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Numerical Methods of High-Order Accuracy for Nonlinear Boundary Value Problems
V. Monotone Operator Theory*

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§ 1. Introduction

The purpose of this paper is to study the Galerkin or projectional method for approximating the solutions of a wide class of nonlinear elliptic boundary value problems, cf. [2, 7—11, 18, 22, 25, 26, 35, 36, 38], and [39]. We study in § 2 the Galerkin method for approximating the solutions of a class of abstract monotone operator equations in reflexive Banach spaces, as originally considered by Zarantonello [42] and Minty [27].

In § 3, we give sufficient conditions which guarantee that a particular nonlinear elliptic boundary value problem in n-dimensions be equivalent to an operator equation satisfying the hypotheses of § 2. In § 4, a priori bounds are determined for the solution of a model semi-linear problem (cf. Eq. (4.1)-(4.2)), such equations having been studied in [9, 13, 23], and [32]. With these a priori bounds, the semi-linear problem is then redefined, so that the results of § 2 and 3 are applicable.

In § 5, we apply Galerkin's method to polynomial-type subspaces, as considered by Harrick [15]. In § 6, we apply the results of [2] concerning bivariate piecewise Hermite polynomial subspaces in rectangular polygons to two-dimensional nonlinear elliptic boundary value problems satisfying the conditions of § 3. In particular, new results for the model semi-linear equation, treated in [9, 13, 23], and [32], are obtained, and these compare favorably with results of [13] and [32].

Finally, in § 7, we discuss nonlinear two-point boundary value problems, and generalize the results of [10] in several directions.

§ 2. Monotone Operator Theory

Let B be a reflexive Banach space over the real field and let B^* be the dual space of B. We will denote by $\|\cdot\|$ (resp. $\|\cdot\|^*$) the norm in B (resp. in B^*) and by $(\cdot,\cdot)$ the pairing between B and B^*, i. e., if $v^* \in B^*$ and $u \in B$, then the value of the functional v^* at u is (v^*, u).

Let T be a (possibly nonlinear) mapping from B into B^* satisfying the following two hypotheses:

(H_1): T is *finitely continuous*, i. e., T is continuous from finite-dimensional subspaces of B into B^* with the weak-star topology of B^*. In other words, given

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any finite-dimensional subspace B^k of B and any sequence $\{u_n\}_{n=1}^{\infty}$ of elements of B^k which converges to an element $u \in B^k$, the sequence $\{(Tu_n, v)\}_{n=1}^{\infty}$ converges to (Tu, v) for any $v \in B$,

(H_2): T is *strongly monotone*, i. e., there exists a continuous and strictly increasing function $c(r)$ on $[0, +\infty)$ with $c(0)=0$ and $\lim_{r\to+\infty} c(r) = +\infty$ such that

$$|(Tu - Tv, u - v)| \geqq c(\|u-v\|)\cdot\|u-v\| \quad \text{for all} \quad u, v \in B. \tag{2.1}$$

We consider the following problem (Problem P): determine $u \in B$ such that

$$Tu = 0, \tag{2.2}$$

or equivalently such that

$$(Tu, v) = 0 \quad \text{for all} \quad v \in B. \tag{2.3}$$

Similarly, given a finite-dimensional subspace B^k of B, we consider the following approximate problem (Problem P^k): determine $u_k \in B^k$ such that

$$(Tu_k, v) = 0 \quad \text{for all} \quad v \in B^k. \tag{2.4}$$

We now state the following result, due to Browder [5]:

Lemma 2.1. Let T satisfy (H_1) and (H_2). Then Problem P has a unique solution u. Similarly, given any finite-dimensional subspace B^k of B, the corresponding Problem P^k has a unique solution u_k.

To have an estimate between the solution u of Problem P and the solution u_k of Problem P_k, we need additional hypotheses on the mapping T (cf. Theorem 2.1). These in turn will allow us to obtain sufficient conditions guaranteeing the convergence of the u_k's to the solution u (cf. Corollary 2.1). We begin with

Theorem 2.1. Let T satisfy (H_1), (H_2), and

(H_3): T is *bounded*, i. e., T maps bounded subsets of B into bounded subsets of B^* (with respect to the strong topology of B^*). Then, given any finite-dimensional subspace B^k of B, there exists a constant K, independent of B^k, such that

$$c(\|u_k - u\|)\cdot\|u_k - u\| \leqq K \inf\{\|w-u\|; \quad w \in B^k\}. \tag{2.5}$$

Similarly, let T satisfy (H_1),

(H_2'): condition (H_2) holds with $c(r) \equiv \alpha r$, $\alpha > 0$, i. e.,

$$|(Tu - Tv, u - v)| \geqq \alpha(\|u-v\|)^2 \quad \text{for all} \quad u, v \in B, \tag{2.6}$$

and

(H_3'): T is Lipschitz continuous with respect to the strong topology of B^* for bounded arguments (a special case of hypothesis (H_3)), i. e., given $M > 0$, there exists a constant $C(M)$, depending only upon M, such that

$$\|Tu - Tv\|^* \leqq C(M)\|u-v\| \quad \text{for all} \quad u, v \in B \quad \text{with} \quad \|u\|, \|v\| \leqq M. \tag{2.7}$$

Then, given any finite-dimensional subspace B^k of B, there exists a constant K' independent of B^k, such that

$$\|u_k - u\| \leqq K' \inf\{\|w-u\|; \quad w \in B^k\}. \tag{2.8}$$

Proof. We begin by showing that (H_2) implies that the same a priori bound holds for both the solution u and the "approximate" solutions u_k. We have, by using (2.1) and (2.4)

$$c(\|u_k\|)\,\|u_k\| \leqq |(Tu_k - T0, u_k)| = |(T0, u_k)| \leqq \|T0\|^* \|u_k\|,$$

and thus $c(\|u_k\|) \leqq M_0$, with $M_0 = \|T0\|^*$. Clearly, the same bound is valid for u.

Let w be now an arbitrary element of B^h. Then by (2.3) and (2.4), we have $(Tu_k - Tu, u_k - w) = 0$ since $\{u_k - w\} \in B^h \subset B$. Thus from (2.1),

$$c(\|u_k - u\|)\,\|u_k - u\| \leqq |(Tu_k - Tu, u_k - u)| = |(Tu_k - Tu, w - u)| \leqq \|Tu_k - Tu\|^* \|w - u\|. \tag{2.9}$$

If T is bounded, then $\|Tu_k - Tu\|^*$ is bounded independently of B^h and the conclusion of (2.5) follows, since w is arbitrary. Similarly, if T satisfies (H_2') and (H_3'), the conclusion of (2.8) follows with $K' = C(M_0)/\alpha$, by (2.9). Q.E.D.

As an immediate consequence, we have:

Corollary 2.1. Let $\{B^h\}_{k=1}^{\infty}$ be a sequence of finite-dimensional subspaces of B with the property that

$$\lim_{k \to +\infty} \{\inf \{\|w - u\|; \quad w \in B^h\}\} = 0, \tag{2.10}$$

where u is the unique solution of Problem P. If T satisfies (H_1), (H_2), (H_3) (including as a special case (H_1), (H_2'), (H_3')), then

$$\lim_{k \to +\infty} \{\|u_k - u\|\} = 0, \tag{2.11}$$

where u_k, $k = 1, 2, \ldots$, are the unique solutions of Problem P_h.

§ 3. Nonlinear Elliptic Partial Differential Equations

The object of this section is to show how the problem of approximating the (generalized) solution of a certain class of nonlinear boundary value problems can be put into the framework of § 2.

Let there be given a bounded open subset Ω of R^n, $n \geqq 1$, whose boundary $\partial\Omega$ is such that the Sobolev Imbedding Theorem (cf. for example [28, p. 72] and [41, p. 174]) holds. For example, this will be the case if $\partial\Omega$ is Lipschitz continuous. Using the standard multi-index notation (cf. [41, p. 27]), we then consider the following formal (real) $2m$-th order Dirichlet problem: Find a solution u of

$$\sum_{|\alpha| \leqq m} (-1)^{|\alpha|} D^\alpha \{A_\alpha(x, u, \ldots, D^m u)\} = 0, \quad x \in \Omega, \tag{3.1}$$

$$D^\beta u(x) = 0, \quad x \in \partial\Omega \quad \text{for all} \quad |\beta| \leqq m-1, \tag{3.2}$$

where $A_\alpha(x, u, \ldots, D^m u)$ denotes a function which can depend upon x and *any* $D^\gamma u$ with $|\gamma| \leqq m$.

For a given p, $1 < p < +\infty$, we shall consider the Sobolev space

$$W^{m,p}(\Omega) = \{u;\, D^\alpha u \in L^p(\Omega) \quad \text{for all} \quad |\alpha| \leqq m\},$$

where the derivatives $D^\alpha u$ in this definition are to be taken in the sense of the theory of distributions. Because $1 < p < +\infty$, $W^{m,p}(\Omega)$ is a reflexive Banach space with respect to the norm

$$\|u\|_{m,p} = \left\{ \sum_{|\alpha| \leq m} \int_{\Omega} |D^{\alpha} u|^p \, dx \right\}^{1/p}. \tag{3.3}$$

We will denote by $W_0^{m,p}(\Omega)$ the closure in $W^{m,p}(\Omega)$ of the space of inf differentiable functions with compact support in Ω, with respect to the of (3.3). Thus, the space

$$B \equiv W_0^{m,p}(\Omega)$$

equipped with the norm of (3.3) is a reflexive Banach space, since it is a subspace of a reflexive Banach space.

For $u, v \in B$, we formally define the "quasibilinear" form

$$a(u, v) = \sum_{|\alpha| \leq m} \int_{\Omega} A_{\alpha}(x, u, \ldots, D^m u) D^{\alpha} v \, dx. \tag{3.4}$$

Then (cf. for example [4, Definition p. 864]), u is said to be a *generalized sol* of (3.1)-(3.2), relative to the space B, if and only if

$$a(u, v) = 0 \quad \text{for all} \quad v \in B. \tag{3.5}$$

In Theorem 3.1 below, we give a sufficient set of hypotheses (essentially a the admissible growth of the functions $A_{\alpha}(x, u, \ldots, D^m u)$ with respect to arguments $u, \ldots, D^m u$) which guarantee that for a fixed $u \in B$, the qua linear form $a(u, v)$ is a bounded linear functional for $v \in B$. This will in turn a us to define a mapping T acting from B into the dual space B^* of B. In T rem 3.2, we will show that this mapping is bounded and finitely continuous in Theorem 3.3 and Corollary 3.1, we will apply the results of § 2 to the n ping T. We begin with the following improvement of a result of BROWDER [6

Theorem 3.1. Let the functions $A_{\alpha}(x, u, \ldots, D^m u)$ appearing in (3.1) measurable in $x \in \Omega$, and continuous in their other arguments $D^{\gamma} u$, $|\gamma| \leq m$, almost all $x \in \Omega$.

Let $\infty > p > 1$, and let $g(r)$ be a nonnegative continuous function on $[0, +$ such that

$$\begin{aligned} |A_{\alpha}(x, u, \ldots, D^m u)| \leq & \left\{ g\left(\sum_{|\beta| < m-(n/p)} |D^{\beta} u| \right) \right\} \\ & \cdot \left\{ 1 + \sum_{|\beta| = m-(n/p)} |D^{\beta} u| + \sum_{m-(n/p) < |\beta| \leq m} |D^{\beta} u|^{q_{\beta}/p_{\alpha}} \right\}, \end{aligned} \tag{3.6}$$

for all $|\alpha| \leq m$, almost all $x \in \Omega$, and all $D^{\gamma} u$, $|\gamma| \leq m$, where

$$q_{\beta} = \frac{np}{n - pm + p|\beta|}, \quad \text{and}$$

$$\frac{1}{p_{\alpha}} = \begin{cases} 1, & \text{if } |\alpha| < m - \dfrac{n}{p}, \\ 1 - \dfrac{1}{p} + \dfrac{m - |\alpha|}{n}, & \text{if } |\alpha| \geq m - \dfrac{n}{p}. \end{cases} \tag{3.7}$$

Then, for a given $u \in B = W_0^{m,p}(\Omega)$, there exists a constant $K = K_u$, depend only upon u, such that

$$|a(u, v)| \leq K_u \|v\|_{m,p} \quad \text{for all} \quad v \in B. \tag{3.8}$$

Proof. For $w \in L^r(\Omega)$, $1 \leqq r \leqq +\infty$, we denote by $\|w\|_r$ the corresponding norm. Throughout this proof, K will designate a generic constant.

Let u be any element in B. By the Sobolev Imbedding Theorem (cf. [28, p. 72]), $D^\beta u \in C^0(\bar{\Omega})$ for all $|\beta| < m - \frac{n}{p}$, and further there exists a constant K such that $\|D^\beta u\|_\infty \leqq K\|u\|_{m,p}$, for all such β's. Thus, the quantity $\left\{g\left(\sum_{|\beta|<m-(n/p)} |D^\beta u|\right)\right\}$ appearing in (3.6) is a bounded continuous function on $\bar{\Omega}$, so that

$$\begin{aligned}\left|\int_\Omega A_\alpha(x, u, \dots, D^m u)\, D^\alpha v\, dx\right| &\leqq \left\{\left\|g\left(\sum_{|\beta|<m-(n/p)} |D^\beta u|\right)\right\|_\infty\right\} \\ &\cdot\left\{\int_\Omega |D^\alpha v|\, dx + \sum_{|\beta|=m-(n/p)} \int_\Omega |D^\beta u|\,|D^\alpha v|\, dx \right. \\ &\left. + \sum_{m-(n/p)<|\beta|\leqq m} \int_\Omega |D^\beta u|^{q_\beta/p_\alpha} |D^\alpha v|\, dx\right\}.\end{aligned} \tag{3.9}$$

We can bound each integral in the right-hand side of the inequality of (3.9) as follows. First

$$\int_\Omega |D^\alpha v|\, dx \leqq (\text{Meas}\,\Omega)^{1-(1/p)} \|D^\alpha v\|_p \leqq K\|v\|_{m,p}. \tag{3.10}$$

Next, if $|\beta| = m - \frac{n}{p}$, $D^\beta u \in L^q(\Omega)$ for all real $q \geqq 1$ (cf. [28, Theorem 3.7, p. 72]), and since $q = p/(p-1) \geqq 1$, we have from Hölder's inequality

$$\int_\Omega |D^\beta u|\,|D^\alpha v|\, dx \leqq \{\|D^\beta u\|_{p/(p-1)}\} \|D^\alpha v\|_p \leqq K\|v\|_{m,p}. \tag{3.11}$$

Finally, if $m - \frac{n}{p} < |\beta| \leqq m$, $D^\beta u \in L_{q_\beta}$ with q_β defined as in (3.7) (cf. [28, Theorem 3.6, p. 72]). Assume first that $|\alpha| < m - \frac{n}{p}$, so that the quantity $1/p_\alpha$ as defined in (3.7) is equal to unity. Since then $D^\alpha v \in L^\infty(\Omega)$ and $\|D^\alpha v\|_\infty \leqq K\|v\|_{m,p}$, it follows that

$$\int_\Omega |D^\beta u|^{q_\beta/p_\alpha} |D^\alpha v|\, dx \leqq \{\|D^\beta u\|_{q_\beta}\}^{q_\beta} \|D^\alpha v\|_\infty \leqq K\|v\|_{m,p}. \tag{3.12}$$

If $|\alpha| \geqq m - n/p$, then $D^\alpha v \in L_{q_\alpha}$ with $q_\alpha = np/(n - pm + p|\alpha|)$, and $\|D^\alpha v\|_{q_\alpha} \leqq K\|v\|_{m,p}$. Since in that case $1/p_\alpha + 1/q_\alpha = 1$, it follows that

$$\int_\Omega |D^\beta u|^{q_\beta/p_\alpha} |D^\alpha v|\, dx \leqq \{\|D^\beta u\|_{q_\beta}\}^{q_\beta/p_\alpha} \|D^\alpha v\|_{q_\alpha} \leqq K\|v\|_{m,p}. \tag{3.12'}$$

To complete the proof, it suffices to observe that the inequalities (3.9) to (3.12″) imply that of (3.8). Q.E.D.

As a consequence of Theorem 3.1, the quantity $a(u, v)$ as defined in (3.4) is for each $u \in B$ a continuous linear functional of $v \in B$, and we can write

$$a(u, v) = (Tu, v) \quad \text{for all} \quad u, v \in B, \tag{3.13}$$

which defines a mapping T from $B = W_0^{m,p}(\Omega)$ into B^* (which is usually denoted $W^{-m,p'}(\Omega)$, with $p' = p/(p-1)$; cf. [24, p. 19]). We next have:

Theorem 3.2. With the same assumptions as in Theorem 3.1, the mapping T as defined in (3.13) is bounded and finitely continuous.

Proof. First, it follows from inequalities (3.9) to (3.12″) and arguments similar to those used in the proof of Theorem 3.1 that there exists a nonnegative continuous function $h(r)$ defined on $[0, +\infty)$ such

$$|(Tu, v)| = |a(u, v)| \leqq h(\|u\|_{m,p}) \|v\|_{m,p} \quad \text{for all} \quad u, v \in B. \tag{3.14}$$

Thus the mapping T is bounded.

Let now B^k be a k-dimensional subspace of B, and let $\{u_j\}_{j=1}^{\infty}$ be a sequence of elements in B^k which converges to an element $u \in B^k$. We must show that $\{(Tu_j, v) = a(u_j, v)\}_{j=1}^{\infty}$ converges to $(Tu, v) = a(u, v)$, for any $v \in B$. Since the sequence $\{u_j\}_{j=1}^{\infty}$ converges in B, there exists a subsequence, still denoted by $\{u_j\}_{j=1}^{\infty}$ for convenience, such that $\{D^{\alpha} u_j\}_{j=1}^{\infty}$ converges a. e. in Ω to $D^{\alpha} u$ as $j \to \infty$, for each $|\alpha| \leqq m$. Likewise, the functions $A_{\alpha}(x, u_j, \ldots, D^m u_j)$ converge a. e. in Ω to $A_{\alpha}(x, u, \ldots, D^m u)$, since these functions are continuous in their arguments $u, \ldots, D^m u$ for almost all x in Ω. Then by Lebesgue's Dominated Convergence Theorem, it will follow that $\{a(u_j, v)\}_{j=1}^{\infty}$ converges to $a(u, v)$ as $n \to \infty$, if we can show that the absolute value of the integrand $\left\{\sum_{|\alpha| \leqq m} A_{\alpha}(x, u_j, \ldots, D^m u_j) D^{\alpha} v\right\}$ is bounded a. e. in Ω by an $L^1(\Omega)$-function, independently of j.

Let w_i, $1 \leqq i \leqq k$, be a basis in B^k. Thus any element $w \in B^k$ has a unique representation in the form $w = \sum_{i=1}^{k} a_i w_i$, and $\|w\| = \max\{|a_i|;\ 1 \leqq i \leqq k\}$ is a norm on B^k. In particular, the u_j's and u have unique representations $u_j = \sum_{i=1}^{k} a_i^j w_i$, and $u = \sum_{i=1}^{k} a_i w_i$, respectively. Since all norms are equivalent on a finite-dimensional space, there exists a constant K such that $\|w\| \leqq K \|w\|_{m,p}$, for all $w \in B^k$. In particular then, there exists a constant K' such that

$$|a_i^j|,\ |a_i| \leqq K' \quad \text{for all} \quad 1 \leqq i \leqq k \quad \text{and all } j. \tag{3.15}$$

We now have, by (3.6) and (3.15), that for each α with $|\alpha| \leqq m$,

$$\begin{aligned}
|A_{\alpha}(x, u_j, \ldots, D^m u_j) D^{\alpha} v| &\leqq \Big\{g\Big(\sum_{|\beta| < m-(n/p)} |D^{\beta} u_j|\Big) D^{\alpha} v\Big\} \\
&\quad \cdot \Big\{1 + \sum_{|\beta| = m-(n/p)} \Big|D^{\beta}\Big(\sum_{i=1}^{k} a_i^j w_i\Big)\Big| + \sum_{m-(n/p) < |\beta| \leqq m} \Big|D^{\beta}\Big(\sum_{i=1}^{k} a_i^j w_i\Big)\Big|^{q_{\beta}/p_{\alpha}}\Big\} \\
&\leqq \Big\{g\Big(\sum_{|\beta| < m-(n/p)} |D^{\beta} u_j|\Big) D^{\alpha} v\Big\} \\
&\quad \cdot \Big\{1 + K'' \sum_{|\beta| = m-(n/p)} \Big|D^{\beta}\Big(\sum_{i=1}^{k} w_i\Big)\Big| + K'' \sum_{m-(n/p) < |\beta| \leqq m} \Big|D^{\beta}\Big(\sum_{i=1}^{k} w_i\Big)\Big|^{q_{\beta}/p_{\alpha}}\Big\},
\end{aligned}$$

where K'' is a constant independent of j.

As in the proof of Theorem 3.1, the quantity in the right-hand side of the above inequality is an $L^1(\Omega)$-function, which is moreover independent of j.

We have thus proved the weak convergence of a subsequence of the original sequence $\{Tu_j\}_{j=1}^{\infty}$. In fact, the whole subsequence weakly converges to Tu, since the limit is unique. Q.E.D.

The following theorem, as well as Corollary 3.1, summarizes the application of the results of this section and those of Theorem 2.1 and Corollary 2.1 of § 2.

Theorem 3.3. If the coefficient functions $A_\alpha(x, u, \ldots, D^m u)$ satisfy the same hypotheses as in Theorem 3.1, and if the mapping T defined by (3.13) is strongly monotone, i. e., there exists a continuous and strictly increasing function $c(r)$ on $[0, +\infty)$ with $c(0)=0$, and $\lim_{r\to+\infty} c(r)=+\infty$ such that

$$(3.16) \qquad |(Tu-Tv, u-v)| \geq c(\|u-v\|_{m,p})\,\|u-v\|_{m,p} \quad \text{for all} \quad u, v \in B,$$

then the nonlinear Dirichlet problem (3.1)-(3.2) has a unique generalized solution u in $B=W_0^{m,p}(\Omega)$, where $1<p<+\infty$.

If B^k is any finite-dimensional subspace of B, then the corresponding Problem P^k (as defined in (2.4)) has a unique solution $u_k \in B^k$, and there exists a constant K independent of B^k such that

$$(3.17) \qquad c(\|u_k-u\|_{m,p})\,\|u_k-u\|_{m,p} \leq K \inf\{\|w-u\|_{m,p};\ \ w\in B^k\}.$$

Similarly, if condition (3.16) holds with $c(r)\equiv\alpha r$, $\alpha>0$, and T is Lipschitz continuous (hypothesis (H_3') of Theorem 2.1), there exists a constant K' independent of B^k such that

$$(3.18) \qquad \|u_k-u\|_{m,p} \leq K' \inf\{\|w-u\|_{m,p};\ \ w\in B^k\}.$$

Corollary 3.1. Let $\{B^k\}_{k=1}^\infty$ be a sequence of finite-dimensional subspaces of B with the property that

$$(3.19) \qquad \lim_{k\to+\infty}\{\inf\|w-u\|_{m,p};\ w\in B^k\}=0,$$

where u is the unique generalized solution of (3.1)-(3.2). Then, with the same assumptions upon the mapping T as in Theorem 3.3,

$$\lim_{k\to+\infty}\{\|u_k-u\|_{m,p}\}=0.$$

To conclude this section, we make two remarks. First, when the generalized solution u is sufficiently smooth, the quantity $\inf\{\|u-w\|_{m,p};\ w\in B^k\}$ appearing in (3.17) or (3.18) can be bounded for particular choices of subspaces B^k, as will be shown in §§ 5, 6, and 7, thus yielding an error estimate for the numerical approximation of (3.1)-(3.2). Second, it is easily seen that the growth conditions imposed upon the coefficient functions $A_\alpha(x, u, \ldots, D^m u)$ by the inequalities of (3.6) are very restrictive. A general method will be described in detail in the next section for a model problem, for studying nonlinear Dirichlet problems, for which the coefficient functions do not satisfy conditions (3.6). The method consists of finding a priori bounds for the solution (and also in some cases for its derivatives, as in the problem of (7.26)-(7.27)) in the L_∞-norm, and using these bounds, to *modify* (3.1) so that the new coefficient functions $A_\alpha(x, u, \ldots, D^m u)$ satisfy (3.6), in such a way that the unique solution of the modified nonlinear Dirichlet problem coincides with that of (3.1)-(3.2).

§ 4. A Priori Bounds

In this section, we show for a model problem (cf. (4.1)-(4.2)) how to apply the method briefly described above to a class of problems not directly covered by our previous analysis. For ease of exposition, we consider the following "model problem": Find a solution u of

(4.1) $$\Delta u(x) \equiv \sum_{i=1}^{n} \frac{\partial^2 u(x)}{\partial x_i^2} = f(x, u), \quad x \in \Omega,$$

(4.2) $$u(x) = 0, \quad x \in \partial\Omega.$$

The following result was proved (in the case $n=2$) by LEVINSON [23] and we now sketch his proof.

Theorem 4.1. Let Ω be a bounded open subset of R^n, $n \geqq 1$. Assume that $f(x, u) \in C^0(\bar{\Omega} \times R)$, and that

(4.3) $$\liminf_{|u| \to +\infty} \frac{f(x, u)}{u} \geqq 0, \quad \text{uniformly in} \quad x \in \bar{\Omega}.$$

Then if $\varphi(x)$ is a classical solution of (4.1)-(4.2), i. e., $\varphi(x) \in C^0(\bar{\Omega}) \cap C^2(\Omega)$, there exists a constant M_1 which can be computed a priori such that

(4.4) $$\|\varphi\|_\infty \leqq M_1.$$

Proof. Since Ω is a bounded subset of R^n, let $a > 0$ be so determined that $\cos(a x_1) - \sin^2(a x_1) > \frac{1}{2}$, for all $x = (x_1, \ldots, x_n) \in \bar{\Omega}$. By (4.3), there exists $M_1 > 0$ such that $|u| \geqq M_1$ implies $f(x, u)/u \geqq -a^2/4$, for all $x \in \bar{\Omega}$. Consider the function $\Psi(x) \equiv \varphi(x) \exp(-\cos(a x_1))$. Then, $\|\Psi\|_\infty \leqq M_1$, for otherwise, there would exist an $\bar{x} \in \Omega$ such that $|\Psi(\bar{x})| = \|\Psi\|_\infty > M_1$. Suppose $\bar{x}$ corresponds to a maximum (a similar argument would hold for a minimum). A direct computation shows that $\Delta \Psi(\bar{x}) > a^2 M_1/4$, which is impossible at a maximum. Hence, $\|\varphi\|_\infty \leqq \|\Psi\|_\infty \leqq M_1$. Q.E.D.

The following result was proved for the case $n=1$ by LEES [21], and for the case $n=2$ by CIARLET [9].

Theorem 4.2. Let Ω be a bounded open subset of R^n, $n \geqq 1$. Let $\partial\Omega$, the boundary of Ω, be smooth in the sense that the boundary value problem

$$\Delta u(x) = -1, \quad x \in \Omega,$$
$$u(x) = 0, \quad x \in \partial\Omega,$$

has a classical solution $\Psi(x)$. Assume that $f(x, u) \in C^0(\bar{\Omega} \times R)$ and let

$$\varrho \equiv \|\Psi\|_\infty, \quad \mathcal{M} \equiv \sup\{|f(x, 0)|; \quad x \in \bar{\Omega}\}.$$

Assume further that $f(x, u)$ is a continuously differentiable function with respect to u for each $x \in \bar{\Omega}$, and that

(4.5) $$\frac{\partial f}{\partial u}(x, u) \geqq \gamma > -\frac{1}{\varrho} \quad \text{for all} \quad x \in \bar{\Omega} \text{ and all real } u,$$

for some constant $\gamma \leqq 0$. If $\varphi(x)$ is a classical solution of (4.1)-(4.2), then

(4.6) $$\|\varphi\|_\infty \leqq M_2,$$

where $M_2 = \mathcal{M} \varrho/(1 + \gamma \varrho)$.

Proof. Let us first consider the case $\gamma=0$. Using the Maximum Principle (cf. [12]), it follows that $\Psi(x)>0$, $x\in\Omega$. Given $\varepsilon>0$, let

$$w_\varepsilon(x)\equiv(\mathcal{M}+\varepsilon)\Psi(x)-\varphi(x).$$

Then $w_\varepsilon(x)\geqq 0$, $x\in\bar{\Omega}$. If not, there would exist an $\bar{x}\in\Omega$ such that $w_\varepsilon(x)=\inf\{w_\varepsilon(\bar{x});\ x\in\bar{\Omega}\}<0$. This implies that $\varphi(\bar{x})>0$ and $f(\bar{x},\varphi(\bar{x}))\geqq f(\bar{x},0)$. Consequently,

$$\Delta w_\varepsilon(\bar{x})=-(\mathcal{M}+\varepsilon)-f(\bar{x},\varphi(\bar{x}))\leqq-\varepsilon-\{\mathcal{M}+f(x,0)\}\leqq-\varepsilon<0,$$

a contradiction. Since $w_\varepsilon(x)\geqq 0$, $x\in\bar{\Omega}$, for all $\varepsilon>0$, it follows that $\varphi(x)\leqq\mathcal{M}\varrho$, $x\in\bar{\Omega}$, and we could similarly prove that $\varphi(x)\geqq-\mathcal{M}\varrho$, $x\in\bar{\Omega}$, proving (4.6) for the case $\gamma=0$.

If $\gamma<0$, we can consider $\varphi(x)$ as the unique solution of $\Delta u(x)=f^*(x,u)$, $x\in\Omega$, and $u(x)=0$, $x\in\partial\Omega$, with

$$f^*(x,u)\equiv f(x,u)-\gamma u+\gamma\varphi(x).$$

The function $f^*(x,u)$ satisfies a condition such as (4.5) with $\gamma^*=0$. Hence, as $\gamma<0$,

$$\|\varphi\|_\infty\leqq\varrho\{\sup\{|f^*(x,0)|;\quad x\in\bar{\Omega}\}\}\leqq\mathcal{M}\varrho-\gamma\varrho\|\varphi\|_\infty,$$

from which the conclusion follows. Q.E.D.

Let us now transform the boundary value problem (4.1)-(4.2). For a given $M>0$, we introduce the real-valued continuously differentiable function $\xi_M(u)$ defined for all real u by

$$(4.7)\qquad \xi_M(u)\equiv\begin{cases}M+1-\exp(M-u), & M<u,\\ u, & |u|\leqq M,\\ -M-1+\exp(M+u), & u<-M.\end{cases}$$

We then have the following result.

Theorem 4.3. With the same assumptions as in Theorems 4.1 and 4.2, respectively, consider the boundary value problems

$$(4.8)\qquad \Delta u(x)=f_i(x,u),\qquad x\in\Omega,\quad i=1,2,$$

$$(4.9)\qquad u(x)=0,\qquad x\in\partial\Omega,$$

where

$$(4.10)\qquad f_i(x,u)\equiv f(x,\xi_{M_i}(u)),\qquad i=1,2,$$

and where M_1 and M_2 are the constants of (4.4) and (4.6), respectively. Then, $\varphi(x)$ is a classical solution of (4.1)-(4.2) if and only if it is a classical solution of (4.8)-(4.9) for $i=1,2$, respectively.

Proof. Assume first that the assumptions of Theorem 4.1 hold. If $\varphi(x)$ is a classical solution of (4.1)-(4.2), then $\|\varphi\|_\infty\leqq M_1$, with M_1 being as in the proof of Theorem 4.1. Since $f(x,u)\equiv f_1(x,u)$ when $|u|\leqq M_1$, $\varphi(x)$ is also a classical solution of (4.8)-(4.9) with $i=1$. Conversely, if $\varphi(x)$ is a classical solution of (4.8)-(4.9), $\|\varphi\|_\infty$ is bounded a priori by the same constant M_1, for as in the proof of Theorem 4.1, $|u| \geqq M_1$ implies $f_1(x, u)/u \geqq -a^2/4$, since then $f_1(x, u) = f(x, v)$ for some v with $M_1 \leqq |v| \leqq |u|$, and $0 < v/u \leqq 1$. Thus

$$\frac{f_1(x, u)}{u} = \frac{f(x, v)}{v} \cdot \frac{v}{u} \geqq -a^2/4, \quad \text{since} \quad \frac{f(x, v)}{v} \geqq -a^2/4.$$

Hence, $\varphi(x)$ is also a classical solution of (4.1)-(4.2).

Assume next the assumptions of Theorem 4.2 hold. Then on one hand, $\sup\{|f(x, 0)|;\ x \in \bar{\Omega}\} = \sup\{|f_2(x, 0)|;\ x \in \bar{\Omega}\}$, and on the other hand

$$\frac{\partial f_2}{\partial u}(x, u) = \frac{\partial f}{\partial u}(x, \xi_{M_2}(u)) \cdot \xi'_{M_2}(u) \geqq \gamma > -\frac{1}{\varrho}, \quad \text{since} \quad 0 < \xi'_{M_2}(u) \leqq 1, \quad \text{and} \quad \gamma \leqq 0;$$

hence the two problems are again equivalent since the a priori upper bounds for $\|\varphi\|_\infty$ are identical. Q.E.D.

As a consequence, observe that

$$(4.11) \qquad |f_i(x, u)| \leqq \sup\{|f(x, u)|;\ x \in \bar{\Omega};\ |u| \leqq M_i + 1\}, \quad i = 1, 2, \quad \text{for all } x \in \bar{\Omega}, \text{ and all real } u,$$

since $\|\xi_{M_i}\|_\infty \leqq M_i + 1$, $i = 1, 2$, and that

$$(4.12) \qquad \left|\frac{\partial f_i}{\partial u}(x, u)\right| \leqq \sup\left\{\left|\frac{\partial f}{\partial u}(x, u)\right|;\ x \in \bar{\Omega};\ |u| \leqq M_i + 1\right\}, \quad i = 1, 2,$$

since $\|\xi'_{M_i}\|_\infty \leqq 1$, $i = 1, 2$. Hence both $|f_i(x, u)|$ and $\left|\frac{\partial f_i}{\partial u}(x, u)\right|$, $i = 1, 2$, are *bounded* for all $x \in \bar{\Omega}$ and all real u. The point is that this is now a type of nonlinearity which is a special case of (3.6), whereas this was not necessarily the case for the original problem (4.1)-(4.2) with the assumptions of Theorems 4.1 or 4.2. Actually, in the special case $p = 2$, the admissible growth hypothesis for $|f(x, u)|$, as given by (3.6), is the following:

$$|f(x, u)| \leqq \begin{cases} g(u), & \text{if } n = 1, \\ K(1 + |u|), & \text{if } n = 2, \\ K(1 + |u|^{(n+2)/(n-2)}), & \text{if } n \geqq 3, \end{cases}$$

and clearly this was not necessarily implied by (4.3) or (4.5).

We now introduce the positive quantity

$$(4.13) \qquad \Lambda = \inf_{\substack{w \in W_0^{1,2}(\Omega) \\ w \neq 0}} \frac{\int_\Omega \left\{\sum_{i=1}^n [w_{x_i}]^2\right\} dx}{\int_\Omega w^2\, dx},$$

which is the smallest eigenvalue of the Laplacian Δ over the domain Ω. Barta [1] has shown that

$$(4.14) \qquad -\frac{1}{\varrho} \geqq -\Lambda,$$

where ϱ is defined as in Theorem 4.2. We can now prove

Theorem 4.4. Let Ω be a bounded open subset of R^n, $n \geqq 1$, such that the Sobolev Imbedding Theorem holds. Consider solving the modified boundary value problems (4.8)-(4.9), $i=1, 2$, with the same assumptions as in Theorems 4.1 and 4.2, respectively, together with the further assumption in the case $i=1$ (which corresponds to Theorem 4.1) that $f(x, u)$ is a continuously differentiable function with respect to u for each $x \in \bar{\Omega}$ and that

$$\frac{\partial f}{\partial u}(x, u) \geqq \gamma > -\Lambda \quad \text{for all} \quad x \in \bar{\Omega}, \quad \text{all real } u, \tag{4.15}$$

for some constant $\gamma \leqq 0$, where Λ is the positive quantity defined in (4.13). Then, either of the modified boundary value problems (4.8)-(4.9), $i=1, 2$, has a unique generalized solution u in the space $B = W_0^{1,2}(\Omega)$. If the data are sufficiently smooth so that this generalized solution is also a classical solution, then it is also a classical solution of the corresponding original problem (4.1)-(4.2).

Finally, for both modified problems (4.8)-(4.9), $i=1, 2$, the corresponding approximate Problem P^h has a unique solution u_h in each finite-dimensional subspace B^h of B, and there exists a constant K', independent of the subspace B^h, such that

$$\|u_h - u\|_{1,2} \leqq K' \inf\{\|w-u\|_{1,2}; \quad w \in B^h\}. \tag{4.16}$$

Proof. The new functions $f_i(x, u)$, $i=1, 2$, now satisfy growth properties compatible with those of (3.6). Thus, by Theorem 3.1, the quasibilinear forms

$$a_i(u, v) = \int_\Omega \left\{ \sum_{j=1}^n u_{x_j} v_{x_j} + f_i(x, u) v \right\} dx = (T_i u, v), \qquad i=1, 2,$$

both define mappings T_i, $i=1, 2$, acting from B into B^* which are bounded and finitely continuous by Theorem 3.2. We now show that the assumption of (4.15), which is also valid in the case $i=2$ by (4.14), implies that the mappings T_i, $i=1, 2$, are *strongly monotone* with $c(r) \equiv \alpha r$, $\alpha = (\Lambda+\gamma)/(\Lambda+1) > 0$ (Hypothesis (H_2') of Theorem 2.1). Dropping the index i for convenience, we have for all $u, v \in B$,

$$\begin{aligned}
|(Tu - Tv, u-v)| &= |a(u, u-v) - a(v, u-v)| \\
&= \int_\Omega \left\{ \sum_{j=1}^n [u_{x_j} - v_{x_j}]^2 + (f(x, u) - f(x, v))(u-v) \right\} dx \\
&\geqq \int_\Omega \left\{ \sum_{j=1}^n [u_{x_j} - v_{x_j}]^2 + \gamma [u-v]^2 \right\} dx, \quad \text{by (4.15)}, \\
&\geqq \left(\frac{\Lambda+\gamma}{\Lambda+1} \right) (\|u-v\|_{1,2})^2 \quad \text{by (4.13)}.
\end{aligned}$$

Finally, we show that the mapping T is Lipschitz continuous (Hypothesis (H_3') of Theorem 2.1). We have for any $w \in B$, and all $u, v \in B$,

$$\begin{aligned}
|(Tu - Tv, w)| &= |a(u, w) - a(v, w)| \\
&= \left| \int_\Omega \left\{ \sum_{j=1}^n [u_{x_j} - v_{x_j}] w_{x_j} + (f(x, u) - f(x, v)) w \right\} dx \right| \\
&\leqq \left(n + \sup\left\{ \left| \frac{\partial f}{\partial u}(x, u) \right|; \quad x \in \bar{\Omega}, \quad u \in R \right\} \right) \|u-v\|_{1,2} \|w\|_{1,2},
\end{aligned}$$

by repeated application of the Cauchy-Schwarz inequality. Hence,

$$\|Tu - Tv\|^* = \sup_{w \neq 0} \frac{|(Tu - Tv, w)|}{\|w\|} \leqq C \|u - v\|_{1,2};$$

with $C \equiv \left(n + \sup\left\{\left|\frac{\partial f}{\partial u}(x, u)\right|;\ x \in \bar{\Omega},\ u \in R\right\}\right)$, where C is finite for both modified problems. Thus, the conclusions of Theorem 4.4 follow by Theorem 2.1. Q.E.D.

§ 5. Polynomial-Type Subspaces

In this section, we examine the use of "polynomial-type" subspaces in the projectional method described in § 2. We start by stating an easy consequence of a fundamental result of KANTOROVICH and KRYLOV [18, p. 276].

Theorem 5.1. Let Ω be a bounded open domain in R^n, $n \geqq 1$, with $\partial\Omega$ its boundary; t and s positive integers with $t > s$, and let $\varphi(x_1, \ldots, x_n) \equiv \varphi(x)$ be a function defined in an open domain, D, containing $\bar{\Omega}$ which satisfies the following conditions:

(i) $\varphi(x) = 0$ if and only if $x \in \partial\Omega$,

(ii) $\varphi \in C^t(\bar{\Omega})$, i. e., $D^\alpha \varphi \in C^0(\bar{\Omega})$ for all $|\alpha| \leqq t$,

(iii) for any α with $|\alpha| = t$, there exists a constant K such that $|D^\alpha \varphi(x) - D^\alpha \varphi(y)| \leqq K \|x - y\|$ for all $x, y \in \bar{\Omega}$, where $\|\cdot\|$ denotes Euclidean distance in R^n, and

(iv) grad $\varphi(x) \neq 0$ for all $x \in \partial\Omega$.

Then, the set of all functions of the form $\{(\varphi(x))^s p_j(x)\}_{j=1}^\infty$, where $p_j(x)$ is a polynomial of degree at most j in each variable, x_i, $1 \leqq i \leqq n$, is dense in $W_0^{s,r}(\Omega)$, for all $r \geqq 1$.

HARRICK [15] has shown that given a smooth function $u(x) \in W_0^{s,r}(\Omega)$, one can actually asymptotically bound quantities such as

$$\inf\{\|D^\beta(u(x) - (\varphi(x))^s p_j(x))\|_\infty;\ p_j(x)\}, \quad \text{for all } |\beta| \leqq t. \tag{5.1}$$

We now state his result. For convenience, let $P_j(R^n, \varphi^s)$ denote the space of all products $\varphi^s p_j$ where p_j is any polynomial in n variables of degree at most j in each variable.

Theorem 5.2. Let Ω and φ satisfy the hypotheses of Theorem 5.1. If $u(x) \in C^t(\bar{\Omega})$ and $D^\xi u(x) = 0$ for all $x \in \partial\Omega$, $|\xi| \leqq s-1$, then there exists a sequence of functions $\{(\varphi(x))^s \tilde{p}_j(x)\}_{j=1}^\infty$ such that $\varphi^s \tilde{p}_j \in P_j(R^n, \varphi^s)$, and there exists a positive constant K such that

$$\|D^\beta(u(x) - (\varphi(x))^s \tilde{p}_j(x))\|_\infty \leqq K\left(\frac{\omega_t\left(u; \frac{1}{j}\right)}{j^{t-|\beta|}}\right), \quad \text{for all } |\beta| \leqq t, \tag{5.2}$$

for all $m \geqq 1$, where

$$\omega_t(u;\varepsilon) \equiv \max_{|\alpha|\leqq t}\left\{\max_{\substack{\|x-y\|\leqq\varepsilon\\ x,y\in\bar{\Omega}}} |D^\alpha u(x) - D^\alpha u(y)|\right\}.$$

For the case $n=1$, this result is obtained independently in [10, Theorem 5].

Combining the previous two theorems with the results of § 3, we obtain

Theorem 5.3. If the coefficients of the differential Eq. (3.1) satisfy the hypotheses of Theorem 3.3 with $c(r)=\alpha r$, $\alpha>0$, and T is Lipschitz continuous, and φ and Ω satisfy the hypotheses of Theorem 5.1, then the nonlinear problem (3.1)-(3.2) has a unique generalized solution $u(x)$ in $W_0^{m,p}(\Omega)$, and the approximate problem over $P_j(R^n, \varphi^{m-1})$ has a unique solution $u_j(x)$ for each $j\geqq 1$. If $u(x)$ is of class $C^t(\Omega)$, $t\geqq m$, then there exists a positive constant K such that

$$\|u_j - u\|_{m,p} \leqq K \frac{\omega_t\left(u;\frac{1}{j}\right)}{j^{t-m}}, \quad j\geqq 1. \tag{5.3}$$

§ 6. Piecewise Hermite Polynomials Subspaces and Two-Dimensional Problems

In this section, we apply the results of §§ 2, 3, and 4 for two-dimensional problems to subspaces of piecewise Hermite polynomials. Moreover, new error estimates are obtained from recent results in the theory of piecewise-polynomial interpolation, cf. [2]. We start by briefly recalling these results.

Let $E\equiv[a,b]\times[c,d]$, be any rectangle in the plane with sides parallel to the coordinate axes and consider arbitrary partitions in each coordinate direction of E:

$$\begin{aligned}\Delta&: a=x_0<x_1<\cdots<x_{N+1}=b,\\ \Delta'&: c=y_0<y_1<\cdots<y_{N'+1}=d,\end{aligned} \tag{6.1}$$

where N and N' are nonnegative integers. We say that $\varrho\equiv\Delta\times\Delta'$ defines a *partition* on E, and we define

$$\begin{aligned}\bar{\Delta} &= \max_{0\leqq i\leqq N}(x_{i+1}-x_i); \quad &\bar{\Delta}' &= \max_{0\leqq j\leqq N'}(y_{j+1}-y_j),\\ \underline{\Delta} &= \min_{0\leqq i\leqq N}(x_{i+1}-x_i); \quad &\underline{\Delta}' &= \min_{0\leqq j\leqq N'}(y_{j+1}-y_j).\end{aligned} \tag{6.1'}$$

Definition 6.1. For a positive integer s and a partition ϱ of E, let $H^{(s)}(\varrho;E)$ be the set of all real-valued piecewise-polynomial functions $w(x,y)$ defined on E such that $D^{(i,j)}w\equiv\frac{\partial^{i+j}w}{\partial x^i\,\partial y^j}\in C^0(E)$ for all $0\leqq i,j\leqq s-1$, and such that $w(x,y)$ is a polynomial of degree at most $2s-1$ in both x and y in each subrectangle $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$ defined on E by ϱ.

Definition 6.2. Given a real-valued function $f(x,y)\in C^{s-1,s-1}(E)$, i.e., $D^{(p,q)}f(x,y)$ is continuous in E for all $0\leqq p,q\leqq s-1$, let its $H^{(s)}(\varrho;E)$-interpolate be an element $f_{s,\varrho}$ of $H^{(s)}(\varrho;E)$ such that

$$D^{(i,j)}f(x_k,y_l) = D^{(i,j)}f_{m,\varrho}(x_k,y_l) \tag{6.2}$$

for all $0\leqq k\leqq N+1$, $0\leqq l\leqq N'+1$, and all $0\leqq i,j\leqq s-1$.

We remark that the $H^{(s)}(\varrho; E)$-interpolate of any function in $C^{s-1,s-1}(E)$ is *uniquely* determined and is *local* (cf. [2]), i. e., the $H^{(s)}(\varrho; E)$-interpolate of f is $[x_i, x_{i+1}]\times[y_j, y_{j+1}]$ is completely determined from the specific values $D^{(\cdot)}f(x_k, y_l)$ where $k=i$ and $i+1$, and $l=j$ and $j+1$.

In this section, we consider regions Ω whose closure $\bar{\Omega}$ is any *rectangular polygon*, i. e., any polygon whose sides are parallel to one of the coordinate axis in the plane, such as an L-shaped region. We remark that any rectangular polygon can be expressed as a union of rectangles $\bigcup_{i=1}^{k} E_i$ such that $E_i \cap E_j$, $1\le i, j\le k$, is either void or a subset of an edge of E_i and an edge of E_j. In this case, we say that the rectangular polygon is *composed* of the rectangles E_i.

Definition 6.3. Let $\bar{\Omega}$ be a rectangular polygon, composed of the rectangles $E_i=[a_i, b_i]\times[c_i, d_i]$, $1\le i\le k$, in the (x, y)-plane and $\mathscr{F}$ be a collection of partitions of $\bar{\Omega}$, i. e., each $\varrho\equiv\Delta\times\Delta'$ of $\mathscr{F}$ defines a partition $\Delta(i)\times\Delta'(i)$ of each rectangle E_i of T. Then, the collection $\mathscr{F}$ is said to be *regular* if and only if there exist three positive constants σ, τ, η such that:

$$\Delta(i)\ge\sigma\bar{\Delta}(i),\quad \Delta'(i)\ge\sigma\bar{\Delta}'(i), \tag{6.3}$$

and

$$\eta\le\frac{\bar{\Delta}'(i)}{\bar{\Delta}(i)}\le\tau, \tag{6.4}$$

for all $1\le i\le k$, and all $\varrho\in\mathscr{F}$.

Definition 6.4. For any positive integer p and any extended real number r with $1\le r\le+\infty$, let $S^{p,r}(\bar{\Omega})$ be the set of all real-valued functions $f(x,y)$ defined on $\bar{\Omega}$ such that

$$D^{(p-i,i)}f\in L^r(\bar{\Omega})\quad\text{for all}\quad 0\le i\le p, \tag{6.5}$$

and

$$D^{(i,j)}f\in C^0(\bar{\Omega})\quad\text{for all}\quad 0\le i+j<p. \tag{6.6}$$

The following result was proved [2, Theorem 6].

Theorem 6.1. Let $\bar{\Omega}$ be a rectangular polygon composed of the rectangles $E_i=[a_i, b_i]\times[c_i, d_i]$, $1\le i\le k$, in the (x, y)-plane, and $\mathscr{F}$ be a regular collection of partitions of $\bar{\Omega}$. If $f\in S^{p,r}(\bar{\Omega})$ where $p\ge 2s$ and $f_{s,\varrho}\in H^{(s)}(\varrho;\bar{\Omega})$ is the $H^{(s)}(\varrho; E_i)$-interpolate of f on each E_i, $1\le i\le k$, then setting $\nu\equiv\max_{1\le i\le k}\bar{\Delta}(i)$, there exists a constant M such that:

$$\|D^{(h,l)}(f-f_{s,\varrho})\|_{L^r(\bar{\Omega})}\le M\nu^{2s-h-l} \tag{6.7}$$

for all $\varrho\in\mathscr{F}$ and all $0\le h, l\le s$ with $0\le h+l\le 2s-1$.

Combining this result with the results of § 3, we have

Theorem 6.2. Let $\bar{\Omega}$ be a rectangular polygon composed of the rectangles $E_i=[a_i, b_i]\times[c_i, d_i]$, $1\le i\le k$, and $\mathscr{F}$ be any collection of partitions of $\bar{\Omega}$. If the coefficients of the differential Eq. (3.1) satisfy the hypotheses of Theorem 3.1, then the nonlinear Dirichlet problem (3.1)-(3.2) has a unique generalized solution $u \in W_0^{m,2}(\Omega)$, and the approximate problem over $H^{(s)}(\varrho, \bar{\Omega})$, $s \geqq m$, has a unique solution, u_ϱ, for each $\varrho \in \mathscr{F}$. If $u \in S^{t,p}(\bar{\Omega})$ where $t \geqq 2s \geqq 2m$, there exists a positive constant K such that

(6.8) $$c(\|u_\varrho - u\|_{m,p}) \|u_\varrho - u\|_{m,p} \leqq K \nu^{2s-m} \quad \text{for all} \quad \varrho \in \mathscr{F}.$$

If, in addition, condition (3.16) holds with [illegible] > 0, and the associated monotone mapping T is Lipschitz continuous, then there exists a positive constant K' such that

(6.9) $$\|u_\varrho - u\|_{m,p} \leqq K' \nu^{2s-m} \quad \text{for all} \quad \varrho \in \mathscr{F}.$$

As in § 4, we now consider the "model problem"

(6.10) $$\Delta u(x, y) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y, u), \quad x \in \Omega,$$

with

(6.11) $$u(x, y) = 0, \quad x \in \partial\Omega,$$

which is the two-dimensional form of (4.1)-(4.2).

Combining Theorem 4.4 and Theorem 6.1, we obtain

Theorem 6.3. Let $\bar{\Omega}$ be a rectangular polygon composed of the rectangles $E_i = [a_i, b_i] \times [c_i, d_i]$, $1 \leqq i \leqq k$, and let $\mathscr{F}$ be a regular collection of partitions of $\bar{\Omega}$. If the function $f(x, y, u)$ of (6.10) satisfies the hypotheses of Theorem 4.4, then the problem (6.10)-(6.11) has a unique generalized solution u in $W_0^{1,2}(\Omega)$, and the approximate problem over $H^{(s)}(\varrho, \bar{\Omega})$ has a unique solution u_ϱ for each $\varrho \in \mathscr{F}$. If $u \in S^{t,2}(\bar{\Omega})$ where $t \geqq 2s$, then there exists a positive constant K such that

(6.12) $$\|u_\varrho - u\|_{1,2} \leqq K \nu^{2s-1} \quad \text{for all} \quad \varrho \in \mathscr{F}.$$

We remark that the result of (6.12) for the special case $s = 1$, i. e., $u \in S^{2,2}(\bar{\Omega})$ and

(6.12') $$\|u_\varrho - u\|_{1,2} \leqq K \nu,$$

is the analog of results in the linear case for discrete approximation of u by NITSCHE and NITSCHE [29] and KELLOGG [19] and [20]. For related results for two-dimensional discrete approximations, see also [3, 8, 13, 31, 32], and [37].

For results of computations in two dimensions for particular cases of Theorem 5.3, both for linear and nonlinear problems, we cite the work of [16].

§ 7. Two-Point Boundary Value Problems

In this section, we apply the results of §§ 2, 3, and 4 to various classes of two-point boundary value problems, including the semilinear even order problems discussed in [10], semi-linear odd order problems, and systems of semi-linear even order problems.

We consider the differential equation (cf. [10])

(7.1) $$\sum_{j=0}^{m} (-1)^j D^j [p_j(x) D^j u(x)] + f(x, u(x)) = 0, \quad 0 < x < 1, \quad m \geqq 1,$$

subject to the boundary conditions

$$(7.2)\qquad D^k u(0) = D^k u(1) = 0, \quad 0 \leqq k \leqq m-1.$$

From Theorem 3.3, the Sobolev Imbedding Theorem (cf. [41, p. 174]) and [28, Theorem 1.2, p. 199], we obtain the next result which generalizes Theorem of [10].

Theorem 7.1. If

(i) the coefficients $p_j(x)$, $j=0,\ldots,m$, are real-valued and measurable on $x\in[0,1]$,

(ii) there exists a constant $c>0$ such that $\int_0^1\left(\sum_{j=0}^m p_j(x)\,[D^j w(x)]^2\right)dx \geqq c\|w\|^2_{m,2}$ for all $w\in W_0^{m,2}[0,1]$ and hence

$$\Lambda \equiv \inf_{\substack{w\in W_0^{m,2}[0,1]\\ w\neq 0}} \frac{\int_0^1 \sum_{j=0}^m p_j(x)\,[D^j w(x)]^2\,dx}{\int_0^1 [w(x)]^2\,dx} > 0$$

and

(iii) $f(x,u)$ is a real-valued function defined on $[0,1]\times R$ such that $f(x,u(x))\in L^2[0,1]$ for all $u\in W_0^{m,2}[0,1]$, and such that there exists a constant γ with

$$(7.3)\qquad \frac{f(x,u)-f(x,v)}{u-v} \geqq \gamma > -\Lambda \quad \text{for almost all} \quad x\in[0,1]$$

and all $-\infty<u, v<+\infty$ with $u\neq v$, and for each $c>0$ there exists $M(c)$ such that

$$(7.4)\qquad \frac{f(x,u)-f(x,v)}{u-v} \leqq M(c) < \infty \quad \text{for almost all} \quad x\in[0,1]$$

and all $-\infty<u, v<+\infty$, with $u\neq v$ and $|u|\leqq c$, $|v|\leqq c$, then the problem (7.1)-(7.2) has a unique generalized solution u over $W_0^{m,2}[0,1]$. Moreover, if $\{S_n\}_{n=1}^\infty$ is a sequence of finite-dimensional subspaces of $W_0^{m,2}[0,1]$ such that $\lim_{n\to\infty}(\inf\{\|g-u\|_{m,2};\ g\in S_n\})=0$, then the approximate problem P_n has a unique solution u_n for each $n\geqq 1$, and there exist positive constants K and K' such that

$$(7.5)\qquad \|D^j(u_n-u)\|_\infty \leqq K\|u_n-u\|_{m,2} \leqq K' \inf\{\|w_n-u\|_{m,2};\quad w_n\in S_n\}$$

for all $0\leqq j\leqq m-1$, and all $n\geqq 1$.

We now apply the results of [38] on L-splines to Theorem 7.1 to obtain upper bounds for the errors for approximate solutions of (7.1)-(7.2) in the finite-dimensional subspaces $\mathrm{Sp}(L,\pi,z)$. To first briefly explain the nature of L-splines, let L be any r-th order linear differential operator of the form

$$(7.6)\qquad L[u(x)] = \sum_{j=0}^r a_j(x)\,D^j u(x), \quad r\geqq 1,$$

where we assume that the coefficient function $a_j(x)$ is in $C^j[0,1]$ for all $0\leqq j\leqq r$, and $a_r(x)\geqq\omega>0$ for all $x\in[0,1]$, as in [38]. Next, let Δ: $0=x_0<x_1<\cdots<x_{N+1}=1$ be any partition of the interval $[0,1]$ and let $z=(z_1,\ldots,z_N)$, the *incidence vector* associated with Δ, be an N-vector with positive integer components z_i with $1\leqq z_i\leqq r$, for all $1\leqq i\leqq N$. Then, $\mathrm{Sp}(L,\Delta,z)$ denotes [38] the collection

of all real-valued functions $s(x)$, called *L-splines*, defined on $[0, 1]$ such that

$$L^*L[s(x)]=0 \qquad \text{for } x\in(x_i, x_{i+1}) \text{ for each } i, \quad 0\leqq i\leqq N,$$

$$(7.7)\qquad D^k s(x_i-)=D^k s(x_i+) \qquad \text{for all } 0\leqq k\leqq 2r-1-z_i, \quad 1\leqq i\leqq N,$$

where $L^*[v(x)]\equiv\sum_{j=0}^{r}(-1)^j D^j[a_j(x)\,v(x)]$ denotes the formal adjoint of L. As an important special case, if $L[u]=D^r u$ with $\hat{z}_1=\hat{z}_2=\cdots=\hat{z}_N=1$, the elements of $\mathrm{Sp}(D^r, \Delta, \hat{z})$ are then simply the *natural spline functions*, and $\mathrm{Sp}(D^r, \Delta, \hat{z})$ becomes $\mathrm{Sp}^{(r)}(\Delta)$ in the notation of [10]. Similarly, when $L[u]=D^r u$ and $\tilde{z}_1=\tilde{z}_2=\cdots=\tilde{z}_N=r$, the elements of $\mathrm{Sp}(D^r, \Delta, \tilde{z})$ are then simply the piecewise *Hermite-polynomial functions*, and $\mathrm{Sp}(D^r, \Delta, \tilde{z})$ becomes $H^{(r)}(\Delta)$ in the notation of [10] and [40].

Given a function $f(x)\in C^{r-1}[0, 1]$, where r is the order of the differential operator L of (7.6), there are various ways in which one might interpolate f in $\mathrm{Sp}(L, \Delta, z)$. As a particular case, if there is an element $s(x)\in\mathrm{Sp}(L, \Delta, z)$ such that

$$(7.8)\qquad D^k s(x_i)=D^k f(x_i), \quad 0\leqq k\leqq z_i-1, \quad 1\leqq i\leqq N,$$

and

$$(7.9)\qquad D^k s(x_i)=D^k f(x_i), \quad 0\leqq k\leqq r-1, \quad i=0 \text{ and } i=N+1,$$

we say that $s(x)$ is an $\mathrm{Sp}(L, \Delta, z)$-*interpolate of* $f(x)$ *of Type I*. It can be shown that, for any partition Δ and any associated incidence vector z, an $\mathrm{Sp}(L, \Delta, z)$-interpolate of $f(x)$ of Type I always *exists* and is in fact *unique*. Thus, given any parameters $\alpha_i^{(k)}$, $0\leqq k\leqq z_i-1$, $0\leqq i\leqq N+1$ (where we define for convenience $z_0=z_{N+1}=r$), there exists a unique function $u(x)\in\mathrm{Sp}(L, \Delta, z)$ with

$$(7.10)\qquad D^k u(x_i)=\alpha_i^{(k)}, \quad 0\leqq k\leqq z_i-1, \quad 0\leqq i\leqq N+1,$$

and we denote by $\mathrm{Sp}^{\mathrm{I}}(L, \Delta, z)$ the finite-dimensional subspace of $\mathrm{Sp}(L, \Delta, z)$ of all such functions.

With the notation $\bar{\Delta}\equiv\max_{0\leqq i\leqq N}(x_{i+1}-x_i)$ for the partition Δ: $0=x_0<\cdots<x_{N+1}=1$, consider now *any* partition Δ of $[0, 1]$, and *any* associated incidence vector z. Based on an extension of [38, Theorems 7 and 9], it is known [33] that if $f(x)$ is of class $W^{r,2}[0, 1]$, $r\geqq 1$, there exists a constant M such that for any partition Δ and any associated incidence vector z,

$$(7.11)\qquad \|D^j(f-s)\|_2\leqq M(\bar{\Delta})^{r-j}\|Lf\|_2, \quad 0\leqq j\leqq r,$$

where $s(x)$ is the unique $\mathrm{Sp}^{\mathrm{I}}(L, \Delta, z)$-interpolate of $f(x)$. Similarly, if $f(x)$ is of class $W^{2r,2}[0, 1]$, $r\geqq 1$, we have

$$(7.12)\qquad \|D^j(f-s)\|_2\leqq M(\bar{\Delta})^{2r-j}\|L^*Lf\|_2, \quad 0\leqq j\leqq r.$$

Applying this to the generalized solution $u(x)$ of (7.1)-(7.2) results in

Theorem 7.2. Let hypotheses (i), (ii), and (iii) of Theorem 7.1 be satisfied, let $\{\Delta_n\}_{n=1}^{\infty}$ be any sequence of partitions of $[0, 1]$ with $\lim_{n\to\infty}\bar{\Delta}_n=0$, let $\{z^{(n)}\}_{n=1}^{\infty}$ be any associated sequence of incidence vectors, let L be a differential operator of the form (7.6) with $r \geq m$, and let $S_n \equiv \mathrm{Sp}_0^{\mathrm{I}}(L, \Delta_n, z^{(n)})$, i. e., those functions in $\mathrm{Sp}^{\mathrm{I}}(L, \Delta_n, z^{(n)})$ satisfying the homogeneous conditions of (7.2). Then, the approximate problem P_n over S_n has a unique solution u_n for each $n \geq 1$, and if u, the generalized solution of (7.1)-(7.2), is of class $W^{t,2}[0, 1]$ with $t \geq r$, then there exist positive constants K and K' such that

$$\text{(7.13)} \quad \|D^j(u_n - u)\|_\infty \leq K \|u_n - u\|_{m,2} \leq K'(\bar{\Delta}_n)^{r-m}, \quad 0 \leq j \leq m-1, \quad n \geq 1.$$

If the generalized solution u is of class $W^{t,2}[0, 1]$ with $t \geq 2r$, then there exists a positive constant K'' such that

$$\text{(7.14)} \quad \|D^j(u_n - u)\|_\infty \leq K \|u_n - u\|_{m,2} \leq K''(\bar{\Delta}_n)^{2r-m}, \quad 0 \leq j \leq m-1, \quad n \geq 1.$$

We remark that if the coefficient functions $p_j(x)$ and $f(x, u)$ of (7.1) are sufficiently smooth, then the generalized solution $u(x)$ of (7.1)-(7.2) is of class $W^{2m,2}[0, 1] \cap W_0^{m,2}[0, 1]$, and (7.14) is applicable with $t = 2m$. Further results which improve the exponent of $\bar{\Delta}$ for upper bounds for $D^j(u_n - u)$ in the *uniform* norm can be found in [33].

Finally, for numerical results of actual computations based on L-spline subspaces, we refer the reader to [10, 16, 17], and [33]. For other computational results, see also [21].

We now consider the third order two-point boundary value problem

$$\text{(7.15)} \qquad -D^3 u(x) = f(x, u(x), Du(x)), \quad 0 < x < 1,$$

with

$$\text{(7.16)} \qquad u(0) = Du(0) = Du(1) = 0.$$

To discuss this problem, we need the theory of K-positive definite operators, which we now recall, cf. [34].

Let H be a separable real Hilbert space with inner-product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$, and consider the problem of solving

$$\text{(7.17)} \qquad A u = f(u),$$

where A is a linear unbounded mapping on a dense domain $\mathscr{D}(A) \subset H$ into H such that there exists a (linear) continuously $\mathscr{D}(A)$-invertible closed mapping K, i. e., $\mathscr{D}(K) \subset \mathscr{D}(A)$ and the range of K, considered as a mapping on $\mathscr{D}(A)$, $R_{\mathscr{D}(A)}(K)$, is dense in H and K has a bounded inverse on $R_{\mathscr{D}(A)}(K)$, and there exists a positive constant α such that

$$\text{(7.18)} \qquad (Au, Ku)_H \geq \alpha \|Ku\|_H^2, \quad \text{for all} \quad u \in \mathscr{D}(A).$$

If A satisfies all of the above conditions, it is said to be *K-positive definite*. We assume that f is a (possibly nonlinear) mapping of $\mathscr{D}(A)$ into H. Clearly, $Au = f(u)$ if and only if

$$\text{(7.19)} \qquad (Au, Kv)_H = (f(u), Kv)_H \quad \text{for all} \quad v \in \mathscr{D}(A).$$

Define a pre-Hilbert space structure on $\mathscr{D}(A)$ by means of the "inner product" $[u, v] \equiv (Au, Kv)_H$. The corresponding norm is defined by $\|u\|_{H_K} = [u, u]$

From (7.18), we have that

$$\|u\|_{H_K} \leqq \alpha^{\frac{1}{2}} \|Ku\|_H \quad \text{for all} \quad u \in \mathscr{D}(A). \tag{7.20}$$

We now complete $\mathscr{D}(A)$ to the Hilbert space H_K with respect to the norm $\|\cdot\|_{H_K}$. PETRYSHYN [34] has proved the following result:

Lemma 7.1. $\mathscr{D}(A)$ is dense in H_K, H_K is a subspace of H, K can be extended to a bounded linear mapping of H_K to H, and (7.20) is valid for all $u \in H_K$.

Taking the absolute value of the right-hand side of (7.19) and using the Cauchy-Schwarz inequality and (7.20), we have

$$|(f(u), Kv)_H| \leqq \|f(u)\|_H \|Kv\|_H \leqq \alpha^{-\frac{1}{2}} \|f(u)\|_H \|v\|_{H_K}.$$

Hence, by the Riesz Representation Theorem in Hilbert space, there exists a unique element $F[u]$ in H_K such that $(f(u), Kv)_H = [F(u), v]$ for all $v \in H_K$, and the problem of solving $Au = f(u)$ reduces to solving $[u, v] = [F(u), v]$ for all v in H_K, or $[u - F(u), v] = 0$ for all $v \in H_K$.

As a consequence of Lemma 2.1 and Theorem 2.1, we thus have

Theorem 7.3. If the associated mapping F is finitely continuous, and there exists a positive constant β such that

$$[(I-F)u - (I-F)v, u - v] \geqq \beta \|u - v\|^2_{H_K} \quad \text{for all} \quad u, v \in H_K,$$

i.e., $I - F$ is strongly monotone in H_K, then the problem (7.17) has a unique generalized solution u in H_K. Moreover, if $\{S_n\}_{n=1}^{\infty}$ is a sequence of finite-dimensional subspaces of H_K such that $\lim_{n\to\infty} \{\inf_{w \in S_n} \|w - u\|_{H_K}\} = 0$, then the approximate problem P_n has a unique solution, u_n, for each $n \geqq 1$, and if F is bounded (resp. Lipschitz continuous), there exists a positive constant M such that

$$\|u_n - u\|_{H_K} \leqq M (\inf \{\|w_n - u\|_{H_K} : w_n \in S_n\})^{\frac{1}{2}}, \tag{7.21}$$

resp.

$$\|u_n - u\|_{H_K} \leqq M (\inf \{\|w_n - u\|_{H_K} : w_n \in S_n\}), \tag{7.21'}$$

and in both cases $\lim_{n\to\infty} \|u_n - u\|_{H_K} = 0$.

In the particular case of (7.15)-(7.16), let $H \equiv L^2[0, 1]$, $A = -D^3$, and $K = D$. The domain $\mathscr{D}(A)$ includes the set of $C^3[0, 1]$ functions satisfying the boundary conditions (7.16), and hence $\mathscr{D}(A)$ is dense in $L^2[0, 1]$, and $\mathscr{D}(K)$ is the set of $C^1[0, 1]$ functions u such that $u(0) = 0$. With these definitions, PETRYSHYN has shown in [34] the result of

Lemma 7.2. A is K-positive definite, and

$$[u, v] = (Au, Kv)_{L^2[0,1]} = -\int_0^1 D^3 u(x)\, Dv(x)\, dx = \int_0^1 D^2 u(x) D^2 v(x)\, dx$$

for all u and v in $\mathscr{D}(A)$.

As a corollary of Theorem 7.3, we have

Theorem 7.4. Let $f(x, \theta, \varphi)$ of (7.15) be measurable with respect to $x \in [0, 1]$ and Lipschitz continuous with respect to θ and φ for almost all $x \in [0, 1]$, and let there exist an α such that

$$(7.22)\qquad \pi^2 > \alpha \geqq \frac{f(x, \vartheta, \varphi) - f(x, \vartheta', \varphi')}{\varphi - \varphi'}$$

for almost all $x \in [0, 1]$, $-\infty < \vartheta, \vartheta' < \infty < \varphi, \varphi' < \infty$. Then, (7.15)-(7.16) has a unique generalized solution u in H_D, and if $\{S_n\}_{n=1}^{\infty}$ is a sequence of finite-dimensional subspaces of H_D such that $\lim_{n\to\infty} \{\inf_{w\in S_n} \|w - u\|_{H_D}\} = 0$, then the approximate problem P_n' has a unique solution, u_n, for each $n \geqq 1$, and there exist positive constants K and K' such that

$$(7.23)\qquad \|D^j(u - u_n)\|_{\infty} \leqq K\|u_n - u\|_{H_D} \leqq K'(\inf\{\|w_n - u\|_{H_D}; \quad w_n \in S_n\}), \qquad j = 0, 1,$$

for all $n \geqq 1$, and $\lim_{n\to\infty} \|u_n - u\|_{H_D} = 0$.

Proof. Using the Sobolev Embedding Theorem [41, p. 174], one can verify that the associated mapping F is Lipschitz continuous. Hence by Theorem 5.1, it suffices to show that $L - F$ is strongly monotone.

But, if $u, v \in H_D$,

$$\begin{aligned}[u - v, u - v] - [Fu - Fv, u - v] &= \int_0^1 (D^2 u - D^2 v)^2 dx - \int_0^1 f(x, w, Du) \\ &\quad - f(x, v, Dv)(Du - Dv)\, dx \\ &\geqq \int_0^1 (D^2 u - D^2 v)^2 dx - \alpha \int_0^1 (Du - Dv)^2 dx \\ &\geqq \left(1 - \frac{\max(\alpha, 0)}{\pi^2}\right) \int_0^1 (D^2 u - D^2 v)^2 dx \\ &= \left(1 - \frac{\max(\alpha, 0)}{\pi^2}\right) \|u - v\|_{H_D}^2,\end{aligned}$$

where we have used the Rayleigh-Ritz inequality, cf. [14, p. 184]. Q.E.D.

Choosing the subspaces S_n to be subspaces of L-splines and using (7.11) and (7.12) we obtain

Theorem 7.5. Let the hypotheses of Theorem 7.4 concerning (7.15) be satisfied, let $\{\Delta_n\}_{n=1}^{\infty}$ be any sequence of partitions of $[0, 1]$ with $\lim_{n\to\infty} \bar{\Delta}_n = 0$, let $\{z^{(n)}\}_{n=1}^{\infty}$ be any associated sequence of incidence vectors, let L be a differential operator of the form (7.6) with $r \geqq 2$, and let $S_n \equiv Sp_0^1(L, \Delta_n, z^{(n)})$, i. e., those functions in $Sp^1(L, \Delta_n, z^{(n)})$ satisfying the homogeneous boundary conditions (7.16). The approximate problem P_n' over S_n has a unique solution, u_n, and if the unique generalized solution u of (7.15)-(7.16) is of class $W^{t,2}[0, 1]$, $t \geqq r$, there exist positive constants K and K' such that

$$(7.24)\qquad \|D^j(u_n - u)\|_{\infty} \leqq K\|u_n - u\|_{H_D} \leqq K'(\bar{\Delta}_n)^{r-2}, \qquad j = 0, 1, \quad n \geqq 1.$$

If the generalized solution u is of class $W^{t,2}[0, 1]$ with $t \geqq 2r$, then there exists a positive constant K'' such that

$$(7.25)\qquad \|D^j(u_n - u)\|_{\infty} \leqq K\|u_n - u\|_{H_D} \leqq K''(\bar{\Delta}_n)^{2r-2}, \qquad j = 0, 1, \quad n \geqq 1.$$

Next, we consider the second-order problem

(7.26) $$-D^2u(x)+f(x,u,Du)=0, \quad 0<x<1,$$

with

(7.27) $$u(0)=u(1)=0.$$

Theorem 3.1 in this special case $m=n=1$ and $p=2$ yields

Theorem 7.6. Let $f(x,\theta,\varphi)$ be measurable with respect to $x\in[0,1]$, continuous with respect to θ and φ for almost all x, let there exist a continuous, nonnegative function, g, on $[0,+\infty)$ such that

(7.28) $$|f(x,\theta,\varphi)|\leq g(|\theta|)\{1+|\varphi|^2\}$$

for almost all $x\in[0,1]$, $-\infty<\theta, \varphi<\infty$, and let

(7.29) $$[f(x,\theta_1,\varphi_1)-f(x,\theta_2,\varphi_2)](\theta_1-\theta_2)\geq a[\theta_1-\theta_2]^2-b[\varphi_1-\varphi_2][\theta_1-\theta_2],$$

for almost all $x\in[0,1]$, $-\infty<\theta_1,\theta_2,\varphi_1,\varphi_2<\infty$, where

(7.30) $$\frac{\max(-a,0)}{\pi^2}+\frac{|b|}{4}<1.$$

Then, (7.26)-(7.27) has a unique generalized solution, u, in $W_0^{1,2}[0,1]$. Moreover, if $\{S_n\}_{n=1}^{\infty}$ is a sequence of finite-dimensional subspaces of $W_0^{1,2}[0,1]$ such that $\lim_{n\to\infty}(\inf\{\|w_n-u\|_{1,2};\ w_n\in S_n\})=0$, then the approximate problem P_n over S_n has a unique solution, u_n, for each $n\geq 1$, and there exists a positive constant K such that

(7.31) $$\|u_n-u\|_\infty\leq\tfrac{1}{2}\|u_n-u\|_{1,2}\leq K(\inf\{\|w_n-u\|_{1,2};\ w_n\in S_n\})^{\frac{1}{2}},$$

and $\lim_{n\to\infty}\|u_n-u\|_{1,2}=0$.

Proof. By Theorem 3.3, it suffices to show that the mapping associated with (7.26) is strongly monotone with respect to $W_0^{1,2}[0,1]$. To show this, consider

$$\int_0^1(Du-Dv)^2dx+\int_0^1(f(x,u,Du)-f(x,v,Dv))(u-v)\,dx$$

$$\geq\int_0^1(Du-Dv)^2dx+a\int_0^1(u-v)^2dx-b\int_0^1(Du-Dv)(u-v)\,dx$$

$$\geq\left(1-\frac{\max(-a,0)}{\pi^2}-\frac{|b|}{4}\right)\int_0^1(Du-Dv)^2dx,$$

where we have used the Rayleigh-Ritz inequality [14, p. 184] and Opial's inequality [30], i. e., for any $w(x)\in W_0^{1,2}[0,1]$,

$$\int_0^1|Dw(x)\,w(x)|\,dx\leq\tfrac{1}{4}\int_0^1(Dw(x))^2dx,$$

which completes the proof. Q.E.D.

We remark that Opial's inequality, though not stated originally for functions in $W_0^{1,2}[0,1]$, is, using the Sobolev Imbedding Theorem (cf. [24, p. 26]), however, valid for $W_0^{1,2}[0,1]$.

In order to eliminate the previous growth hypotheses on $f(x,\theta,\varphi)$ and to improve the general error estimate, we employ the method used in § 4, i.e., we obtain an a priori bounded on classical solutions (7.26)-(7.27) and then appropriately modify the right-hand side.

Theorem 7.7. Let $f(x,\theta,\varphi)\in C^1([0,1]\times R\times R)$, let there exist two numbers a and b such that

$$\frac{\partial f}{\partial \theta}(x,\theta,\varphi)\geqq a,\quad 0\leqq x\leqq 1,\quad -\infty<\theta,\varphi<\infty \tag{7.32}$$

and

$$\left|\frac{\partial f}{\partial \varphi}(x,\theta,\varphi)\right|\leqq b,\quad 0\leqq x\leqq 1,\quad -\infty<\theta,\varphi<\infty, \tag{7.33}$$

with $k\equiv\frac{\max(-a,0)}{\pi^2}+\frac{b}{4}<1$. If $\mathscr{M}\equiv\sup_{0\leqq x\leqq 1}|f(x,0,0)|$ and $\mathscr{N}\equiv\sup_{\substack{0\leqq x\leqq 1\\ |\theta|\leqq \mathscr{M}/2\pi(1-k)}}|f(x,\theta,0)|$ and if u is a classical solution of (7.26)-(7.27), then

$$\|u\|_\infty\leqq\frac{1}{2}\|u\|_D\leqq\frac{\mathscr{M}}{2\pi(1-k)}\equiv B_0, \tag{7.34}$$

and

$$\|Du\|_\infty\leqq\left(4\mathscr{N}^2+(2+4b^2)\frac{\mathscr{M}^2}{\pi^2(1-k)^2}\right)^{\frac{1}{2}}\equiv B_1. \tag{7.35}$$

Proof. $\|u\|_D^2\equiv\int\limits_0^1(Du(x))^2dx=-\int\limits_0^1 f(x,u(x),Du(x))\,u(x)\,dx$

$$=-\int\limits_0^1 f(x,0,0)\,u(x)\,dx-\int\limits_0^1\frac{\partial f}{\partial\theta}(x,\xi(x)\,u(x),\xi(x)\,Du(x))\,(u(x))^2dx$$

$$-\int\limits_0^1\frac{\partial f}{\partial\varphi}(x,\xi(x)\,u(x),\xi(x)\,Du(x))\,Du(x)\,u(x)\,dx,$$

for some $\xi(x)$, $0<\xi(x)<1$. Using the Raleigh-Ritz, Opial, and Cauchy-Schwarz inequalities, we obtain

$$\|u\|_D^2\leqq\frac{\mathscr{M}}{\pi}\|u\|_D+\frac{\max(-a,0)}{\pi^2}\|u\|_D^2+\frac{b}{4}\|u\|_D^2,$$

which implies (7.34). Next, we derive an a priori bound for $Du(x)$. Let $x,y\in[0,1]$. Since

$$Du(x)=Du(y)+\int\limits_y^x D^2u(\xi)\,d\xi=Du(y)+\int\limits_y^x f(\xi,u(\xi),Du(\xi))\,d\xi,$$

and $|r_1+r_2|^2\leqq 2r_1^2+2r_2^2$ for any real constants r_1 and r_2, then

$$|Du(x)|^2\leqq 2|Du(y)|^2+2\left|\int\limits_y^x\{f(\xi,u(\xi),0)+\frac{\partial f}{\partial\varphi}(\xi,u(\xi),t(\xi)\,Du(\xi))\right.$$

$$\left.\cdot Du(\xi)\}\,d\xi\right|^2.$$

for some $t(\xi)$, $0<t(\xi)<1$. Letting $\mathcal{N}\equiv \sup\limits_{\substack{0\le x\le 1\\ |\theta|\le \mathcal{M}/2\pi(1-h)}} |f(x, \theta, 0)|$, we obtain

$$|Du(x)|^2 \le 2|Du(y)|^2+4\mathcal{N}^2+4b^2\left(\int_y^x |Du(\xi)|\,d\xi\right)^2$$
$$\le 2|Du(y)|^2+4\mathcal{N}^2+4b^2\|u\|_D^2.$$

Integrating with respect to y from 0 to 1, we obtain

$$|Du(x)|^2 \le 4\mathcal{N}^2+(2+4b^2)\|u\|_D^2 \le 4\mathcal{N}^2+(2+4b^2)\frac{\mathcal{M}^2}{\pi^2(1-h)^2} \qquad \text{Q.E.D.}$$

We now consider the modified version of (7.26)-(7.27):

(7.36) $$-D^2u(x)+\bar{f}(x, u, Du)=0, \quad 0<x<1,$$

with

(7.37) $$u(0)=u(1)=0,$$

where $\bar{f}(x, u, Du)=f(x, \xi_{B_0}(u), \xi_{B_1}(Du))$, ξ_{B_0}, ξ_{B_1} being defined in (4.7). As in Theorem 4.3 and 4.4, we have

Theorem 7.8. $u(x)$ is a classical solution of (7.36)—(7.37) if and only if it is a classical solution of (7.26)-(7.27). Moreover, $\bar{f}(x, \theta, \varphi)$ is a $C^1([0,1]\times R\times R)$ function satisfying

(7.38) $$|\bar{f}(x, \theta, \varphi)| \le B_2, \quad 0\le x\le 1, \quad -\infty<\theta, \varphi<\infty,$$

where $B_2\equiv \sup\limits_{\substack{0\le x\le 1\\ |\theta|\le B_0+1\\ |\varphi|\le B_1+1}} |f(x, \theta, \varphi)|$.

(7.39) $$a\le \frac{\partial \bar{f}}{\partial \theta}(x, \theta, \varphi)\le B_3, \quad 0\le x\le 1, \quad -\infty<\theta, \varphi<\infty$$

where $B_3\equiv \sup\limits_{\substack{0\le x\le 1\\ |\theta|\le B_0+1\\ |\varphi|\le B_1+1}} \left(\frac{\partial f}{\partial \theta}(x, \theta, \varphi)\right)$, and

(7.40) $$\left|\frac{\partial \bar{f}}{\partial \varphi}(x, \theta, \varphi)\right|\le b, \quad 0\le x\le 1, \quad -\infty<\theta, \varphi<\infty.$$

From Theorem 7.8 we see that if suffices to consider the approximation of the solutions of (7.36)-(7.37).

Theorem 7.9. The mapping associated with (7.36)—(7.37) is Lipschitz continuous and strongly monotone in $W_0^{1,2}[0, 1]$. Thus, (7.36)-(7.37) has a unique generalized solution, u, in $W_0^{1,2}[0, 1]$. Moreover, if $\{S_n\}_{n=1}^{\infty}$ is a sequence of finite-dimensional subspaces of $W_0^{1,2}[0, 1]$ such that $\lim\limits_{n\to\infty}(\inf\{\|w_n-u\|_{1,2};\ w_n\in S_n\})=0$, then the approximate problem P_n has a unique solution, u_n, for each $n\ge 1$, and there exists a positive constant K such that

(7.41) $$\|u_n-u\|_\infty \le \tfrac{1}{2}\|u_n-u\|_{1,2}\le K(\inf\{\|w_n-u\|_{1,2};\ w_n\in S_n\})$$

for all $n\ge 1$, and $\lim\limits_{n\to\infty}\|u_n-u\|_{1,2}=0$.

Proof. To show that the associated mapping is Lipschitz continuous in $W_0^{1,2}[0,1]$, let $u, v \in W_0^{1,2}[0,1]$. Then,

$$\int_0^1 (Du - Dv)^2 dx + \int_0^1 \left(\tilde{f}(x, u, Du) - \tilde{f}(x, v, Dv)\right)(u - v)\, dx \le \left(1 + \frac{|B_3|}{\pi^2} + \frac{b}{4}\right)\int_0^1 (Du - Dv)^2 dx,$$

where we have used the Rayleigh-Ritz and Opial inequalities. Likewise, to show that the mapping is strongly monotone in $W_0^{1,2}[0,1]$, let $u, v \in W_0^{1,2}[0,1]$. Then

$$\begin{aligned}\int_0^1 (Du - Dv)^2 dx &+ \int_0^1 \left(\tilde{f}(x, u, Du) - \tilde{f}(x, v, Dv)\right)(u - v)\, dx \\ &= \int_0^1 (Du - Dv)^2 dx + \int_0^1 \left\{\frac{\partial f}{\partial u}(x, \theta u, \theta Du)(u - v)^2 \right. \\ &\quad \left. + \frac{\partial f}{\partial Du}(x, \theta u, \theta Du)(Du - Dv)(u - v)\right\} dx \\ &\ge \left(1 - \frac{\max(-a, 0)}{\pi^2} - \frac{b}{4}\right)\int_0^1 (Du - Dv)^2 dx,\end{aligned}$$

where $0 < \theta(x) < 1$. Q.E.D.

Applying the results of (7.11) and (7.12), we have

Theorem 7.10. Let the hypotheses of Theorem 7.7 be satisfied, let $\{\Delta_n\}_{n=1}^{\infty}$ be any sequence of partitions of $[0, 1]$ with $\lim_{n\to\infty} \bar{\Delta}_n = 0$, let $\{z^{(n)}\}_{n=1}^{\infty}$ be any associated sequence of incidence vectors, L be a differential operator of the form (7.6) with $r \ge 1$, and let $S_n \equiv \mathrm{Sp}_0^1(L, \Delta_n, z^{(n)})$. The approximate problem P_n over S_n, has a unique solution, u_n, and if the unique generalized solution u of (7.36)-(7.37) is of class $W^{t,2}[0,1]$, $t \ge r$, there exists a positive constant K such that

$$\|u_n - u\|_\infty \le \tfrac{1}{2}\|u_n - u\|_{1,2} \le K(\bar{\Delta}_n)^{r-1}, \quad n \ge 1. \tag{7.42}$$

If the generalized solution u is of class $W^{t,2}[0,1]$ with $t \ge 2r$, then there exists a positive constant K' such that

$$\|u_n - u\|_\infty \le \tfrac{1}{2}\|u_n - u\|_{1,2} \le K'(\bar{\Delta}_n)^{2r-1}, \quad n \ge 1. \tag{7.43}$$

Finally, we consider a basic example of a coupled system of nonlinear two-point boundary value problems, i. e.,

$$\begin{aligned} &\mathscr{L}[u_1] + f_1(x, u_1, \ldots, u_q) = 0 \\ &\mathscr{L}[u_2] + f_2(x, u_1, \ldots, u_q) = 0, \quad 0 < x < 1, \\ &\quad\vdots \\ &\mathscr{L}[u_q] + f_q(x, u_1, \ldots, u_q) = 0 \end{aligned} \tag{7.44}$$

where $\mathscr{L}[u_i]=\sum_{j=0}^{m}(-1)^j D^j[p_j(x)D^j u_i(x)]+f(x, u_i(x))$, $1\leq i\leq q$, $m\geq 1$, subject to the boundary conditions

$$D^k u_i(0)=D^k u_i(1)=0, \quad 0\leq k\leq m-1, \quad 1\leq i\leq q. \tag{7.45}$$

We can write (7.44)-(7.45) in vector form, i.e., putting $u(x)=(u_1(x), \ldots, u_q(x))$ and $f(x, u)=(f_1(x, u), \ldots, f_q(x, u))$ we have

$$\mathscr{L}[u]=f(x, u), \qquad 0<x<1, \tag{7.46}$$

$$D^k u(0)=D^k u(1)=0, \quad 0\leq k\leq m-1. \tag{7.47}$$

To apply the theory of strongly monotone operators, we consider the mapping associated with (7.46)-(7.47) in the Hilbert space $H=\prod_{i=1}^{q}[W_0^{m,2}[0, 1]]_i$. The following result is the analogue of Theorem 7.1 for coupled systems.

Theorem 7.11. Let $\mathscr{L}$ satisfy the hypotheses of Theorem 7.1, let f_i be continuously differentiable with respect to $u_1, \ldots, u_q$ for each $1\leq i\leq q$, and let there exist a γ such that $\lambda_j\left[\frac{(Jf)+(Jf)^*}{2}\right]\geq\gamma>-\Lambda$, for all $1\leq j\leq q$, where Jf is the Jacobian of f and λ_j is its j-th eigenvalue, and Λ is the fundamental eigenvalue of $\mathscr{L}$. Then the problem (7.46)-(7.47) has a unique generalized solution, u, in H. Moreover, if $\{S_n\}_{n=1}^{\infty}$ is a sequence of finite-dimensional subspaces of H such that $\lim_{n\to\infty}(\inf\{\|w_n-u\|_H;\ w_n\in S_n\})=0$, then the approximate problem P_n has a unique solution, u_n, for each $n\geq 1$, there exists a positive constant K such that

$$\|u_n-u\|_H\leq K(\inf\{\|w_n-u\|_H; \quad w_n\in S_n\}) \quad \text{for all} \quad n\geq 1, \tag{7.48}$$

and $\lim_{n\to\infty}\|u_n-u\|_H=0$.

As in the case of (7.1)-(7.2), we may use the subspaces of L-splines and results essentially the same as Theorem 7.2 are true.

Theorem 7.12. Let the hypotheses of Theorem 7.11 be satisfied, let $\{\Delta_n\}_{n=1}^{\infty}$ be any sequence of partitions of $[0, 1]$ with $\lim_{n\to\infty}\bar{\Delta}_n=0$, let $\{z^{(n)}\}_{n=1}^{\infty}$ be any associated sequence of incidence vectors, let L be a differential operator of the form (7.6) with $r\geq m$, and let $S_n\equiv\prod_{i=1}^{q}[\mathrm{Sp}_0^{\mathrm{I}}(L, \Delta_n, z^{(n)})]_i$. The approximate problem P_n over S_n, has a unique solution, u_n, and if the components u_i of the generalized solution u of (7.44)-(7.45) are of class $W^{t,2}[0, 1]$, $1\leq i\leq q$, $t\geq r$, then there exists a positive constant K such that

$$\|u_n-u\|_H\leq K(\bar{\Delta}_n)^{r-m}, \qquad n\geq 1. \tag{7.49}$$

If the components u_i of the generalized solution u are of class $W^{t,2}[0, 1]$, $1\leq i\leq q$, $t\geq 2r$, then there exists a positive constant K' such that

$$\|u_n-u\|_H\leq K'(\bar{\Delta}_n)^{2r-m}, \qquad n\geq 1. \tag{7.50}$$

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