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Numerical Methods of High-Order Accuracy for Nonlinear Boundary Value Problems

IV. Periodic Boundary Conditions*

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§ 1. Introduction

We shall consider here the numerical approximation of the solution of the following real nonlinear boundary value problem

$$(1.1)\qquad \mathscr{L}[u(x)] = f(x, u(x)), \quad 0 < x < 1,$$

with periodic boundary conditions

$$(1.2)\qquad D^k u(0) = D^k u(1), \quad 0 \leqq k \leqq 2n-1, \quad D \equiv \frac{d}{dx},$$

where the differential operator $\mathscr{L}$ is defined by

$$(1.3)\qquad \mathscr{L}[u(x)] = \sum_{j=0}^{n} (-1)^{j+1} D^j [p_j(x) D^j u(x)], \quad n \geqq 1.$$

The coefficient functions $p_j(x)$ are assumed to be of class $C^j[0, 1]$ with periodic boundary behavior:

$$(1.4)\qquad D^k p_j(0) = D^k p_j(1), \quad 0 \leqq k \leqq j-1, \quad 0 \leqq j \leqq n.$$

The material presented here is an extension of the results of [6, 7, 18] to such periodic boundary conditions (1.2), and it is convenient to follow the notations and assumptions given there. Here, in analogy with [6], let S denote the linear space of all real-valued functions $w(x)$ defined on $[0, 1]$ such that $w(x) \in C^{n-1}[0, 1]$ with $D^{n-1}w(x)$ absolutely continuous and $D^n w(x) \in L^2[0, 1]$, and such that $w(x)$ is $2n-1$ times continuously differentiable in neighborhoods of $x = 0$ and $x = 1$ of $[0, 1]$ with

$$(1.5)\qquad D^k w(0) = D^k w(1), \quad 0 \leqq k \leqq 2n-1.$$

Further, we assume that there exist two real constants $K > 0$ and β such that

$$(1.6)\qquad \|w\|_{L^\infty[0,1]} \equiv \sup_{x\in[0,1]} |w(x)| \leqq K \left\{ \int_0^1 \left[\sum_{j=0}^{n} p_j(x) \left(D^j w(x)\right)^2 + \beta \left(w(x)\right)^2 \right] dx \right\}^{\frac{1}{2}}$$

for all $w \in S$.

Next, we introduce the finite quantity (see Lemma 1)

$$(1.7)\qquad \Lambda \equiv \inf_{\substack{w\in S \\ w \neq 0}} \frac{\int_0^1 \left[\sum_{j=0}^{n} p_j(x) \left(D^j w(x)\right)^2 \right] dx}{\int_0^1 [w(x)]^2 \, dx}.$$

Although this fact is not used, it turns out that Λ is a lower bound for the eigenvalues of the associated eigenvalue problem $\mathscr{L}[u(x)]+\lambda u(x)=0$, $0<x<1$, subject to the boundary conditions of (1.2).

We finally assume that the function $f(x, u)$ is real, continuous in both variables, i.e., $f(x, u)\in C^0([0, 1]\times R)$, that there exists a constant γ such that

$$(1.8) \qquad \frac{f(x,u)-f(x,v)}{u-v} \geqq \gamma > -\Lambda,$$

for all $x\in[0, 1]$, and all $-\infty<u, v<+\infty$, with $u\neq v$, and for each $c>0$, there exists a number $M(c)$ such that

$$(1.8') \qquad u\neq v, \quad |u|\leqq c, \quad |v|\leqq c \quad \text{implies} \quad \frac{f(x,u)-f(x,v)}{u-v}\leqq M(c)<\infty$$

for all $x\in[0, 1]$.

One of our main goals is to study the effects of applying the classical Rayleigh-Ritz procedure (cf. [6] and [10]) to a variational formulation [9] of (1.1)—(1.2) by minimizing over finite dimensional subspaces of S which for periodic problems are most naturally taken as subspaces of trigonometric polynomials. In so doing, we extend the results of [6] to periodic problems and obtain new error estimates which improve upon known results in the literature for the Galerkin Method, which is equivalent to the Rayleigh-Ritz procedure for the class of problems under consideration (cf. [3—5, 16, 17]).

Another of our goals is to show that these techniques can, from a numerical point of view, be efficiently applied on modern high-speed digital computers. To illustrate these theoretical results, numerical results for particular examples of (1.1)—(1.2) will be discussed.

§ 2. Rayleigh-Ritz Method

In this section, we discuss the Rayleigh-Ritz method for the problem (1.1) to (1.2). The proofs of the theorems of this section are exact analogues of the proofs given in [6] for the application of the Rayleigh-Ritz method to nonlinear two-point boundary value problems with Dirichlet boundary conditions. We begin with (cf. [6, Lemma 1])

Lemma 1. With the assumption of (1.6), then

$$(2.1) \qquad \Lambda \equiv \inf_{\substack{w\in S\\ w\neq 0}} \frac{\int_0^1 \left\{\sum_{j=0}^n p_j(x)\,[D^j w(x)]^2\right\} dx}{\int_0^1 [w(x)]^2\,dx} \geqq \frac{1}{K^2}-\beta > -\infty.$$

We make the essential hypothesis that (1.1)—(1.2) has a classical solution $\varphi(x)$. Then, we have (cf. [6, Theorem 1]).

Theorem 1. With the assumptions of (1.4), (1.6), and (1.8), let $\varphi(x)$ be a classical solution of (1.1)—(1.2). Then $\varphi(x)$ *strictly* minimizes the following functional

$$(2.2) \qquad F[w]\equiv\int_0^1\left\{\tfrac{1}{2}\sum_{j=0}^n p_j(x)\,(D^j w(x))^2+\int_0^{w(x)} f(x,\eta)\,d\eta\right\}dx$$

over the space S, and $\varphi(x)$ is thus the unique classical solution of (1.1)—(1.2).

Consider now *any* finite dimensional subspace S_M of S of dimension M, and let $\{w_i(x)\}_{i=1}^M$ be M linearly independent functions from the subspace. The analogue of Theorem 1, concerning the minimization of the functional $F[w]$ over the subspace S_M, is given in (cf. [6, Theorem 2])

Theorem 2. With the assumptions of (1.4), (1.6), and (1.8), there exists a unique function $\hat{w}_M(x)$ in the finite dimensional subspace S_M of S which minimizes the functional $F[w]$ over S_M.

To find this unique element $\hat{w}_M(x)=\sum_{i=1}^M \hat{u}_i w_i(x)$ in S_M which minimizes $F[w]$ over S_M, we must solve the M nonlinear equations

$$(2.3)\qquad \int_0^1 \left\{\sum_{j=0}^n p_j(x)\left(\sum_{k=1}^M u_k D^j w_k(x)\right) D^j w_i(x) + f\left(x, \sum_{k=1}^M u_k w_k(x)\right) w_i(x)\right\} dx = 0, \quad 1 \leqq i \leqq M$$

for the M unknowns $u_1, u_2, \ldots, u_M$, which arise from

$$\frac{\partial F\left[\sum_{i=1}^M u_i w_i\right]}{\partial u_i} = 0, \qquad 1 \leqq i \leqq M.$$

Letting $\boldsymbol{u} \equiv (u_1, u_2, \ldots, u_M)^T$, the equations of (2.3) can be written in matrix form as

$$(2.4)\qquad A\boldsymbol{u} + \boldsymbol{g}(\boldsymbol{u}) = \boldsymbol{0},$$

where $A=(a_{i,k})$ is a real $M \times M$ matrix, and $\boldsymbol{g}(\boldsymbol{u}) = (g_1(\boldsymbol{u}), g_2(\boldsymbol{u}), \ldots, g_M(\boldsymbol{u}))^T$ is a column vector, both being determined by

$$(2.5)\qquad a_{i,k} = \int_0^1 \left\{\sum_{j=0}^n p_j(x) D^j w_i(x) D^j w_k(x)\right\} dx, \qquad 1 \leqq i, k \leqq M,$$

and

$$(2.6)\qquad g_k(\boldsymbol{u}) = \int_0^1 f\left(x, \sum_{i=1}^M u_i w_i(x)\right) w_k(x)\, dx, \qquad 1 \leqq k \leqq M.$$

This nonlinear matrix equation (2.4) can then be efficiently solved for example by Gauss-Seidel iteration, which is known in this case to be convergent [12, 13]. For more computational details, we refer the reader to [6] and [8].

If we have a sequence $\{S_{M_i}\}_{i=1}^\infty$ of not necessarily nested finite dimensional subspaces of S, we consider the problem of when the associated sequence $\{\hat{w}_{M_i}(x)\}_{i=1}^\infty$ converges to $\varphi(x)$, the unique solution of (1.1)—(1.2), in the uniform norm. If α is a real constant, define

$$(2.7)\qquad \|w\|_\alpha = \left\{\int_0^1 \left[\sum_{j=0}^n p_j(x)\,(D^j w(x))^2 + \alpha\,(w(x))^2\right] dx\right\}^{\frac{1}{2}}$$

for all $w \in S$. Then, recalling the constant γ of (1.8), we have (cf. [6, Lemma 2])

Lemma 2. If $\alpha > -\Lambda$, then $\|w\|_\alpha$ and $\|w\|_\gamma$ are equivalent norms on S. Moreover, the inequality of (1.6) is valid for all $w \in S$ with β replaced by any γ' with $\gamma' > -\Lambda$.

As a consequence of this lemma, we can write for the particular constant γ of (1.8) that

$$\|w\|_{L^\infty} \leqq K\|w\|_\gamma \quad \text{for all} \quad w \in S, \tag{2.8}$$

which is an analogue of the original assumption of (1.6). In what is to follow, we shall regard K and γ as *fixed* constants satisfying (1.8) and (2.8).

The following fundamental result gives us an error estimate (cf. [6, Theorem 3]).

Theorem 3. Let $\varphi(x)$ be the (classical) solution of (1.1)—(1.2), subject to the assumptions of (1.4), (1.6), and (1.8), let S_M be any finite dimensional subspace of S, and let $\hat{w}_M(x)$ be the unique function which minimizes $F[w]$ over S_M. Then, there exists a constant C which is independent of the choice of S_M, such that the following error bound is valid:

$$\|\hat{w}_M - \varphi\|_{L^\infty} \leqq K\|\hat{w}_M - \varphi\|_\gamma \leqq C \inf_{w \in S_M} \|w - \varphi\|_\gamma. \tag{2.9}$$

As an immediate consequence of Theorem 3, we have

Theorem 4. Let $\varphi(x)$ be the (classical) solution of (1.1)—(1.2), subject to the assumptions of (1.4), (1.6), and (1.8), let $\{S_{M_i}\}_{i=1}^\infty$ be any sequence of (not necessarily nested) finite dimensional subspaces of S, and let $\{\hat{w}_{M_i}(x)\}_{i=1}^\infty$ be the sequence of functions obtained by minimizing $F[w]$ respectively over the subspace S_{M_i}. If $\lim_{i\to\infty}\left\{\inf_{w \in S_{M_i}} \|w-\varphi\|_\gamma\right\} = 0$, then $\{\hat{w}_{M_i}(x)\}_{i=1}^\infty$ converges *uniformly* to $\varphi(x)$.

In place of $\lim_{i\to\infty}\left\{\inf_{w \in S_{M_i}} \|w-\varphi\|_\gamma\right\} = 0$, we could of course make the stronger hypothesis in Theorem 4 that $\lim_{i\to\infty}\left\{\inf_{w \in S_{M_i}} \|w-g\|_\gamma\right\} = 0$ for *every* $g \in S$. In practice, where the solution $\varphi(x)$ is in general not known a priori, this latter hypothesis is more readily checked, and is in fact valid for the subspaces considered in § 3 and 4.

§ 3. Trigonometric and Algebraic Polynomial Subspaces

As our first example of subspaces S_{M_i} satisfying the sufficient conditions of Theorem 4, let N be any nonnegative integer, and let $T^{(N)}$ be the collection of all real trigonometric polynomials $w(x)$ of order N, i.e.,

$$w(x) = \frac{a_0}{2} + \sum_{j=1}^{N} \left(a_j \cos(2\pi j x) + b_j \sin(2\pi j x)\right), \tag{3.1}$$

where the coefficients a_j and b_j are real. It is clear that $T^{(N)}$ is a subspace of S of dimension $2N+1$.

We now discuss the error of best least squares approximation by elements of $T^{(N)}$. For any function $f(x) \in L^2[0, 2\pi]$, let

$$\omega_2(f;\delta) \equiv \sup_{|h|\leqq\delta}\left\{\frac{1}{2\pi}\int_0^{2\pi} |f(x+h) - f(x)|^2\,dx\right\}^{\frac{1}{2}} \tag{3.2}$$

denote the *L^2-modulus of continuity*, where $f(x)$ is periodically extended outside the interval $[0, 2\pi]$. We remark that $\omega_2(f;\delta)$ is a nondecreasing function of δ, and that $\lim_{\delta\to 0}\omega_2(f;\delta) = 0$ for any $f(x) \in L^2[0, 2\pi]$. The following Jackson-like result is an easy extension of a result found in Alexits [2].

Theorem 5. For s a positive integer, assume that $u(x) \in C_p^{s-1}[0, 2\pi]$, i.e., $u(x) \in C^{s-1}[0, 2\pi]$ and $D^j u(0) = D^j u(2\pi)$ for all $0 \leqq j \leqq s-1$, and assume that $D^{s-1} u(x)$ is absolutely continuous with $D^s u(x) \in L^2[0, 2\pi]$. Then, for each positive integer N, there exists a trigonometric polynomial $\tilde{t}_n(x) \in T^{(N)}$ such that

$$\|D^j(u - \tilde{t}_N)\|_{L^2[0,2\pi]} \leqq \frac{K\,\omega_2(D^s u;\, 1/N)}{N^{s-j}} \quad \text{for all} \quad 0 \leqq j \leqq s, \tag{3.3}$$

where K is a constant dependent only on s.

Proof. We prove the case for $s=1$; the cases $s>1$ follow similarly by integration by parts. If $\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ denotes the Fourier series for $u(x)$, then integration by parts gives

$$a_n = \frac{-1}{\pi n} \int_0^{2\pi} Du(x) \sin nx\, dx \quad \text{and} \quad b_n = \frac{1}{\pi n} \int_0^{2\pi} Du(x) \cos nx\, dx, \qquad n \geqq 1,$$

where we have used the hypothesis that $u(0) = u(2\pi)$. Denoting the Fourier series for $Du(x) \in L^2[0, 2\pi]$ by $\sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx)$, the constant term being necessarily zero since $u(x)$ is periodic, we see that $\alpha_n = n b_n$ and $\beta_n = -n a_n$, $n \geqq 1$, and thus

$$|\alpha_n|^2 + |\beta_n|^2 = n^2\{|a_n|^2 + |b_n|^2\}, \qquad n \geqq 1, \tag{3.4}$$

and the best least squares approximation by trigonometric polynomials of order N for $Du(x)$ is just $g_N(x) \equiv \sum_{n=1}^{N} (\alpha_n \cos nx + \beta_n \sin nx)$. By PARSEVAL's relation, we have

$$\frac{1}{\pi} \int_0^{2\pi} (Du(x) - g_N(x))^2\, dx = \sum_{n=N+1}^{\infty} (|\alpha_n|^2 + |\beta_n|^2). \tag{3.5}$$

On the other hand, we have from ALEXITS [2, p. 270] that

$$\sum_{n=N+1}^{\infty} (|\alpha_n|^2 + |\beta_n|^2) \leqq \frac{16}{3} \omega_2^2\left(Du;\, \frac{1}{N}\right). \tag{3.6}$$

Hence, if we define $\tilde{t}_N(x)$ by $\frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx)$, we see that $D\tilde{t}_N(x) \equiv g_N(x)$, and combining (3.5) and (3.6) then yields

$$\|D(u - \tilde{t}_N)\|_{L^2[0,2\pi]} \leqq \sqrt{\frac{16\pi}{3}}\, \omega_2\left(Du;\, \frac{1}{N}\right)$$

the special case $j = s = 1$ of (3.3). If we now compare $u(x)$ and $\tilde{t}_N(x)$, it follows from (3.4) that

$$\frac{1}{\pi} \int_0^{2\pi} (u(x) - \tilde{t}_N(x))^2\, dx = \sum_{n=N+1}^{\infty} (|a_n|^2 + |b_n|^2) = \sum_{n=N+1}^{\infty} \frac{(|\alpha_n|^2 + |\beta_n|^2)}{n^2}$$
$$\leqq \frac{1}{N^2} \sum_{n=N+1}^{\infty} (|\alpha_n|^2 + |\beta_n|^2) \leqq \frac{16}{3N^2} \omega_2^2\left(Du;\, \frac{1}{N}\right),$$

which gives the remaining inequality of (3.3) for the case $j=0$, and $s=1$. Q.E.D.

Corollary. With the hypotheses of Theorem 5, for each positive integer N there exists a trigonometric polynomial $\tilde{t}_N(x)$ of order N such that

$$(3.3') \qquad \|D^j(u-\tilde{t}_N)\|_{L^2[0,2\pi]} \leqq \frac{K'\|D^s u\|_{L^2[0,5\pi]}}{N^{s-j}} \quad \text{for all} \quad 0 \leqq j \leqq s,$$

where K' is a constant dependent only on s.

Proof. By definition, $\omega_2(f;\delta) \leqq \frac{2}{\sqrt{2\pi}}\|f\|_{L^2[0,2\pi]}$ for all $\delta \geqq 0$. Q.E.D.

We now apply the result of Theorem 5 and its corollary to the solution $\varphi(x)$ of (1.1)—(1.2), where we necessarily have that $\varphi(x) \in C_p^{2n-1}[0,1]$, and $D^{2n}\varphi \in L^2[0,1]$. The following result follows from the basic inequalities of (2.9) of Theorem 3.

Theorem 6. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4), (1.6), and (1.8), be of class $C_p^{s-1}[0,1]$ with $D^s\varphi \in L^2[0,1]$, where $s \geqq 2n$, and let $\hat{t}_N(x)$ be the unique function which minimizes the functional $F[w]$ over $T^{(N)}$. Then, there exist constants M and M' depending only on s and γ such that

$$(3.7) \qquad \|\hat{t}_N - \varphi\|_{L^\infty} \leqq K\|\hat{t}_N - \varphi\|_\gamma \leqq \frac{M}{N^{s-n}}\,\omega_2\left(D^s\varphi;\frac{1}{N}\right) \leqq \frac{M'}{N^{s-n}}\|D^s\varphi\|_{L^2[0,1]}.$$

If $u(x) \in C_p^\infty[0,2\pi]$ is analytic in some open set of the complex plane containing the interval $[0,2\pi]$, we can of course apply the result of Theorem 5 for any $s \geqq 0$. But, an even stronger result, indicating exponential convergence, is possible. As in Theorem 7 of [6], the following result can be established from a classical result of BERNSTEIN (cf. [11, p. 158]).

Theorem 7. If $u(x) \in C_p^\infty[0,2\pi]$ is analytic in some open set of the complex plane containing the interval $[0,2\pi]$, then there exist a constant μ with $0 \leqq \mu < 1$ and a sequence of trigonometric polynomials $\{\tilde{t}_N(x)\}_{N=0}^\infty$ with $\tilde{t}_N(x) \in T^{(N)}$ such that for each nonnegative integer n,

$$(3.8) \qquad \overline{\lim_{N\to\infty}}\left(\|D^k(u-\tilde{t}_N)\|_{L^\infty[0,2\pi]}\right)^{1/N} \leqq \mu \quad \text{for all} \quad 0 \leqq k \leqq n.$$

If $u(x)$ is moreover an entire function, i.e., $u(x)$ can be extended to a function $u(z)$ which is analytic for all complex z, then for each nonnegative integer n,

$$(3.8') \qquad \lim_{N\to\infty}\left(\|D^k(u-\tilde{t}_N)\|_{L^\infty[0,2\pi]}\right)^{1/N} = 0 \quad \text{for all} \quad 0 \leqq k \leqq n.$$

Applying this result to the solution $\varphi(x)$ of (1.1)—(1.2), we have

Theorem 8. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4), (1.6), and (1.8), be of class $C_p^\infty[0,1]$ and be analytic in some open set in the complex plane containing the interval $[0,1]$, and let $\hat{t}_N(x)$ be the unique function which minimizes the functional $F[w]$ over $T^{(N)}$. Then, there exists a constant μ with $0 \leqq \mu < 1$ such that

$$(3.9) \qquad \overline{\lim_{N\to\infty}}\,(\|\hat{t}_N - \varphi\|_\gamma)^{1/N} = \mu,$$

and consequently from Theorem 3,

$$(3.9') \qquad \overline{\lim_{N\to\infty}}\,(\|\hat{t}_N - \varphi\|_{L^\infty})^{1/N} \leqq \mu.$$

If $\varphi(x)$ is moreover an entire function, the constant μ can be chosen to be zero.

With respect to the solution of the nonlinear matrix problem (2.4) associated with the subspace $T^{(N)}$ of S, we mention that in special, but nevertheless interesting cases, it is easy to deduce explicitly an *orthonormal* basis for $T^{(N)}$, i.e., one for which the matrix entries $a_{i,j}$ of the matrix A of (2.5) satisfy $a_{i,j}=\delta_{i,j}$. For example, for $n=1$, $p_1(x)\equiv 1$, $p_0(x)\equiv 1$ in (1.3), the functions

$$\left\{1;\left(\sqrt{\frac{2}{1+4\pi^2 n^2}}\cos(2\pi n x),\ \sqrt{\frac{2}{1+4\pi^2 n^2}}\sin(2\pi n x)\right)_{n=1}^{N}\right\}$$

form such a basis. This choice of basis makes the numerical solution of the nonlinear matrix problem (2.4) considerably simpler.

As our second example of subspaces S_{M_i} satisfying the sufficient conditions of Theorem 4, let N be a positive integer with $N\geqq 4n-1$, and let $P_p^{(N)}$ be the collection of all real algebraic polynomials of degree N which satisfy the periodicity condition of (1.5). It is clear that the elements of $P_p^{(N)}$ are polynomials which can be represented in the form

$$\text{(3.10)}\quad \begin{aligned} &b_0 h_0(x)+b_1 h_1(x)+\cdots+b_{2n-1}h_{2n-1}(x)\\ &\qquad +x^{2n}(1-x)^{2n}\{a_0+a_1 x+\cdots+a_{N-4n}x^{N-4n}\}, \end{aligned}$$

where $h_i(x)$, $0\leqq i\leqq 2n-1$, is the unique (Hermite) polynomial of degree $4n-1$ such that

$$\text{(3.11)}\qquad D^j h_i(0)=D^j h_i(1)=\delta_{i,j},\qquad 0\leqq j\leqq 2n-1.$$

It is clear that $P_p^{(N)}$ is a subspace of S of dimension $N-2n+1$.

We now discuss the error in best approximation by elements of $P_p^{(N)}$.

Theorem 9. If $u(x)\in C^t[0,1]$, $t\geqq 2n$, and $u(x)$ satisfies the boundary conditions of (1.2), then for each positive integer $N\geqq\max(t,4n-1)$, there exists an algebraic polynomial $\tilde{p}_N(x)\in P_p^{(N)}$ such that

$$\text{(3.12)}\quad \|D^j(u-\tilde{p}_N)\|_{L^\infty}\leqq\frac{K}{(N-2n)^{t-2n}}\,\omega\left(D^t\left(u-\sum_{i=0}^{2n-1}(D^i u(0))\,h_i\right);\frac{1}{N-2n}\right)$$

for all $0\leqq j\leqq 2n$, where K is a constant dependent only on t and n, and ω is the (usual) modulus of continuity.

Proof. Fixing $N\geqq\max(t,4n-1)$, the proof is accomplished by applying Theorem 5 of [6] to the function $u(x)-\sum_{i=0}^{2n-1}(D^i u(0)(\,h_i(x)$, whose first $2n-1$ derivatives are zero at $x=0$ and $x=1$. Q.E.D.

Corollary. With the hypotheses of Theorem 9, then for each positive integer $N\geqq\max(t,4n-1)$, there exists an algebraic polynomial $\tilde{p}_N(x)\in P_p^{(N)}$ such that

$$\text{(3.13)}\quad \|D^j(u-\tilde{p}_N)\|_{L^\infty}\leqq\frac{K'\left\|D^t\left(u-\sum_{i=0}^{2n-1}(D^i u(0))\,h_i\right)\right\|_{L^\infty}}{(N-2n)^{t-2n}}\quad\text{for all}\quad 0\leqq j\leqq 2n,$$

where K' is a constant dependent only on t and n.

Proof. By definition, $\omega(f,\delta)=\sup_{|h|\leqq\delta}|f(x+h)-f(x)|\leqq 2\|f\|_{L_\infty}$. Q.E.D.

We remark that the result of Theorem 9 provides no estimates such as those of (3.3) for polynomials $\tilde{p}_N(x)$ of low degree, i.e., for $N < \max(t, 4n-1)$.

We now apply the result of Theorem 9 and its corollary to the solution $\varphi(x)$ of (1.1)—(1.2), where we necessarily have that $\varphi(x) \in C^{2n}[0, 1]$. The following result follows from the basic inequalities of (2.9) of Theorem 3.

Theorem 10. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4), (1.6), and (1.8), be of class $C^t[0, 1]$ where $t \geqq 2n$, and let $\hat{p}_N(x)$ be the unique function which minimizes the functional $F[w]$ over $P_p^{(N)}$ where $N \geqq \max(t, 4n-1)$. Then, there exists constants M and M' depending only on t and γ such that

$$
\begin{aligned}
\|\hat{p}_N - \varphi\|_{L^\infty} &\leqq K \|\hat{p}_N - \varphi\|_\gamma \\
&\leqq \frac{M}{(N-2n)^{t-2n}}\, \omega\left(D^t\left(\varphi - \sum_{i=0}^{2n-1} (D^i \varphi(0))\, h_i\right);\ \frac{1}{N-2n}\right) \\
&\leqq \frac{M'}{(N-2n)^{t-2n}} \left\| D^t\left(\varphi - \sum_{i=0}^{2n-1} (D^i \varphi(0))\, h_i\right)\right\|_{L^\infty}.
\end{aligned}
\tag{3.14}
$$

If $u(x)$ is actually analytic in some open set of the complex plane containing the interval $[0, 1]$, we can apply the result of Theorem 9 for any $t \geqq 2n$. However, an even stronger result, indicating exponential convergence, is possible. As in Theorem 7 of [6], which is based on a classical result of BERNSTEIN (cf. [11, p. 162]), the following can be established.

Theorem 11. Let $u(x)$ be analytic in some open set of the complex plane containing the interval $[0, 1]$. Then, there exists a constant μ with $0 \leqq \mu < 1$, and a sequence of algebraic polynomials $\{\tilde{p}_N(x)\}_{N=4n-1}^{\infty}$ with $\tilde{p}_N(x) \in P_p^{(N)}$ such that

$$
\overline{\lim_{N\to\infty}} \left(\|D^k(u - \tilde{p}_N)\|_{L^\infty}\right)^{1/N} \leqq \mu \quad \text{for all} \quad 0 \leqq k \leqq n. \tag{3.15}
$$

Applying this result to the solution $\varphi(x)$ of (1.1)—(1.2), we then have

Theorem 12. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4), (1.6), and (1.8), be analytic in some open set of the complex plane containing the interval $[0, 1]$, and $\hat{p}_N(x)$ be the unique function which minimizes the functional $F[w]$ over $P_p^{(N)}$ where $N \geqq 4n-1$. Then, there exists a constant μ with $0 \leqq \mu < 1$ such that

$$
\overline{\lim_{N\to\infty}} \left(\|\hat{p}_N - \varphi\|_\gamma\right)^{1/N} = \mu, \tag{3.15}
$$

and consequently from Theorem 3,

$$
\overline{\lim_{N\to\infty}} \left(\|\hat{p}_N - \varphi\|_{L^\infty}\right)^{1/N} \leqq \mu. \tag{3.15'}
$$

§ 4. *L*-Splines and *G*-Splines

To give other examples of subspaces S_{M_i} satisfying the sufficient conditions of Theorem 4, we consider now subspaces of L-splines, which include the Hermite and natural spline subspaces as special cases (cf. [15]). For each positive integer m, let $K_2^m[0, 1]$ denote the collection of all real-valued functions $u(x)$ defined on

$[a, b]$ such that $u(x) \in C^{m-1}[a, b]$ and such that $D^{m-1}u(x)$ is absolutely continuous with $D^m u(x) \in L^2[0, 1]$. Assuming $m \geqq n$, let L be the m-th order differential operator defined by

$$L[u(x)] = \sum_{j=0}^{m} a_j(x) D^j u(x) \tag{4.1}$$

for any $u(x) \in C^m[a, b]$, where $a_j(x) \in K_2^m[a, b]$ for all $0 \leqq j \leqq m$, and $a_m(x) \geqq \omega > 0$ for all $x \in [0, 1]$. Let π: $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$ denote a partition of $[0, 1]$, let $z = (z_1, z_2, \ldots, z_N)$ denote an incidence vector associated with π where the components z_i are positive integers satisfying $1 \leqq z_i \leqq m$ for all $1 \leqq i \leqq N$, and let L^*, the formal adjoint of L, be defined by $L^*[v(x)] = \sum_{j=0}^{m} (-1)^j D^j \{a_j(x) v(x)\}$. Then, the collection of all real-valued functions $s(x)$ such that $s(x) \in K_2^{2m}[x_i, x_{i+1}]$ for each $0 \leqq i \leqq N$, satisfying

$$L^* L[s(x)] = 0 \quad \text{for almost all} \quad x \in (x_i, x_{i+1}), \quad 0 \leqq i \leqq N, \tag{4.2}$$

and

$$D^k s(x_i-) = D^k s(x_i+) \quad \text{for} \quad 0 \leqq k \leqq 2m - 1 - z_i, \quad 1 \leqq i \leqq N, \tag{4.3}$$

with

$$D^k s(0) = D^k s(1) \quad \text{for all} \quad 0 \leqq k \leqq 2n - 1, \tag{4.4}$$

will be denoted by $Sp_p(L, \pi, z)$. Because $m \geqq n$, then $Sp_p(L, \pi, z)$ is readily seen to be a finite-dimensional subspace of S. As a special case, the choice $L = L^* = D^m$ is such that each element $s(x)$ of $Sp_p(L, \pi, z)$ is a piecewise-polynomial function in $[0, 1]$, i.e., $s(x)$ is a polynomial of degree $2m - 1$ in each interval $[x_i, x_{i+1}]$, $0 \leqq i \leqq N$. If, in addition, the components z_i of the incidence vector z are all chosen to be m, we obtain in particular the (smooth) periodic Hermite space $H_p^{(m)}(\pi)$. Such subspaces were used in our practical computations, to be described in § 5.

Given any real-valued function $f(x) \in C^{m-1}[0, 1]$ such that $D^k f(0) = D^k f(1)$ for $0 \leqq k \leqq 2n - 1$, and any partition π of $[0, 1]$ and any associated incidence vector z, there exists a (unique) element (cf. [15, Theorem 3]) $\tilde{s}(x) \in Sp_p(L, \pi, z)$ such that

$$\begin{aligned} D^k \tilde{s}(x_i) &= D^k f(x_i), \quad 0 \leqq k \leqq z_i - 1, \quad 1 \leqq i \leqq N, \\ D^k \tilde{s}(x_i) &= D^k f(x_i), \quad 0 \leqq k \leqq m - 1, \quad i = 0 \text{ and } i = N + 1. \end{aligned} \tag{4.5}$$

This corresponds to interpolation of Type I in [15]. Similarly, if $f(x) \in C_p^{2m-1}[0, 1]$ and $a_j(x) \in C_p^{m-1}[0, 1]$ for all $0 \leqq j \leqq m$, then for any sufficiently fine partition π of $[0, 1]$, i.e., for $\bar{\pi} \equiv \max_{0 \leqq i \leqq N} |x_{i+1} - x_i|$ sufficiently small, and any associated incidence vector z, there exists a (unique) element $\bar{s}(x) \in Sp_p(L, \pi, z)$ such that

$$\begin{aligned} &D^k \bar{s}(x_i) = D^k f(x_i), \quad 0 \leqq k \leqq z_i - 1, \quad 1 \leqq i \leqq N \\ &D^k \bar{s}(x_i) = D^k f(x_i), \quad 0 \leqq k \leqq z_i - 1, \quad i = 0, \quad \text{and} \\ &D^k \bar{s}(0) = D^k \bar{s}(1), \quad z_0 \leqq k \leqq 2m - 1, \end{aligned} \tag{4.6}$$

for any positive integer z_0 with $1 \leqq z_0 \leqq m$. This corresponds to interpolation of Type IV in [15]. For brevity, we shall in the following consider only the interpolation of (4.5). Note that the classical solution $\varphi(x)$ of (1.1)—(1.2) for $2n \geqq m-1$ then always possesses a unique interpolate $\tilde{s}(x)$ in $Sp(L, \pi, z)$ satisfying (4.5) for *any* choice of partition π and associated incidence vector z.

It follows from Theorem 9 of [15] that if $\varphi(x) \in K_2^{2m}[0, 1]$, then, for any partition π of [0, 1] and any associated incidence vector z, there exists a constant M, dependent only on j and m, such that

$$(4.7) \qquad \|D^j(\varphi - \tilde{s})\|_{L^2[0,1]} \leqq M(\bar{\pi})^{2m-j} \|L^* L \varphi\|_{L_2[0,1]} \quad \text{for any} \quad 0 \leqq j \leqq m,$$

where $\varphi(x)$ is the classical solution of (1.1)—(1.2), and $\tilde{s}(x)$ is its unique interpolation (in the sense of (4.5)) in $Sp_p(L, \pi, z)$. As a direct consequence of this inequality and the basic inequalities of (2.9) of Theorem 3, we have

Theorem 13. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4), (1.6), and (1.8), be of class $K_2^{2m}[0, 1]$, and let $\hat{s}(x)$ be the unique function which minimizes the functional $F[w]$ over $Sp_p(L, \pi, z)$. Then, there exists a constant M depending only on m and γ such that

$$(4.8) \qquad \|\hat{s} - \varphi\|_{L^\infty} \leqq K \|\hat{s} - \varphi\|_\gamma \leqq M(\bar{\pi})^{2m-n} \|L^* L \varphi\|_{L^2}.$$

As a direct consequence of the inequality of (4.8), we have the

Corollary. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4), (1.6), and (1.8), be of class $K_2^{2m}[0, 1]$, let $\{\pi_i\}_{i=1}^\infty$ be any sequence of partitions of [0, 1] with $\lim_{i \to \infty} \bar{\pi}_i = 0$, and let $\{z_i\}_{i=1}^\infty$ be any sequence of associated incidence vectors. If $\hat{s}_i(x)$ is the unique function which minimizes the functional $F[w]$ over $Sp_p(L, \pi_i, z_i)$, then $\{\hat{s}_i(x)\}_{i=1}^\infty$ converges uniformly to $\varphi(x)$.

In the special case that $L[u(x)] = D^m u(x)$ for $x \in [0, 1]$, the previous results may be further generalized. As before, let π: $0 = x_0 < x_1 < \cdots < x_{N+1} = 1$ denote a partition of [0, 1], and let $E = (e_{i,j})$ denote an $N \times m$ *incidence matrix*, $1 \leqq i \leqq N$, $0 \leqq j \leqq m-1$, having entries of 0's or 1's, with at least one nonzero entry in each row of E. Further, let e denote the collection of (i, j) such that $e_{i,j} = 1$. Then, from the results of [1, 14], and [15], let $Sp_p(m, \pi, E)$ denote the collection of all real-valued functions $s(x)$, *called periodic g-splines of order m* for π and E, defined on [0, 1] such that

$s(x)$ is a polynomial of degree at most $2m-1$ in each subinterval (x_i, x_{i+1}), $0 \leqq i \leqq N$,

(4.9) $s(x) \in C^{m-1}[0, 1]$ and if $e_{i,j} = 0$, then $D^{2m-j-1}s(x)$ is continuous at x_i, i.e., $(i, j) \notin e$ implies that $D^{2m-j-1}s(x_i-) = D^{2m-j-1}s(x_i+)$, and

$D^k s(0) = D^k s(1)$ for all $0 \leqq k \leqq 2n-1$.

Given any real-valued function $f(x) \in C^{m-1}[0, 1]$ such that $D^k f(0) = D^k f(1)$ for $0 \leqq k \leqq 2n-1$, any partition π of [0, 1] and any incidence matrix E, there exists a (unique) element (cf. [15, Theorem 15]) $\tilde{s}(x) \in Sp_p(m, \pi, E)$ such that

$$(4.10) \qquad \begin{aligned} D^j \tilde{s}(x_i) &= D^j f(x_i) \quad \text{for all} \quad (i, j) \in e, \\ D^k \tilde{s}(x_i) &= D^k f(x_i), \quad 0 \leqq k \leqq m-1, \quad i = 0 \text{ and } i = N+1. \end{aligned}$$

This corresponds to interpolation of Type I in [15]. As before, other types of interpolation are possible, and we refer the reader to [15].

Let $\{\pi_i\}_{i=1}^{\infty}$ and $\{E^{(i)}\}_{i=1}^{\infty}$ be a sequence of partitions of [0, 1] and an associated sequence of incidence matrices such that if π_i: $0=x_0^{(i)}<x_1^{(i)}<\cdots<x_{N_i+1}^{(i)}=1$, then we assume as in [1] and [15] that there exists a positive constant c such that for each k with $0\leqq k\leqq N_i+1$ there exists an integer $j=j(k)$ such that $e_{j,0}^{(i)}=1$ and

$$(4.11)\qquad |x_k^{(i)}-x_j^{(i)}|\leqq c\,\bar{\pi}_i \quad \text{for all} \quad i\geqq 1, \quad \text{all} \quad 0\leqq k\leqq N_i+1.$$

It then follows from Theorem 21 of [15] that if $\varphi(x)\in K_2^{2m}[0,1]$, then there exists a constant M, dependent only on j and m, such that

$$(4.12)\qquad \|D^j(\varphi-\tilde{s}_i)\|_{L^2}\leqq M(\bar{\pi}_i)^{2m-j}\|D^{2m}\varphi\|_{L^2} \quad \text{for any} \quad 0\leqq j\leqq m,$$

where $\varphi(x)$ is the classical solution of (1.1)—(1.2), and $\tilde{s}_i(x)$ is its unique interpolation (in the sense of (4.10)) in $Sp_p(m,\pi_i,E_i)$. As a direct consequence of this inequality and the basic inequalities of (2.9) of Theorem 3, we have

Theorem 14. Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4), (1.6), and (1.8), be of class $K_2^{2m}[0,1]$, and let $\hat{s}_i(x)$ be the unique function which minimizes the functional $F[w]$ over $Sp_p(m,\pi_i,E_i)$, where the partitions $\{\pi_i\}_{i=1}^{\infty}$ and associated incidence matrices $\{E^{(i)}\}_{i=1}^{\infty}$ satisfy the hypothesis of (4.11). Then, there exists a constant M, depending only on m and γ, such that

$$(4.13)\qquad \|\hat{s}_i-\varphi\|_{L^\infty}\leqq K\|\hat{s}_i-\varphi\|_\gamma\leqq M(\bar{\pi}_i)^{2m-n}\|D^{2m}\varphi\|_{L^2} \quad \text{for all} \quad i\geqq 1.$$

If, in addition, $\lim_{i\to\infty}\bar{\pi}_i=0$, then the sequence $\{\hat{s}_i(x)\}_{i=1}^{\infty}$ converges uniformly to $\varphi(x)$.

§ 5. Numerical Results

In this section, we discuss the numerical results we have obtained for some concrete examples by using particular subspaces described in §§ 3—4 in the Rayleigh-Ritz procedure. Let us however first summarize the results of §§ 3—4 by comparing the *asymptotic* error estimates for the subspaces $T^{(N)}$ and $Sp_p(L,\pi,\mathfrak{z})$ in terms of the total number of parameters associated with each of the subspaces.

Let $\varphi(x)$, the solution of (1.1)—(1.2), subject to the conditions of (1.4), (1.6), and (1.8), be of class $C_p^{s-1}[0,1]$ with $D^s\varphi\in L^2(0,1)$, where $s\geqq 2n$. If $\hat{t}_N(x)$ is the unique function which minimizes $F[w]$ over the subspace $T^{(N)}$, then (cf. Theorem 6)

$$(5.1)\qquad \|\hat{t}_N-\varphi\|_{L^\infty}=\mathcal{O}\left\{\frac{1}{(d_N-1)^{s-n}}\right\} \quad \text{as} \quad N\to\infty, \quad d_N\equiv 2N+1,$$

where d_N is the dimension of $T^{(N)}$. Next, consider a sequence of (smooth) periodic Hermite subspaces $\{H_p^{(m)}(\pi_i)\}_{i=1}^{\infty}$ where *m is fixed*. As previously noted, $H_p^{(m)}(\pi_i)$ corresponds to the periodic L-spline subspace $Sp_p(D^m,\pi_i,\mathfrak{z}^{(i)})$ where each component of the associated incidence vector $\mathfrak{z}^{(i)}$, $z_j^{(i)}$, satisfies $z_j^{(i)}=m$. Such subspaces were used in the actual numerical computations to be described. We assume that the partitions π_i: $0=x_0^{(i)}<x_1^{(i)}<\cdots<x_{N_i+1}^{(i)}=1$ satisfy the regularity conditions that

$$(5.2)\qquad \bar{\pi}_i\leqq\frac{K}{N_i+1} \quad \text{for all} \quad i\geqq 1, \quad \text{and} \quad \lim_{i\to\infty}N_i=+\infty.$$

If $\hat{u}_i(x)$ is the unique function which minimizes the functional $F[w]$ over $H_p^{(m)}(\pi_i)$, then (cf. Theorem 13)

$$(5.3)\qquad \|\hat{u}_i-\varphi\|_{L^\infty}=\mathcal{O}\left\{\frac{1}{(d_i-m+2n)^{2m-n}}\right\},\quad d_i=m(N_i+2)-2n,\quad i\to\infty,$$

where d_i is the dimension of $H_p^{(m)}(\pi_i)$. Thus, in the special case that $s=2m$, we observe, surprisingly enough, that these theoretical error estimates are *asymptotically the same.* β

We now consider the numerical solution of particular examples of (1.1)—(1.2). As our first example, consider the linear differential equation

$$(5.4)\qquad \begin{aligned} D^2u(x)=u(x)+\sin(2\pi x)\left\{(-1-4\pi^2)\left(x^3-\frac{4}{3}x^2+\frac{x}{3}\right)+6x-\frac{8}{3}\right\}\\ +4\pi\cos(2\pi x)\left\{3x^2-\frac{8}{3}x+\frac{1}{3}\right\},\quad 0<x<1,\end{aligned}$$

subject to the boundary conditions of

$$(5.5)\qquad u(0)=u(1),\qquad Du(0)=Du(1).$$

For this example, we have $p_1(x)\equiv 1$, $p_0(x)\equiv 0$ in (1.3), and thus (1.4) is satisfied. By virtue of SOBOLEV's inequality in one dimension [19, p. 174], we have that there exists a positive constant K such that

$$(5.6)\qquad \|w\|_{1,2}\equiv\left\{\int_0^1\{(Dw(t))^2+(w(t))^2\}dt\right\}^{\frac{1}{2}}\geq K\|w\|_{L^\infty}\quad\text{for all}\quad w(x)\in S.$$

Thus, we see that the inequality of (1.6) is satisfied for the choice $\beta=+1$. Next, we see that the quantity Λ of (1.7) is necessarily zero for this example, and as $f(x,u)=u+g(x)$, then $f_u\equiv+1$, showing that we can choose $\gamma=+1$. Thus, the inequality of (1.8) is satisfied. The unique solution $\varphi(x)$ of (5.4)—(5.5) is given by

$$(5.7)\qquad \varphi(x)=\left(x^3-\frac{4}{3}x^2+\frac{x}{3}\right)\sin(2\pi x),\quad 0\leq x\leq 1.$$

In this case, $\varphi(x)\in C_p^1[0,1]\cap C^2[0,1]$, and Theorem 6 is applicable with $s=2$ and $n=1$.

The numerical results of minimizing the associated functional $F[w]$ over the trigonometric subspaces $T^{(N)}$ are summarized in Table 5.1. For specific details about the efficient numerical minimization of $F[w]$ over various subspaces, we refer the reader to [6] and [8].

For purposes of comparison, numerical results were also obtained for minimizing the functional $F[w]$ over the (smooth) periodic cubic Hermite subspaces

Table 5.1. *Trigonometric subspaces* $T^{(N)}$

N in $T^{(N)}$	dim $T^{(N)}$	$\|\hat{u}_N-\varphi\|_{L^\infty}$
4	9	$9.63\cdot 10^{-4}$
6	13	$3.98\cdot 10^{-4}$
8	17	$2.17\cdot 10^{-4}$
10	21	$1.36\cdot 10^{-4}$
12	25	$9.35\cdot 10^{-5}$

Table 5.2. *Smooth periodic Hermite subspaces* $H_p^{(2)}(\pi(h))$

h	dim $H^{(2)}(\pi(h))$	$\|\hat{u}_h-\varphi\|_{L^\infty}$
$\frac{1}{5}$	10	$7.48\cdot 10^{-4}$
$\frac{1}{7}$	14	$2.21\cdot 10^{-4}$
$\frac{1}{10}$	20	$6.42\cdot 10^{-5}$
$\frac{1}{16}$	32	$1.10\cdot 10^{-5}$
$\frac{1}{25}$	50	$1.97\cdot 10^{-6}$

$H_p^{(2)}(\pi(h))$, where a uniform partition $\pi(h)$ of $[0, 1]$ with mesh-size h was used. The results are summarized in Table 5.2. For this case, Theorem 13 is applicable with $m=2$ and $n=1$.

As our second example, consider the *nonlinear* differential equation

$$(5.8)\qquad \begin{aligned} D^2 u(x) = u(x) + (u(x))^3 + e^{\sin(2\pi x)}\,[4\pi^2\cos^2(2\pi x) \\ - 4\pi^2 \sin(2\pi x) - e^{2\sin(2\pi x)} - 1], \quad 0 < x < 1, \end{aligned}$$

subject to the boundary conditions of

$$(5.9)\qquad u(0) = u(1), \quad Du(0) = Du(1).$$

As in the previous example, we have $p_1(x) \equiv 1$, $p_0(x) \equiv 0$ in (1.3), and thus (1.4) is again satisfied. From the inequality of (5.6), we again have that the inequality of (1.6) is satisfied for the choice $\beta = +1$, and Λ of (1.7) is again zero. For this example, $f(x, u) = u + u^3 + g(x)$, so that $f_u(x, u) = 1 + 3u^2 \geqq +1$, and again the inequality of (1.8) is satisfied for the choice $\gamma = +1$. The unique solution of (5.8)—(5.9) is given by

$$(5.10)\qquad \varphi(x) = e^{\sin 2\pi x}, \quad 0 \leqq x \leqq 1.$$

Thus, for this case $\varphi(x) \in C_p^\infty[0, 1]$ and $\varphi(x)$ can be extended to an entire function (cf. § 3). Thus, Theorem 8 is applicable with $\mu = 0$, and we have

$$(5.11)\qquad \lim_{N\to\infty} (\|\hat{p}_N - \varphi\|_\infty)^{1/N} = 0.$$

The numerical results of minimizing the associated functional $F[w]$ over the trigonometric subspaces $T^{(N)}$ are summarized in Table 5.3.

Table 5.3. *Trigonometric subspaces* $T^{(N)}$

N in $T^{(N)}$	dim $T^{(N)}$	$\|\hat{u}_N - \varphi\|_{L^\infty}$
3	7	$7.32 \cdot 10^{-3}$
4	9	$6.04 \cdot 10^{-4}$
5	11	$6.22 \cdot 10^{-5}$
6	13	$5.49 \cdot 10^{-6}$

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