W^{1,2}}. \tag{2.24}$$

As a consequence, let $\{S_{M_i}\}_{i=1}^{\infty}$ be any sequence of finite dimensional subspaces of S, and let $\{\hat{w}_{M_i}(x)\}_{i=1}^{\infty}$ be the sequence of functions obtained by minimizing $F[w]$ respectively over the subspaces S_{M_i}. If $\lim_{i \to \infty} \{\inf_{w \in S_{M_i}} \|w - g\|_{\gamma a b}\} = 0$ for all $g \in S$, then $\{\hat{w}_{M_i}(x)\}_{i=1}^{\infty}$ converges uniformly to $\varphi(x)$.

The distinction between the error bounds of (2.23) and (2.24) would appear to be an important one, since approximating φ by w in the Sobolev norm requires only L^2-estimates for $\varphi - w$ and $D(\varphi - w)$. In contrast, approximating φ by w directly in the norm $\|\cdot\|_{\gamma a b}$ requires from (2.8) the additional point estimates $\varphi(0) - w(0)$ and $\varphi(1) - w(1)$. However, for our choices of subspaces of S in §§3 and 4, the error bounds to be deduced are *independent* of the assumption that a and b of (1.8) are nonnegative.

§ 3. Polynomial Subspaces

In considering particular finite dimensional subspaces of S, we mention first that in contrast to the treatment in [4], the basis elements of any finite dimensional subspace of S in the present case need *not* satisfy the boundary conditions of (1.5). This is basically a consequence of the well-known distinction between *essential* and *nonessential* boundary conditions in variational formulations [6, 7, 12]. Because of this, the results we use from approximation theory apply more directly here than in the case of [4].

For N a positive integer, let $P^{(N)}$ denote the linear space of all real polynomials of degree at most N. Clearly, $P^{(N)}$ is a finite dimensional subspace of S of dimension $N+1$. If $\varphi(x)$, the classical solution of (1.4)—(1.5), is of class $C^t[0, 1]$, $t \geqq 2$, let $q_N(x)$ be the unique polynomial of degree $N-1$ of best approximation to $D\varphi(x)$ in $[0, 1]$ in the L^∞-norm. Then, by a classical result of Jackson [11, p. 66], there exists a constant M, dependent only on t, such that

$$\|D\varphi - q_N\|_{L^\infty} \leqq \frac{M}{(N-1)^{t-1}}\, \omega\left(D^t \varphi;\ \frac{1}{N-1}\right), \qquad N \geqq t, \tag{3.1}$$

where ω is the modulus of continuity. Thus, if

$$\tilde{p}_N(x) \equiv \varphi(0) + \int_0^x q_N(x')\,dx', \tag{3.2}$$

then $D\tilde{p}_N(x) = q_N(x)$ and $\tilde{p}_N \in P^{(N)}$. Moreover, it also follows from (3.1) that

$$\|\tilde{p}_N - \varphi\|_{L^\infty} \leqq \frac{M}{(N-1)^{t-1}}\,\omega\left(D^t\varphi;\ \frac{1}{N-1}\right), \qquad N \geqq t. \tag{3.3}$$

As this is a pointwise bound, we can then obtain an upper bound for $\|\tilde{p}_N - \varphi\|_{\gamma a b}$, using the definition of (2.8). This gives us

Theorem 4. Let $\varphi(x)$, the solution of (1.4)—(1.5), subject to the assumptions of (1.8), (1.9), and (1.11), be of class $C^t[0, 1]$ with $t \geqq 2$, and let $\hat{p}_N(x)$ be the unique function which minimizes $F[w]$ over $P^{(N)}$, where $N \geqq t$. Then, there exists a constant M, dependent on t and γ, such that

$$\|\hat{p}_N - \varphi\|_{L^\infty} \leqq K\|\hat{p}_N - \varphi\|_{\gamma a b} \leqq \frac{M}{(N-1)^{t-1}}\,\omega\left(D^t\varphi;\ \frac{1}{N-1}\right) \tag{3.4}$$

for all $N \geqq t$.

If, as in [4], φ is only of class $C^2[0, 1]$, i.e., $t=2$, we deduce from (3.4) that the sequence of polynomials $\{\hat{p}_N(x)\}_{N=2}^{\infty}$ converges at least *linearly* (in $h = 1/(N+1)$) and uniformly to $\varphi(x)$ as $N \to \infty$.

If the solution $\varphi(x)$ of (1.4)—(1.5) is known to be analytic in some open set in the complex plane containing the interval [0, 1], the following stronger form of Theorem 4 can be proved. Its proof, based on a classical result of Bernstein [11, p. 76], is analogous to the proof of Theorem 8 of [4], and is thus omitted.

Theorem 5. Let $\varphi(x)$, the solution of (1.4)—(1.5), subject to the conditions of (1.8), (1.9), and (1.11), be analytic in some open set of the complex plane containing the interval [0, 1], and let $\hat{p}_N(x)$ be the unique function which minimizes $F[w]$ over $P^{(N)}$. Then, there exists a constant μ with $0 \leqq \mu < 1$ such that

$$\overline{\lim_{N\to\infty}}\, (\|\hat{p}_N - \varphi\|_{\gamma a b})^{1/N} = \mu, \tag{3.5}$$

and consequently from (2.11),

$$\overline{\lim_{N\to\infty}}\, (\|\hat{p}_N - \varphi\|_{L^\infty})^{1/N} \leqq \mu. \tag{3.6}$$

We remark that the reciprocal ϱ of the constant μ in (3.5) can be given a precise geometrical interpretation. With $\varrho = 1/\mu$, let A and A' be the semi-axes of the largest ellipse in the complex plane with foci $x_0 = 0$ and $x_1 = 1$, in which φ in analytic. Then, $\varrho = A + A'$ (cf. [11, p. 76]). This will be useful in the discussion of the numerical results of §6.

§ 4. *L*-Splines and *g*-Splines

In this section, we apply the recent results of [14] on *L*-splines and *g*-splines to obtain upper bounds for the errors for approximate solutions of the boundary value problem of (1.4)—(1.5) in the finite dimensional subspaces $Sp(L, \pi, z)$ and

$Sp(m, \pi, E)$. The advantage of this general treatment is that it simultaneously gives error bounds for approximate solutions in the finite dimensional Hermite subspaces $H^{(m)}(\pi)$, as well as in the finite dimensional (natural) spline subspaces $Sp^{(m)}(\pi)$ (cf. [4, §6—7]).

To first briefly explain the nature of L-splines, let L be any m-th order linear differential operator of the form

$$L[u(x)] = \sum_{j=0}^{m} a_j(x)\, D^j u(x), \quad m \geqq 1, \tag{4.1}$$

where we assume that the coefficient functions $a_j(x)$ are sufficiently smooth. For example, it sufficies to have $a_j(x) \in K^{m,2}[0, 1]$ for all $0 \leqq j \leqq m$, where $K^{m,2}[0, 1]$ denotes the collection of all real-valued functions $v(x)$ defined on $[0, 1]$ such that $v(x) \in C^{m-1}[0, 1]$, and such that $D^{m-1}v(x)$ is absolutely continuous with $D^m v \in L^2[0, 1]$. Note that $K^{1,2}[0, 1] = S = W^{1,2}[0, 1]$ in the notation of §1. Next, let π: $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$ be any partition of the interval $[0, 1]$, and let $\underline{z} = (z_1, z_2, \ldots, z_N)$, the *incidence vector* associated with π, be an N-vector with positive integer components z_i with $1 \leqq z_i \leqq m$ for all $1 \leqq i \leqq N$. Then, $Sp(L, \pi, \underline{z})$ denotes [14] the collection of all real-valued functions $s(x)$, *called L-splines*, defined on $[0, 1]$ such that

$$\begin{aligned} &L^* L[s(x)] = 0 && \text{for all } \ x \in (x_i, x_{i+1}) \ \text{ and for each } i, \\ & && 0 \leqq i \leqq N, \\ &D^k s(x_i -) = D^k s(x_i +) && \text{for all } \ 0 \leqq k \leqq 2m - 1 - z_i, \\ & && 1 \leqq i \leqq N, \end{aligned} \tag{4.2}$$

where $L^*[v(x)] = \sum_{j=0}^{m} (-1)^j D^j [a_j(x)\, v(x)]$ denotes the formal adjoint of L. As an important special case, suppose $L[u] = D^m u$. With $\hat{z}_1 = \hat{z}_2 = \cdots = \hat{z}_N = 1$, the elements of $Sp(D^m, \pi, \hat{z})$ are then simply the *natural spline functions*, and $Sp(D^m, \pi, \hat{z})$ becomes $Sp^{(m)}(\pi)$ in the notation of [4]. Similarly, when $L[u] = D^m u$ and $\hat{z}_1 = \hat{z}_2 = \cdots = \hat{z}_N = m$, the elements of $Sp(D^m, \pi, \hat{z})$ are then simply the Hermite piecewise polynomial functions, and $Sp(D^m, \pi, \hat{\underline{z}})$ becomes $H^{(m)}(\pi)$ in the notation of [4].

Given a function $f(x) \in C^{m-1}[0, 1]$, where m is the order of the differential operator L of (4.1), there are various ways in which one might interpolate f in $Sp(L, \pi, \underline{z})$. As a particular case, if there is an element $s(x) \in Sp(L, \pi, \underline{z})$ such that

$$D^k s(x_i) = D^k f(x_i), \quad 0 \leqq k \leqq z_i - 1, \quad 1 \leqq i \leqq N, \tag{4.3}$$

and

$$D^k s(x_i) = D^k f(x_i), \quad 0 \leqq k \leqq m - 1, \quad i = 0 \ \text{ or } \ i = N + 1, \tag{4.4}$$

we say that $s(x)$ is an *$Sp(L, \pi, \underline{z})$-interpolate of $f(x)$ of Type I*. It can be shown [14] that, for any partition π and any associated incidence vector $\underline{z}$, an $Sp(L, \pi, \underline{z})$-interpolate of $f(x)$ of Type I always exists and is in fact unique. Thus, given any parameters $\alpha_i^{(k)}$, $0 \leqq k \leqq z_i - 1$, $0 \leqq i \leqq N + 1$ (where we define for convenience $z_0 \equiv z_{N+1} \equiv m$), there exists a unique function $u(x) \in Sp(L, \pi, \underline{z})$ with

$$D^k u(x_i) = \alpha_i^{(k)}, \quad 0 \leqq k \leqq z_i - 1, \quad 0 \leqq i \leqq N + 1, \tag{4.5}$$

and we denote by $Sp^{\mathrm{I}}(L, \pi, \underline{z})$ the finite-dimensional subspace of $Sp(L, \pi, \underline{z})$ of all such functions.

With the notation $\bar{\pi} \equiv \max_{0 \leq i \leq N} (x_{i+1} - x_i)$ for the partition π: $0 = x_0 < \cdots < x_{N+1} = 1$, consider now any sequence of partitions $\{\pi_i\}_{i=1}^{\infty}$ of $[0, 1]$ with the property that $\lim_{i \to \infty} \bar{\pi}_i = 0$, and consider any sequence of incidence vectors $\{\underline{z}^{(i)}\}_{i=1}^{\infty}$ associated with $\{\pi_i\}_{i=1}^{\infty}$. It is known [14, Theorems 6—9] that, if $f(x)$ is of class $K^{m,2}[0, 1]$, $m \geq 1$, there exists a constant M such that the following inequalities hold:

$$\|D^j(f - s_i)\|_{L^2} \leq M(\bar{\pi}_i)^{m-j} \|Lf\|_{L^2}, \qquad 0 \leq j \leq m, \tag{4.6}$$

and

$$\|D^j(f - s_i)\|_{L^\infty} \leq M(\bar{\pi}_i)^{m-j-\frac{1}{2}} \|Lf\|_{L^2}, \qquad 0 \leq j \leq m-1, \tag{4.7}$$

where $s_i(x)$ is the unique $Sp^{\mathrm{I}}(L, \pi_i, \underline{z}^{(i)})$-interpolate of $f(x)$, for each i. Similarly, if $f(x)$ is of class $K^{2m,2}[0, 1]$, $m \geq 1$, we have:

$$\|D^j(f - s_i)\|_{L^2} \leq M(\bar{\pi}_i)^{2m-j} \|L^* Lf\|_{L^2}, \qquad 0 \leq j \leq m, \tag{4.8}$$

and

$$\|D^j(f - s_i)\|_{L^\infty} \leq M(\bar{\pi}_i)^{2m-j-\frac{1}{2}} \|L^* Lf\|_{L^2}, \qquad 0 \leq j \leq m-1. \tag{4.9}$$

Thus, we can bound the norm $\|s_i - f\|_{\Gamma AB}$, defined in (2.8), by applying the cases $j = 0$, $j = 1$ of (4.6) (resp. (4.8)) to the integral terms of (2.8), and by applying the case $j = 0$ of (4.7) (resp. (4.9)) to the boundary terms of (2.8). Applying this to the solution of $\varphi(x)$ of (1.4)—(1.5) results in

Theorem 6. Let $\{\pi_i\}_{i=1}^{\infty}$ be any sequence of partitions of $[0, 1]$ with $\lim_{i \to \infty} \bar{\pi}_i = 0$, let $\{\underline{z}^{(i)}\}_{i=1}^{\infty}$ be any sequence of corresponding incidence vectors, let L be a differential operator of the form (4.1), and let $\hat{w}_i(x)$ be the unique function which minimizes the functional $F[w]$ of (2.6) over $Sp^{\mathrm{I}}(L, \pi_i, \underline{z}^{(i)})$. If $\varphi(x)$, the solution of (1.4)—(1.5), subject to the conditions of (1.8), (1.9), and (1.11), is of class $K^{m,2}[0, 1]$ with $m \geq 2$, then there exists a positive constant M, independent of i, such that

$$\|\hat{w}_i - \varphi\|_{L^\infty} \leq K \|\hat{w}_i - \varphi\|_{\gamma ab} \leq M(\bar{\pi}_i)^{m-1} \|L\varphi\|_{L^2}, \quad \text{for all} \quad i \geq 1. \tag{4.10}$$

Similarly, if $\varphi(x)$ is of class $K^{2m,2}[0, 1]$ with $m \geq 1$, there exists a positive constant M, independent of i, such that

$$\|\hat{w}_i - \varphi\|_{L^\infty} \leq K \|\hat{w}_i - \varphi\|_{\gamma ab} \leq M(\bar{\pi}_i)^{2m-1} \|L^* L\varphi\|_{L^2}, \quad \text{for all} \quad i \geq 1. \tag{4.11}$$

As a simple application of the above results, suppose that the solution $\varphi(x)$ of the nonlinear problem of (1.4)—(1.5) is only of class $C^2[0, 1]$. We may then choose $m = 2$ in (4.10) and $m = 1$ in (4.11), and we deduce that the sequence $\{\hat{w}_i\}_{i=1}^{\infty}$ of elements in $Sp^{\mathrm{I}}(L, \pi_i, \underline{z}^{(i)})$ converges at least *linearly* in $\bar{\pi}_i$ to $\varphi(x)$, as $i \to \infty$.

In the special case that $L[u(x)] = D^m u(x)$ for $x \in [0, 1]$, the previous results may be further generalized. As before, let π: $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$ denote a partition of [0, 1], and let $E=(e_{i,j})$ denote an $N\times m$ *incidence matrix*, $1\leqq i\leqq N$, $0\leqq j\leqq m-1$, having entries of 0's and 1's, with at least one nonzero entry in each row of E. Further, let e denote the collection of (i, j) such that $e_{i,j}=1$. Then, $Sp(m, \pi, E)$ denotes [14] the collection of all real-valued functions $s(x)$, called *g-splines of order m* for π and E, defined on [0, 1] such that

$$\begin{aligned}&s(x) \text{ is a polynomial of degree at most } 2m-1 \text{ in each subinterval } (x_i, x_{i+1}),\\ &\quad 0\leqq i\leqq N, \text{ i.e., } D^{2m}s(x)=0 \text{ in each subinterval of } \pi, \text{ and}\\ &s(x)\in C^{m-1}[0,1], \text{ and if } e_{i,j}=0, \text{ then } D^{2m-j-1}s(x) \text{ is continuous at } x_i, \text{ i.e.,}\\ &\quad (i,j)\notin e \text{ implies that } D^{2m-j-1}s(x_i-)=D^{2m-j-1}s(x_i+).\end{aligned} \tag{4.12}$$

In analogy with the case of L-splines, we say that, given a function $f(x)\in C^{m-1}[0, 1]$, $s(x)\in Sp(m, \pi, E)$ is an *$Sp(m, \pi, E)$-interpolate of $f(x)$ of Type I* if

$$D^j s(x_i)=D^j f(x_i) \quad \text{for all} \quad (i,j)\in e, \tag{4.13}$$

and

$$D^j s(x_i)=D^j f(x_i), \quad 0\leqq j\leqq m-1 \quad \text{for} \quad i=0 \quad \text{or} \quad i=N+1. \tag{4.14}$$

It can be shown [14] that, for any partition π and any incidence matrix E, an $Sp(m, \pi, E)$-interpolate of $f(x)$ of Type I always exists and is unique. We now simply point out that analogues of Theorem 6 are also valid not only for g-splines of Type I, but also for L-splines and g-splines with more general boundary interpolations than those of (4.4), referred to in [14] as interpolations of Types II and III. We again refer the interested reader to [14] for details.

§ 5. Extensions

In this section, we extend our previous results to cover more general nonlinear boundary value problems with nonlinear boundary conditions.

First, as in [4], it is easy to verify that we may make the following weakened assumptions about $f(x, u)$, without affecting the validity of Theorem 3. We again assume that $f(x, u)\in C^0([0, 1]\times R)$, but in place of (1.11), we assume that there exists a constant γ such that

$$\frac{f(x,u)-f(x,v)}{u-v}\geqq\gamma>-\Lambda, \quad \text{for } x\in[0,1], \text{ and for all } -\infty<u, v<+\infty, \text{ with } u\neq v, \tag{5.1}$$

and for each $c>0$, there exists a number $M(c)$ such that

$$\frac{f(x,u)-f(x,v)}{u-v}\leqq M(c)<+\infty, \quad \text{for all } x\in[0,1], \text{ and for all } |u|\leqq c, \ |v|\leqq c, \text{ with } u\neq v. \tag{5.2}$$

Similarly, difference quotients may be also used to weaken the assumptions on the functions $\psi_0(t)$ and $\psi_1(t)$ of (1.5). Specifically, in place of (1.8), we may assume that $\psi_0(t)\in C^0(R)$ with

$$\frac{\psi_0(t_1)-\psi_0(t_2)}{t_1-t_2}\geqq a, \quad \text{for all real } t_1\neq t_2, \tag{5.3}$$

and that for each $c>0$, there exists a number $M_0(c)$ such that

$$\frac{\psi_0(t_1)-\psi_0(t_2)}{t_1-t_2} \leqq M_0(c) < +\infty, \quad \text{for all} \quad |t_1| \leqq c, \quad |t_2| \leqq c, \quad \text{with} \quad t_1 \neq t_2. \tag{5.4}$$

Obviously, an analogous weakened hypothesis can also be made for $\psi_1(t)$.

Perhaps a more interesting extension concerns the numerical approximation of a boundary problem with radiative-type (or STEFAN-BOLTZMANN) boundary conditions. Consider the radiation problem:

$$D^2 u(x) = f(x), \quad 0<x<1, \quad \text{where} \quad f(x) \leqq 0, \tag{5.5}$$

$$u(0)=0, \quad Du(1) = -\big(u(1)\big)^4. \tag{5.6}$$

This problem combines one *essential* boundary condition (i.e. of Dirichlet-type) with a nonlinear and inessential one. It can be shown that this problem has a unique nonnegative solution (cf. [1, 5]). Hence, if we are only interested in this particular solution, we might as well replace the second boundary condition of (5.6) by

$$Du(1) = -\psi_1\big(u(1)\big), \quad \text{where} \quad \psi_1(t) = \operatorname{Max}\{t^4, 0\}. \tag{5.6'}$$

This leads us to consider what might be called a *generalized radiation problem:*

$$D^2 u(x) = f(x, u), \quad 0<x<1, \tag{5.7}$$

$$u(0)=0; \quad Du(1) = -\psi_1\big(u(1)\big), \tag{5.8}$$

where

$$\frac{\partial f}{\partial u}(x,u) \geqq 0, \quad 0 \leqq x \leqq 1, \quad -\infty < u < +\infty, \tag{5.9}$$

and

$$\psi_1(0)=0, \quad D\psi_1(t) \geqq 0 \quad \text{for all real } t. \tag{5.10}$$

We now describe how to apply a variational scheme to this problem. Since the boundary condition at the origin is of the essential type, it must now be satisfied by the admissible functions. Hence, we let S be the space consisting of all absolutely continuous functions $w(x)$ such that $Dw(x) \in L^2[0,1]$ and such that $w(0)=0$. As in Theorem 1, the unique solution $\varphi(x)$ of (5.7)—(5.8) with the hypotheses of (5.9)—(5.10) minimizes strictly the following functional over the space S:

$$F[w] = \int_0^1 \left\{\tfrac{1}{2}\big(Dw(x)\big)^2 + \int_0^{w(x)} f(x,\eta)\, d\eta\right\} dx + \int_0^{w(1)} \psi_1(\lambda)\, d\lambda. \tag{5.11}$$

As before, the approximation scheme is defined by minimizing the functional $F[w]$ of (5.11) over finite dimensional subspaces S_M of S. Similarly, a convergence theorem similar to Theorem 3 can be established, in the $W^{1,2}[0,1]$-norm.

§ 6. Computational Methods

To our knowledge, little has been published in the literature about numerical methods for solving two-point boundary value problems with nonlinear boundary conditions. Recently, KELLER [9] has suggested a shooting technique, combined with a finite difference approximation, for numerically solving somewhat more general problems in one dimension, but no numerical results or error estimates were presented in [9]. Also, KELLOGG [10] has suggested a very clever numerical technique for solving radiative-type problems of the form (5.5)—(5.6). In essence, a three-point difference approximation to (5.5) is made, and the associated tri-diagonal matrix problem is solved via Gaussian elimination. Because the nonlinearity of the problem (5.5)—(5.6) occurs only on the boundary, just one nonlinear equation in one unknown need be solved in the Gaussian elimination technique. The same numerical efficiency also applies to the use of the Hermite subspace $H^{(1)}(\pi)$ for this special problem.

To illustrate the results of the previous sections, consider the numerical approximation of the solution of

$$D^2 u(x) = (u(x))^3 - (\cos x + 1)^3 - \cos x, \qquad 0 < x < 1, \tag{6.1}$$

with boundary conditions

$$D u(0) = 0; \qquad D u(1) = -\frac{\sin 1 \, (u(1))^3}{(\cos 1 + 1)^3}. \tag{6.2}$$

A solution of (6.1)—(6.2) is readily verified to be

$$\varphi(x) = \cos x + 1. \tag{6.3}$$

For this example, $\psi_0(t) \equiv 0$ and $\psi_1(t) \equiv \frac{\sin 1 \cdot t^3}{(\cos 1 + 1)^3}$, and as such, Eqs. (1.7) and (1.8) are satisfied with $a = b = 0$. It then follows from Lemma 1 that real constants $K > 0$ and β exist such that the inequality of (1.9) is valid for all $w(x) \in S$. Moreover, the constant Λ of (1.10) is positive, and is at least π^2. Thus, as $f_u(x, u) = 3u^2$, then $f_u(x, u) \geqq 0 > -\Lambda$ and we see from Theorem 1 that $\varphi(x)$ of (6.3) is the *unique* solution of (6.1)—(6.2). The associated functional of (2.6) in this case is

$$F[w] = \int_0^1 \left\{\frac{1}{2}(D w(x))^2 + \frac{(w(x))^4}{4} - [(\cos x + 1)^3 + \cos x]\, w(x)\right\} dx + \frac{\sin 1}{(\cos 1 + 1)^3} \frac{(w(1))^4}{4}, \qquad w \in S, \tag{6.4}$$

and this functional is minimized over the finite dimensional subspaces $P^{(N)}$ of polynomials, and over the finite dimensional subspaces of piecewise cubic ($m = 2$) Hermite and spline subspaces $H^{(2)}(\pi(h))$ and $Sp^{(2)}(\pi(h))$, with a uniform mesh $\pi(h)$ on $[0, 1]$. As in [4], the associated nonlinear matrix equations were solved using a nonlinear point successive over-relaxation method [13]. The efficient computational treatment of such techniques is considered in detail in [8]. The numerical results are given in the tables below.

For this case of the polynomial subspaces $P^{(N)}$ of S, we can take advantage of the fact that the solution $\varphi(x)$ of (6.3) is an *entire function*, i.e., it can be extended to be analytic in the whole complex plane. As such, Theorem 5 applies with $\mu = 0$, and consequently

$$\overline{\lim_{N \to \infty}} \left(\|\hat{p}_N(x) - \varphi(x)\|_{L_\infty}\right)^{1/N} = 0. \tag{6.5}$$

For this example, we thus have rapid convergence of the sequence $\{\hat{p}_N(x)\}_{N=1}^{\infty}$ to $\varphi(x)$ as $N\to\infty$, as is corroborated by the numerical results of Table 1.

Table 1. *Polynomial subspaces* $P^{(N)}$

N in $P^{(N)}$	$\dim(P^{(N)})$	$\\|\hat{p}_N-\varphi\\|_{L^\infty}$
0	1	$2.61\cdot10^{-1}$
2	3	$3.50\cdot10^{-3}$
4	5	$1.13\cdot10^{-5}$
6	7	$1.82\cdot10^{-8}$
8	9	$1.49\cdot10^{-11}$

Table 2
Smooth cubic Hermite subspaces $H^{(2)}(\pi(h))$

h	$\dim(H^{(2)}(\pi(h)))$	$\\|\hat{w}_n-\varphi\\|_{L^\infty}$
1	4	$4.04\cdot10^{-4}$
1/2	6	$4.97\cdot10^{-5}$
1/4	10	$4.52\cdot10^{-6}$
1/6	14	$1.03\cdot10^{-6}$
1/8	18	$4.47\cdot10^{-7}$

Table 3
Cubic spline subspaces $Sp^{(2)}(\pi(h))$

h	$\dim(Sp^{(2)}(\pi(h)))$	$\\|\hat{w}_h-\varphi\\|_{L^\infty}$
1/4	7	$5.53\cdot10^{-6}$
1/6	9	$1.13\cdot10^{-6}$
1/8	11	$4.47\cdot10^{-7}$
1/10	13	$2.23\cdot10^{-7}$
1/12	15	$1.19\cdot10^{-7}$

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