Galerkin Methods for the Numerical Solution of Boundary Value Problems

J. C. CAVENDISH
H. S. PRICE
MEMBER AIME
R. S. VARGA

GULF RESEARCH & DEVELOPMENT CO.
PITTSBURGH, PA.

CASE WESTERN RESERVE U.
CLEVELAND, OHIO

ABSTRACT

This paper presents a new technique for solving some of the partial differential equations that are commonly used in simulating reservoir performance. The results of applying this technique to a simple problem show that one obtains accurate pressure values near wells, as well as accurate pressure gradients, which can be explicitly calculated. The method is completely rigorous in that convergence of the discrete numerical solution to the continuous solution for both pressure and pressure gradient is established. High-order, piecewise-polynomial approximations are used near the wells where pressure gradients are steep, while low-order, piecewise-polynomial approximations are used elsewhere to reduce greatly the calculation time. This combination is shown to give a uniformly good approximation to the solution. These approximations, obtained by using a Galerkin process with suitable Hermite subspaces, are shown to be theoretically and numerically superior to the usual approximations obtained from standard finite-difference techniques. Not only are much greater accuracies obtained, but computer times are also greatly reduced.

The application of this technique to multiphase flow problems (e.g., single well coning problems) would have considerable practical interest, but such extensions of this technique with full mathematical rigor have not been made as yet. However, the numerical methods presented here are general, and in principle extend to multidimensional, multiphase flow. Moreover, the preliminary results given in this paper are sufficiently encouraging that we feel the effort in attempting these extensions is justified.

INTRODUCTION

The problem of obtaining accurate pressure distributions and pressure gradients around wells is of considerable importance in the numerical simulation of reservoir performance. The most common approach to solving this problem is to use finite difference techniques (see McCarty and Barfield[9] or Peaceman and Rachford[10]). This approach, however, has many disadvantages, the major one being that many grid points are generally necessary for accurately describing the pressure distribution and the pressure gradient around wells. This need for a fine grid results in large computer times and often in prohibitively high costs.

Besides investigating the method of finite differences, some authors, such as Welge and Weber[17] and Roper, Merchant and Duvall,[13] have considered a combination of analytical and numerical techniques with some success. These approaches, however, are all nonrigorous and quite often cannot even be applied. In this paper, we present a numerical formulation of high-order accuracy, based on the Galerkin method, for solving this problem.

We treat here only the partial differential equation that describes steady-state, single-phase flow. However, the methods presented are general and in principle extend to multidimensional, multiphase flow. Specifically we treat special cases of the problem in two dimensions described by:

$$\frac{\partial}{\partial x}\left[k(x,y)\,\frac{\partial p}{\partial x}\right] + \frac{\partial}{\partial y}\left[k(x,y)\,\frac{\partial p}{\partial y}\right] = f(x,y),\ (x,y) \in G, \quad \ldots\ldots\ldots\ldots\ (1)$$

$$\gamma p(x,y) + \beta\,\frac{\partial p}{\partial n}(x,y) = 0,$$

$$(x,y) \in \partial G, \quad \ldots\ldots\ldots\ldots\ (2)$$

where G is a rectangle (with sides parallel to the coordinate axis) with boundary ∂G, $\partial/\partial n$ denotes the outward normal, and γ and β are non-negative constants such that $\gamma + \beta > 0$.

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For various cases of this general problem, we apply the Galerkin method to obtain approximate solutions using particular finite dimensional subspaces. This method is presented in detail to illustrate its use for obtaining approximate solutions of Eqs. 1 and 2. In addition, a number of specific problems are solved numerically in order to illustrate the computational superiority of these methods.

We begin by treating the special case of Eqs. 1 and 2 where $k(x,y) = 1$, $\beta = 0$, and $f(x,y)$ is sufficiently smooth in $\bar{G} = G \cup \partial G$ to illustrate the basic ideas involved in using the Galerkin method for solving two-dimensional, elliptic, partial-differential equations. Some interesting numerical results are presented that illustrate the theoretically proved (Birkhoff, Schultz and Varga[1]), high-order accuracy obtainable by these methods. In addition, we show that not only can a high-order accuracy approximation be obtained for the solution, but also a high-order approximation for the first partial derivatives. The numerical results presented indicate an even higher order of convergence than can be proven at present.

In order to illustrate the tremendous advantage of these methods over central differences for calculating pressure distributions and pressure gradients around wells, we next consider another special case of Eqs. 1 and 2. Specifically, this is the special case of these two equations where $k(x,y)$ is a piecewise constant, $\beta = 0$ and $f(x,y)$ of Eq. 1 takes the form

$$f(x,y) = \sum_{i=1}^{M} Q(x_i,y_i)\delta(x-x_i)(y-y_i), \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where we are representing point sources by the dirac delta function (δ) (see Sneddon[14]). For this problem, we show how nonsmooth subspaces can be used to great advantage to obtain accurate approximations around the wells with only a very small amount of additional computations. The process of rigorously incorporated analytical solutions into the method is also shown.

Finally, we consider some numerical examples to illustrate the use and advantage of these ideas for problems with point sources. For solving these problems numerically, the Galerkin method requires the evaluation of numerous integrals. Therefore, some results of Herbold[5] and Herbold, Schultz and Varga[6] on the computational aspects of consistent quadrature schemes are also presented.

APPLICATION OF GALERKIN'S METHOD

In this section, we consider the special case of Eqs. 1 and 2 given by:

$$L[p(x,y)] \equiv \frac{\partial^2 p(x,y)}{\partial x^2} + \frac{\partial^2 p(x,y)}{\partial y^2} = f(x,y), \quad (x,y) \in G, \quad \ldots\ldots (4)$$

$$p(x,y) = 0, \quad (x,y) \in \partial G, \quad \ldots\ldots (5)$$

where here G is the open unit square $(0,1) \times (0,1)$, with boundary ∂G. We assume that the functions $f(x,y)$ are capable of being differentiated infinitely in $\bar{G}$ [i.e., $f(x,y) \in C^{\infty}(\bar{G})$]. This simple problem was chosen in order to carefully illustrate the Galerkin method (see, for example, Collatz[4] and Kantorovich and Krylov[8]) for obtaining approximate solutions of two-dimensional elliptic equations.

Let S denote the class of all real-valued, piecewise, continuously differentiable functions, $w(x,y)$, defined on G such that $w(x,y) = 0$ for $(x,y) \in \partial G$. If $\phi(x,y)$ is the unique solution of Eqs. 4 and 5, then evidently $L[\phi]w = fw$ for any w in S. Integrating this expression over G gives

$$\int_0^1 \int_0^1 L[\varphi]w(x,y)dx\,dy = \int_0^1 \int_0^1 f(x,y)w(x,y)dx\,dy, \text{ all } w \in S, \quad (6)$$

or equivalently

$$\int_0^1 \int_0^1 \{\varphi_{xx}(x,y)w(x,y) + \varphi_{yy}(x,y)w(x,y)\}dx\,dy = \int_0^1 \int_0^1 f(x,y)w(x,y)dx\,dy, \text{ all } w \in S. \quad (7)$$

Applying Green's theorem,[18] i.e.,

$$\int_0^1 \int_0^1 (L[\varphi]w + \varphi_x w_x + \varphi_y w_y)dx\,dy = \oint_{\partial G} w \frac{\partial \varphi}{\partial n} ds,$$

and using the fact that $w(x,y) = 0$ on ∂G for any $w \in S$, we have

$$-\int_0^1 \int_0^1 \{\varphi_x(x,y)w_x(x,y) + \varphi_y(x,y)w_y(x,y)\}dx\,dy = \int_0^1 \int_0^1 f(x,y)w(x,y)dx\,dy, \text{ all } w \in S. \quad (8)$$

The Galerkin method can be described as follows.

First, choose any *finite-dimensional* subspace S_M of S. The Galerkin approximation $\hat{w}(x,y)$ to $\phi(x,y)$ in S_M is then defined, in analogy with Eq. 8, as the element $\hat{w}(x,y)$ in S_M, which satisfies

$$-\int_0^1\int_0^1 \{\hat{w}_x(x,y)w_x(x,y) + \hat{w}_y(x,y)w_y(x,y)\}dx\,dy = \int_0^1\int_0^1 f(x,y)w(x,y)dx\,dy, \text{ all } w\in S_M. \quad (9)$$

The important difference is that Eq. 8 is valid for all $w \epsilon S$, while Eq. 9 is to be valid for all $w \epsilon S_M$.

If $\{w_i(x,y)\}_{i=1}^M$ is any basis for S_M, we can express $\hat{w}(x,y)$ as

$$\hat{w}(x,y) = \sum_{i=1}^{M} \hat{\alpha}_i w_i(x,y),$$

and the defining relation (Eq. 9) for the Galerkin approximation $\hat{w}(x,y)$ to $\phi(x,y)$ in S_M can be written as

$$-\int_0^1\int_0^1 \{\hat{w}_x w_{x,i} + \hat{w}_y w_{y,i}\}dx\,dy = \int_0^1\int_0^1 f w_i dx\,dy, \quad 1 \le i \le M. \quad (10)$$

Regarding the $\hat{\alpha}_i$'s as components of the vector

$$\underline{\hat{\alpha}} = (\hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_M)^T,$$

then the *linear* equations of Eq. 10 can be expressed in terms of the $\hat{\alpha}_i$'s as the matrix equation

$$A\underline{\hat{\alpha}} = \underline{t}, \quad (11)$$

where t = vector and $\underline{t} = (t_1, t_2, \ldots, t_M)^T$ has components t_i given by

$$t_i \equiv -\int_0^1\int_0^1 f(x,y)\, w_i(x,y)\, dx\, dy, \quad 1 \le i \le M. \quad (12)$$

The entries of the real $M \times M$ matrix $A = (a_{i,j})$ can defined as*

$$a_{i,j} = \langle w_i, w_j \rangle, \quad 1 \le i,j \le M, \quad (13)$$

where

$$\langle u,v \rangle \equiv \int_0^1\int_0^1 \{u_x v_x + u_y v_y\}dx\,dy. \quad (14)$$

It is evident that the matrix A is real and *symmetric*. Moreover, it can be shown (Appendix B) that the matrix A is also *positive definite*; i.e., the matrix problem of Eq. 11 admits a unique solution and, consequently, the Galerkin approximation $\hat{w}(x,y)$ of $\phi(x,y)$ in S_M is uniquely defined. In summary, then, the Galerkin method consists first of selecting a finite-dimensional subspace S_M of S, then determining a basis for S_M, and then solving the resulting matrix problem (Eq. 11).

SMOOTH HERMITE SPACES $H_0^m(\pi)$.

The smooth Hermite spaces, $H_0^m(\pi)$, $m \ge 1$, are in fact subspaces of S, and thus can be used for determining Galerkin approximations of the solution $\phi(x,y)$ of Eqs. 4 and 5. These subspaces of S are particularly interesting from a numerical point of view since, as will be shown, for suitable choice of the basis functions for $H_0^m(\pi)$, the matrices A (whose elements are defined in Eq. 13) are *sparse* matrices. This leads to efficient methods for solving the corresponding matrix equations of Eq. 11.

Let π_x and π_y be two partitions in each coordinate direction of the unit square:

$$\pi_x: 0 = x_0 < x_1 < \cdots < x_{N_x+1} = 1,$$

$$\pi_y: 0 = y_0 < y_1 < \cdots < y_{N_y+1} = 1,$$

where N_x and N_y are non-negative integers. Then, let the mesh π on $\bar{G} = [0,1] \times [0,1]$ denote the set of all points (x_i, y_j), $0 \le i \le N_x + 1$, $0 \le j \le N_y + 1$. For any positive integer m, $H_0^m(\pi)$ will be the set of all real, piecewise-polynomial functions $w(x,y)$ defined on $[0,1] \times [0,1]$ such that $w(x,y)$ satisfies the boundary conditions (Eq. 5), $w(x,y) \epsilon C^{m-1}([0,1] \times [0,1])$, i.e.,

$$\frac{\partial^{k+\ell} w(x,y)}{\partial x^k \partial y^\ell}$$

is continuous in $[0,1] \times [0,1]$ for all $0 \le k, \ell \le m-1$, and on each cell $R_{i,j} \equiv [x_i, x_{i+1}] \times [y_j, y_{j+1}]$ of $\bar{G}$ defined by π, $w(x,y)$ is a polynomial of degree $2m-1$ in each variable. Therefore, any element $w(x,y) \epsilon H_0^m(\pi)$ is uniquely determined by the $m^2 N_x N_y + 2m(m-1)(N_x + N_y) + 4(m-1)^2$ values

$$\frac{\partial^{k+\ell} w(x_i, y_j)}{\partial x^k \partial y^\ell}, \quad 0 \le k, \ell \le m-1,$$

*$a_{i,j}$ is just the coupling between the i-th and j-th basis functions.

$$1 \le i \le N_x, \quad 1 \le j \le N_y,$$

$$\frac{\partial^{k+\ell} w(x_i, y_j)}{\partial x^k \partial y^\ell}, \quad 1 \le \ell \le m-1, \quad 0 \le k \le m-1,$$

$$1 \le i \le N_x, \quad j = 0, \quad N_y + 1,$$

$$\frac{\partial^{k+\ell} w(x_i, y_j)}{\partial x^k \partial y^\ell}, \quad 0 \le \ell \le m-1, \quad 1 \le k \le m-1,$$

$$1 \le j \le N_y, \quad i = 0, \quad N_x + 1,$$

and

$$\frac{\partial^{k+\ell} w(x_i, y_j)}{\partial x^k \partial y^\ell}, \quad 1 \le k, \ell \le m-1, \quad i = 0,$$

$$N_x + 1, \quad j = 0, \quad N_y + 1.$$

Hence, the dimension M of $H_0^m(\pi)$ is $m^2 N_x N_y - 2m(m-1)(N_x + N_y) + 4(m-1)^2$. The terms entering into this expression for the dimension of $H_0^m(\pi)$ can be described as follows. For each of the $N_x N_y$ interior points of π, there are associated m^2 distinct basis functions. Because of the boundary conditions of Eq. 5, for each of the boundary points of π that is not a corner, there are associated $m(m-1)$ distinct basis functions; for each corner point, there are associated $(m-1)^2$ basis functions.

A basis for elements in this space can be obtained in the following way. First, we define the functions of a single variable $S_{i,k}(x;m)$, $0 \le i \le N_x + 1$, $0 \le k \le m-1$, and $T_{j,\ell}(y;m)$, $0 \le j \le N_y + 1$, $0 \le \ell \le m-1$. Let $S_{i,k}(x;m)$ and $T_{j,\ell}(y;m)$ be piecewise polynomials in $C^{m-1}[0,1]$ and be polynomials of degree $2m-1$ in the subintervals of their respective partitions π_x and π_y. By requiring that

$$\frac{d^\ell}{dx^\ell} S_{i,k}(x_j;m) = \delta_{\ell,k}\delta_{i,j},$$

$$0 \le \ell, k \le m-1, \quad 0 \le i, j \le N_x + 1,$$

and

$$\frac{d^\ell}{dy^\ell} T_{i,k}(y_j;m) = \delta_{\ell,k}\delta_{i,j},$$

$$0 \le k, \ell \le m-1, \quad 0 \le i, j \le N_y + 1,$$

where $\delta_{j,k}$ is the Kronecker delta function; i.e., $\delta_{j,k} = 0$ for all $j \ne k$ and $\delta_{j,j} = 1$, the functions $S_{i,k}(x;m)$ and $T_{j,\ell}(y;m)$ are uniquely determined.[1] The basis is then formed by taking the set of functions

$$S_{i,k}(x;m)T_{j,\ell}(y;m), \quad 0 \le k, \ell \le m-1,$$

$$0 \le i \le N_x + 1, \quad 0 \le j \le N_y + 1,$$

and omitting the boundary functions

$$S_{i,k}(x;m)T_{j,0}(y;m), \quad 0 \le i \le N_x + 1,$$

$$0 \le k \le m-1, \quad j = 0, \quad N_y + 1,$$

and

$$S_{i,0}(x;m)T_{j,\ell}(y;m), \quad 0 \le j \le N_y + 1,$$

$$0 \le \ell \le m-1, \quad i = 0, \quad N_x + 1.$$

Herbold[5] proved that these functions $S_{i,k}(x;m) \cdot T_{j,\ell}(y;m)$ are linearly independent, and do form a basis for $H_0^m(\pi)$.

To illustrate more clearly the preceding discussion, let us consider now two specific smooth Hermite spaces $H_0^1(\pi)$ and $H_0^2(\pi)$, where π is now the *uniform* mesh on the unit square; i.e., π is made up of the points (x_i, y_j), $0 \le i, j \le N+1$, where $h = 1/(N+1)$, $x_i = ih$ and $y_j = jh$.

By definition, each element in $H_0^1(\pi)$ is a polynomial of degree one in each variable in each cell $R_{i,j}$; i.e., in each cell $R_{i,j}$, each element of $H_0^1(\pi)$ is the product of a linear function of x with a linear function of y, the so-called *bilinear functions*. From this, it is clear that a basis for the subspace $H_0^1(\pi)$ is

$$t_i(x)t_j(y), \quad 1 \le i, j \le N,$$

where the functions t_i, $1 \le i \le N$, are defined by:

$$t_i(x) = \begin{cases} \dfrac{x-(i-1)h}{h}, & (i-1)h \le x \le ih, \\ \dfrac{(i+1)h-x}{h}, & ih \le x \le (i+1)h, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{. . . . . (15a)}$$

and are illustrated by Fig. 1. The smooth Hermite space $H_0^1(\pi)$ consists of all bilinear functions defined

on π that satisfy the boundary conditions in Eq. 5. For any element $w(x,y) \in H_0^1(\pi)$, there exist real $a_{i,j}$, $1 \le i,j \le N$, such that

$$w(x,y) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{i,j} t_i(x) t_j(y). \quad (15b)$$

For the uniform partition π of the unit square, we remark in this case that

$$t_i(x) = S_{i,0}(x;l); \quad t_i(y) = T_{i,0}(y;m),$$

which connects the definition (Eq. 15a) with the preceding general description of the basis elements of $H_0^m(\pi)$.

A basis for $H_0^2(\pi)$ is the set of functions

$$S_{i,k}(x) S_{j,\ell}(y), \quad 1 \le i,j \le N, \quad 0 \le k,\ell \le 1,$$

$$S_{i,0}(x) S_{j,0}(y), \quad S_{i,1}(x) S_{j,1}(y),$$

$$0 \le j \le N+1, \quad i = 0, \quad N+1;$$

and

$$S_{i,1}(x) S_{i,0}(y), \quad S_{i,0}(x) S_{j,1}(y),$$

$$1 \le i \le N, \quad j = 0, \quad N+1;$$

where the functions $S_{i,k}$, $0 \le i \le N+1$, $0 \le k \le N+1$, $0 \le k \le 1$, are piecewise cubic polynomials defined by

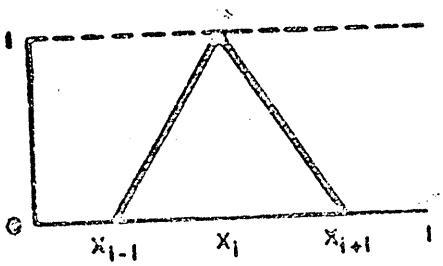
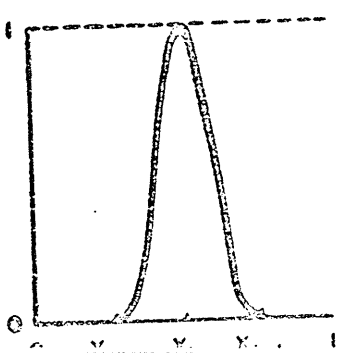


FIG. 1 — THE GRAPH OF $t_i(x)$.



$$S_{i,0}(x) = \begin{cases} -2\left(\frac{x-(i-1)h}{h}\right)^3 + 3\left(\frac{x-(i-1)h}{h}\right)^2, \\ \qquad (i-1)h \le x \le ih, \\ 2\left(\frac{x-ih}{h}\right)^3 - 3\left(\frac{x-ih}{h}\right)^2 + 1, \\ \qquad ih \le x \le (i+1)h, \\ 0, \quad \text{otherwise,} \end{cases} \quad (16)$$

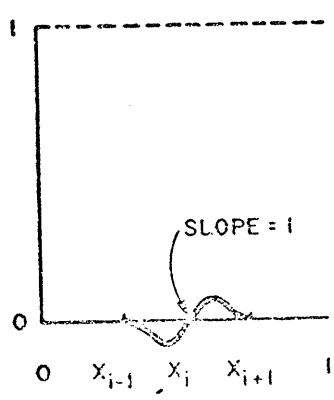
and

$$S_{i,1}(x) = \begin{cases} \left(\frac{x-(i-1)h}{h}\right)^2 (x-ih), \\ \qquad (i-1)h \le x \le ih, \\ (x-ih)\left(\frac{(i+1)h-x}{h}\right)^2, \\ \qquad ih \le x \le (i+1)h, \\ 0, \quad \text{otherwise.} \end{cases} \quad (17)$$

The functions $S_{i,0}(x)$ and $S_{i,1}(x)$ are graphed in Figs. 2 and 3, respectively.

Let us now consider the basis elements $t_i(x)t_j(y)$ of $H_0^1(\pi)$ defined by Eq. 15a and illustrate how the matrix elements $a_{(i,j)(k,\ell)}$ * of Eq. 13 are obtained. If Fig. 4 below represents a portion of the partition or grid, then the basis element for the i,j point is only non-zero over the shaded area.

*We mean by this notation that $a_{(i,j),(k,l)}$ is the coupling between the basis function centered at (x_i, y_j) and the one centered at (x_k, y_ℓ).

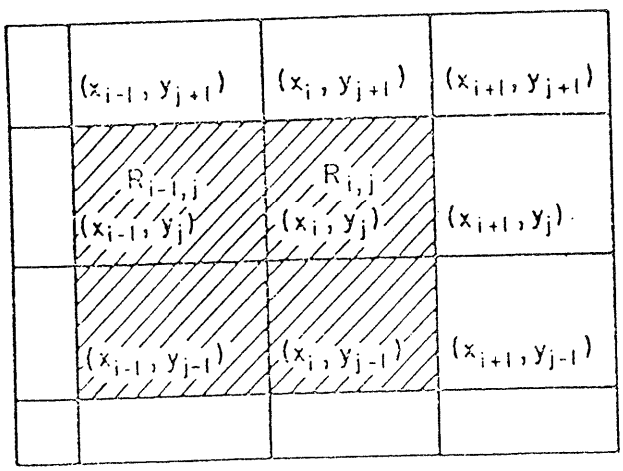


Therefore, the equation for this point from Eq. 11 becomes

$$(A\underline{\alpha})_{(i,j)} = a_{(i,j),(i-1,j+1)}\alpha_{(i-1,j+1)} + a_{(i,j),(i,j+1)}\alpha_{(i,j+1)} + a_{(i,j),(i+1,j+1)}\alpha_{(i+1,j+1)} + a_{(i,j),(i-1,j)}\alpha_{(i-1,j)} + a_{(i,j),(i,j)}\alpha_{(i,j)} + a_{(i,j),(i+1,j)}\alpha_{(i+1,j)} + a_{(i,j),(i-1,j-1)}\alpha_{(i-1,j-1)} + a_{(i,j),(i,j-1)}\alpha_{(i,j-1)} + a_{(i,j),(i+1,j-1)}\alpha_{(i+1,j-1)} = k_{(i,j)}. \quad (18)$$

Now, from Eq. 13,

$$a_{(i,j),(i,j)} = \int_{x_{i-1}}^{x_{i+1}} \int_{y_{j-1}}^{y_{j+1}} [(t_{x,i}(x)\cdot t_j(y))^2 + (t_i(x)t_{y,j}(y))^2]\,dx\,dy, \quad (19)$$



and by symmetry coupled with the definitions (Eq. 15a), Eq. 19 becomes

$$a_{(i,j),(i,j)} = 4\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} \left(\frac{1}{h^2}\right)\left(\frac{y-y_{j-1}}{h}\right)^2 + \left(\frac{1}{h^2}\right)\left(\frac{x-x_{i-1}}{h}\right)^2 dx\,dy = \frac{8}{3}. \quad (20)$$

Again, by symmetry, $a_{(i,j),(i+1,j+1)} = a_{(i,j),(i+1,j-1)} = a_{(i,j),(i-1,j+1)} = a_{(i,j),(i-1,j-1)}$, so we need only to calculate the one element

$$a_{(i,j),(i+1,j+1)} = \int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} [(t_{x,i+1}(x)t_{j+1}(y))(t_{x,i}(x)t_j(y)) + (t_{i+1}(x)t_{y,j+1}(y))(t_i(x)t_{y,j}(y))]dxdy \quad (21)$$

which becomes, using Eq. 15a,

$$a_{(i,j),(i+1,j+1)} = \int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}} [\left(\frac{1}{h}\right)\left(\frac{y-y_j}{h}\right)\cdot\left(-\frac{1}{h}\right)\left(\frac{y_{j+1}-y}{h}\right) + \left(\frac{x-x_i}{h}\right)\left(\frac{1}{h}\right)\left(\frac{x_{i+1}-x}{h}\right)\left(-\frac{1}{h}\right)]\,dxdy = -\frac{1}{3}. \quad (22)$$

Finally, $a_{(i,j),(i-1,j)} = a_{(i,j),(i-1,j)} = a_{(i,j),(i,j-1)} = a_{(i,j),(i,j-1)}$, and this element is given by

$$a_{(i,j),(i-1,j)} = \int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_{j+1}} [(t_{x,i-1}(x)t_j(y))(t_{x,i}(x)t_j(y)) + (t_{i-1}(x)t_{y,j}(y))(t_i(x)t_{y,j}(y))]dx\,dy, \quad (23)$$

which becomes, using Eq. 15a and symmetry,

$$a_{(i,j),(i-1,j)} = 2\int_{x_{i-1}}^{x_i}\int_{y_{j-1}}^{y_j} \left[\left(-\frac{1}{h}\right) \left(\frac{y-y_{j+1}}{h}\right) \left(\frac{1}{h}\right) \left(\frac{y-y_{j-1}}{h}\right) + \left(\frac{x_i-x}{h}\right) \left(\frac{1}{h}\right) \left(\frac{x-x_{i-1}}{h}\right) \left(\frac{1}{h}\right) \right] dx\, dy = -\frac{1}{3}. \quad \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

Therefore, the star for this equation is given in Fig. 5. This is just a nine-point formula and the system of Eqs. 11 can be easily solved by single or double line SOR (see Varga,[15] Chap. 4).

When we compute the system of equations for the basis elements of $H_0^2(\pi)$ defined by Eqs. 16 and 17, we get a more complex matrix. The entries in this matrix can, as in the discussion above, be computed directly from analogous integrations of the basis functions of Eqs. 16 and 17, but we omit the obviously lengthy calculations. If the equations are now ordered so that all the elements at one point are counted before moving to the next grid point, we get a nine-point formula where now instead of single elements at a point we have 4×4 matrices (see Fig. 6). There is still a natural partitioning of the matrix A by lines or double lines, and Block SOR can be shown to converge (see Varga,[15] Chap. 3). The elements of the matrices in Fig. 6 are $a^{(m,n)}_{(i,j),(k,\ell)}$, where this notation means the coupling between the m-th basis function centered at the point (x_i, y_j) and the n-th basis function centered at the point (x_k, y_ℓ).

NONSMOOTH SUBSPACES AND PRESSURE DISTRIBUTIONS AND GRADIENTS AROUND WELLS

In order to consider the problem of calculating accurate pressure distributions and pressure gradients around wells, we present some theoretical results for the problem

$$\frac{\partial}{\partial x}\left[k(x,y)\,\frac{\partial p(x,y)}{\partial x}\right] + \frac{\partial}{\partial y}\left[k(x,y)\,\frac{\partial p(x,y)}{\partial y}\right] = \sum_{i=1}^{r} Q(x_i,y_i)\,\delta(x-x_i, y-y_i), \quad (x,y) \in \partial G, \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

$$p(x,y) = 0, \quad (x,y) \in \partial G, \quad \ldots (26)$$

where G is again the open unit square $(0,1) \times (0,1)$ with boundary ∂G and point sources are represented by the dirac delta function. By using nonsmooth subspaces of S, we will show how very accurate approximations can be obtained around wells and how analytical solutions can be incorporated with the variational approach. The results of this section will be illustrated by numerical examples in a later section.

For the problem described by Eqs. 25 and 26, we define a uniform partition π as in the previous section. We can now make the additional assumptions that

1. $k(x,y)$ is a piecewise constant (i.e., in each grid block $R_{i,j} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $k(x,y)$ is a positive constant), and there exist constants k_1 k_2 such that

$$k_1 \geq k(x,y) \geq k_2 > 0, \ (x,y) \in G. \quad \ldots\ldots (27)$$

2. None of the point sources lie on grid intersections (x_i, y_j), $0 \leq i \leq N+1$, $0 \leq j \leq N+1$.

Proceeding as in the previous section, let $H_0^1(\pi)$ be the smooth subspace spanned by the basis elements $w_{i,j}(x,y) \equiv t_i(x)t_j(y)$ given by Eqs. 15a. We now seek an approximate solution, $\hat{w}(x,y) \in H_0^1(\pi)$, to Eqs. 25 and 26 of the form:

$$\hat{w}(x,y) = \sum_{i=1}^{N}\sum_{j=1}^{N} \hat{a}_{i,j} w_{i,j}(x,y).$$

Using Galerkin's method, the $\hat{a}_{i,j}$ are obtained by solving the following linear system of equations in the $\hat{a}_{i,j}$:

$$0 = \int_0^1\int_0^1 k(x,y)\left[w_{x,i,j}(x,y)\,\hat{w}_x(x,y) + w_{y,i,j}(x,y)\,\hat{w}_y(x,y)\right] - \int_0^1\int_0^1 f(x,y)\,w_{i,j}(x,y)\,dx\,dy,$$

$$1 \leq i,j \leq N. \quad \ldots\ldots\ldots\ldots\ldots (28)$$

$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$
$-\frac{1}{3}$	$+\frac{8}{3}$	$-\frac{1}{3}$
$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$

FIG. 5

$A_{(i,j),(i-1,j+1)}$	$A_{(i,j),(i,j+1)}$	$A_{(i,j),(i+1,j+1)}$
$A_{(i,j),(i-1,j)}$	$A_{(i,j),(i,j)}$	$A_{(i,j),(i+1,j)}$
$A_{(i,j),(i-1,j-1)}$	$A_{(i,j),(i,j-1)}$	$A_{(i,j),(i+1,j-1)}$

FIG. 6

If $\phi(x,y)$ is the solution of Eqs. 25 and 26, then Green's theorem again gives

$$0 = \int_0^1 \int_0^1 k(x,y)[w_{x,i,j}(x,y)\phi_x(x,y) + w_{y,i,j}(x,y)\phi_y(x,y)]\,dx\,dy + \int_0^1 \int_0^1 f(x,y)\,w_{i,j}(x,y)\,dx\,dy,$$

$$1 \leq i,j \leq N. \quad \ldots\ldots\ldots\ldots\ldots (29)$$

Now define the function $\tilde{w}(x,y) \in H_0^1(\pi)$ to be the unique interpolate of $\phi(x,y)$. By this, we mean that if

$$\tilde{w}(x,y) \equiv \sum_{i=1}^{N} \sum_{j=1}^{N} \tilde{\alpha}_{i,j} w_{i,j}(x,y), \quad \ldots (30)$$

then

$$\tilde{\alpha}_{i,j} = \phi(x_i, y_j), \quad \ldots\ldots\ldots (31)$$

In other words, $\tilde{w}(x_i,y_j) = \phi(x_i,y_j)$ for all $1 \leq i,j \leq N$.

Defining the Sobolev norm (Yosida,[19] Page 55) by

$$\|w\|_{1,2} \equiv \{\int_0^1 \int_0^1 [w^2 + (w_x)^2 + (w_y)^2]\,dx\,dy\}^{\frac{1}{2}}, \quad \ldots (32)$$

we show in Appendix C that

$$\|\hat{w} - \tilde{w}\|_{1,2} \leq C_3 h, \quad \ldots\ldots\ldots (33)$$

for some constant C_3, independent of h. By virtue of the triangle inequality for the norm $\|\cdot\|_{1,2}$, we have

$$\|\phi - \hat{w}\|_{1,2} = \|(\phi - \tilde{w}) + (\tilde{w} - \hat{w})\|_{1,2} \leq \|\phi - \tilde{w}\|_{1,2} + \|\hat{w} - \tilde{w}\|_{1,2}. \quad \ldots (34)$$

Again, the fact that $\tilde{w}$ is the $H_0^1(\pi)$-interpolate of ϕ, also gives us (Birkhoff, Schultz and Varga,[1] Theorem 13) $\|\phi - \tilde{w}\|_{1,2} \leq Mh$, where M is independent of h. Combining this with the inequality of Eq. 33 then results in

$$\|\phi - \hat{w}\|_{1,2} \leq C_4 h, \quad \ldots\ldots\ldots (35)$$

where C_4 is independent of h.

The use of nonsmooth subspaces leads to one of the most computationally interesting features of such variational methods. What is meant by a nonsmooth subspace in this paper can be defined as follows. Again, choose a partition π_x and π_y of the two coordinates and consider all the basis elements that span the space $H_0^1(\pi)$. Then in each mesh block $R_{i,j}$ one can add basis elements $w_{i,j}^{\ell,k}(x,y)$ of S such that

$$\int_0^1 \int_0^1 [(w_{x,i,j}^{\ell,k})^2 + (w_{y,i,j}^{\ell,k})^2]\,dx\,dy < \infty, \quad \ldots (36a)$$

and

$$w_{i,j}^{\ell,k}(x,y) \equiv 0, \quad (x,y) \notin R_{i,j}. \quad \ldots (36b)$$

Define this new space to be $H_0^1(\pi, \underline{z})$ where $\underline{z}$ is a vector with elements $z_{i,j}$ which are just the number of basis elements added in the block $R_{i,j}$.

Some examples of functions that may be added to the $R_{i,j}$-th block are

$$w_{i,j}^{\ell,k}(x,y) = (x-x_i)(x_{i+1}-x)(y-y_j)(y_{j+1}-y)x^k y^\ell, \quad \ldots\ldots\ldots (37a)$$

or

$$w_{i,j}^{\ell,k}(x,y) = \sin\frac{k\pi(x-x_i)}{(x_{i+1}-x_i)} \sin\frac{\ell\pi(y-y_j)}{(y_{j+1}-y_j)}. \quad \ldots (37b)$$

This ability to add basis elements to *any* mesh block has the advantage that the approximation may be improved locally without greatly increasing the dimensionality of the whole system. More significant is the fact that these extra basis functions *uncouple* from the rest of the equations (see Appendix A). This means that when computing, one chooses a partition π for the region and finds the element of best approximation $\tilde{u}(x,y) \in H_0^1(\pi)$. This involves solving a system of $N_x N_y$ linear equations iteratively. Then, if we wish to add r basis elements to the $R_{i,j}$-th block, we need now only solve a linear system of r equations for r unknowns. By doing little additional work, the approximation can be improved in a block containing a well. Another interesting result that comes from this uncoupling allows us to include functions on a homogeneous rectangle that actually are solutions of our problem. These functions may be added to any mesh block and will significantly improve the accuracy. This will be better illustrated by the following example.

Consider the problem

where x_0, y_0 is the location of the point source of strength Q. However, $\psi(x,y)$ from Eq. 44 does not satisfy the condition in Eq. 36a, so it cannot be used as a basis element. We can, however, use functions such as those given by Eq. 37b. Let $\hat{w}(x,y) \in H_0^1(\pi)$ be the first approximate to our solution and suppose

$$\hat{w}_1(x,y) = \hat{w}(x,y) + \sum_{n=1}^{r_1} \sum_{m=1}^{r_2} C_{n,m} w_{i,j}^{n,m}(x,y) \quad \ldots\ldots\ldots\ldots\ldots\ldots (45)$$

is the solution we seek where the $w_{i,j}^{n,m}(x,y)$ are given by Eq. 37b. Since the equations for $C_{n,m}$ uncouple from the equations for $\hat{a}_{i,j}$, we need only solve the system of equations

$$\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \Big[\Big(\sum_{n=1}^{r_1} \sum_{m=1}^{r_2} C_{n,m} w_{x,i,j}^{n,m}(x,y) \Big) \cdot w_{x,i,j}^{\ell,k}(x,y) + \Big(\sum_{n=1}^{r_1} \sum_{m=1}^{r_2} C_{n,m} w_{y,i,j}^{n,m}(x,y) \Big) \cdot w_{y,i,j}^{\ell,k}(x,y) \Big] dx\, dy = \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} Q \cdot \delta(x-x_0)(y-y_0) w_{i,j}^{\ell,k}(x,y) = Q w_{i,j}^{\ell,k}(x_0,y_0),$$

$$1 \le \ell \le r_1, \qquad 1 \le k \le r_2. \quad \ldots\ldots (46)$$

Notice that the functions $w_{i,j}^{\ell,k}(x,y)$ given by Eq. 37b form an orthogonal set as well as the functions $w_{y,i,j}^{\ell,k}(x,y)$. Therefore, Eqs. 46 become

$$\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} C_{\ell,k} \Big[(w_{x,i,j}^{\ell,k}(x,y))^2 + (w_{y,j,i}^{\ell,k}(x,y))^2 \Big] dx\, dy = Q w_{i,j}^{\ell,k}(x_0,y_0),$$

$$1 \le \ell \le r_1, \qquad 1 \le k \le r_2,$$

and the $C_{\ell,k}$'s are just the coefficients of

$$\sin \frac{k\pi(x-x_i)}{(x_{i+1}-x_i)} \sin \frac{\ell\pi(y-y_j)}{(y_{j+1}-y_j)}$$

in Eq. 44. Therefore, if we let r_1 and r_2 tend to infinity, we have

$$\hat{w}_1(x,y) = \hat{w}(x,y) + \psi(x,y), \quad \ldots\ldots (47)$$

where $\psi(x,y)$ is the exact solution given by Eq. 44. The uncoupling described in Appendix A works only if our original space is bilinear over the blocks where we wish to add basis elements. Moreover, this uncoupling is not possible for parabolic problems so that these results are not a panacea for all ills. However, in any case, adding functions in particular mesh blocks does not increase the dimensionality of the system very much, so that great improvements are still available for a small amount of additional computer time.

NUMERICAL EXAMPLES

Let us now consider some examples of problems of Eqs. 1 and 2 whose solutions have been approximated by the techniques discussed in the previous sections. For those cases where an exact solution was known, we computed the error in the uniform norm. That is, if $\phi(x,y)$ is the actual solution to the two-dimensional problem (Eqs. 1 and 2), and $\hat{w}(x,y)$ is an approximation to $\phi(x,y)$ obtained from Galerkin's method, then the quantity

$$\|\varphi(x,y)-\hat{w}(x,y)\|_{L^\infty} \equiv \max_{x,y \in G} |\varphi(x,y)-\hat{w}(x,y)|$$

is very accurately approximated by choosing $h = 1/N+1$, where N is a very large positive integer* and calculating $\max_{1 \le i,j \le N} |\phi(ih,jh)-\hat{w}(ih,jh)|$. For the finite difference approximations, the maximum error is calculated only at the grid points; that is, if $v(ih,jh)$ is the finite difference solution, then $\|\phi-v\|_{L^\infty} \equiv \max_{0 \le i,j \le N+1} |\phi(ih,jh)-v(ih,jh)|$.

The first problem (Herbold[5]) we consider is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6xye^x e^y(xy+x+y-3), \quad (x,y) \in G, \quad \ldots\ldots (48)$$

$$u(x,y) = 0, \qquad (x,y) \in \partial G, \quad \ldots (49)$$

where G is the open unit square $(0,1) \times (0,1)$, with boundary ∂G. It is easily seen that the exact solution of Eqs. 48 and 49 is given by

$$u(x,y) = 3e^x e^y(x-x^2)(y-y^2). \quad \ldots (50)$$

*The choice of N here is independent of the choice of h in $H_0^1(\pi)$.

$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y),$$

$$(x,y) \in R_{i,j}, \quad \ldots\ldots\ldots\ldots (38)$$

$$u(x,y) = 0,$$

$$(x,y) \in \Gamma_{i,j} \equiv \partial R_{i,j}. \quad \ldots\ldots (39)$$

Suppose the unique solution, $\psi(x,y)$, of Eqs. 38 and 39 is some known function. Defining $\psi(x,y)$ to be identically zero outside $R_{i,j}$, add $\psi(x,y)$ to the space $H_0^1(\pi)$. For this new subspace of S, we get as our new approximate solution

$$\hat{w}_1(x,y) = \hat{w}(x,y) + \psi(x,y). \quad \ldots (40)$$

If $\phi(x,y)$ is the exact solution to our original problem (Eqs. 25 and 26), then from Eq. 36b we have on $\Gamma_{i,j}$ that

$$\hat{w}_1(x,y) = \hat{w}(x,y).$$

Based on the result of Eq. 35, it can be shown that

$$\max_{(x,y)\in\Gamma_{i,j}} |\varphi(x,y)-\hat{w}_1(x,y)|$$

$$\leq Ch|\log h|. \quad \ldots\ldots\ldots (41)$$

Because $\hat{w}(x,y)$ is a bilinear function, then

$$\nabla^2 \hat{w}(x,y) = 0, \quad (x,y) \in R_{i,j}.$$

This, combined with Eq. 40 and the definitions of $\psi(x,y)$ and $\phi(x,y)$, shows that

$$\nabla^2(\varphi-\hat{w}_1) = 0, \quad (x,y) \in R_{i,j},$$

i.e., $\phi - \hat{w}_1$ is harmonic in $R_{i,j}$. From the Maximum Principle (Petrovsky,[12] Page 169), and inequality (Eq. 41), we have

$$\max_{(x,y)\in R_{i,j}} |\varphi(x,y)-\hat{w}_1(x,y)|$$

$$\leq Ch|\log h|. \quad \ldots\ldots\ldots (42)$$

This says that if we know an exact solution to Eqs. 25 and 26 for the case $K(x,y) = 1$, and $G = R_{i,j}$ with boundary $\Gamma_{i,j}$, then we may add this function to our space $H_0^1(\pi)$ and significantly improve our accuracy in the block $R_{i,j}$.

Let us now apply this result to a problem with wells, but one where the wells are not point sources but are distributed over a small rectangle (one could choose this about 6 in. on a side to correspond to an actual wellbore). Then, by a slight modification of some results of Hovanessian,[7] if $4\epsilon^2$ is the area of the square with center (x_0,y_0), over which Q is distributed, then

$$\psi(x,y) = \frac{4Q}{\pi^2(x_{i+1}-x_i)(y_{j+1}-y_j)} \cdot$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\left(\frac{m}{(x_{i+1}-x_i)}\right)^2+\left(\frac{n}{(y_{j+1}-y_j)}\right)^2} \cdot$$

$$\left(\frac{(x_{i+1}-x_i)}{m\pi\epsilon} \sin\frac{m\pi\epsilon}{(x_{i+1}-x_i)}\right)$$

$$\left(\frac{(y_{j+1}-y_j)}{n\pi\epsilon} \sin\frac{n\pi\epsilon}{(y_{j+1}-y_j)}\right) \cdot$$

$$\left(\sin\frac{m\pi(x_0-x_i)}{(x_{i+1}-x_i)} \sin\frac{n\pi(y_0-y_j)}{(y_{j+1}-y_j)}\right) \cdot$$

$$\left(\sin\frac{m\pi(x-x_i)}{(x_{i+1}-x_i)} \sin\frac{n\pi(y-y_j)}{(y_{j+1}-y_j)}\right) \quad \ldots (43)$$

is a solution to Eqs. 38 and 39 on any homogeneous mesh block $R_{i,j}$ containing a well. Therefore, the result of Eq. 42 says that the approximate solution near the well is at least as accurate as the solution $w(x,y)$ on the boundary of the mesh block containing the well. Since mesh blocks are usually quite large, the boundary is some distance from the well and the solution there is generally very good.

For the case where we treat the wells as point sources, Hovanessian's solution is given by

$$\psi(x,y) = \frac{4Q}{\pi^2(x_{i+1}-x_i)(y_{j+1}-y_j)}$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\left(\frac{m}{(x_{i+1}-x_i)}\right)^2+\left(\frac{n}{(y_{j+1}-y_j)}\right)^2} \cdot$$

$$\sin\frac{m\pi(x_0-x_i)}{(x_{i+1}-x_i)} \sin\frac{n\pi(y_0-y_j)}{(y_{j+1}-y_j)} \cdot$$

$$\sin\frac{m\pi(x-x_i)}{(x_{i+1}-x_i)} \sin\frac{n\pi(y-y_j)}{(y_{j+1}-y_j)}, \quad \ldots (44)$$

The solution of Eqs. 48 and 49 is first approximated by using Galerkin's technique for the smooth Hermite subspace $H_0^1(\pi_N)$, where π_N is the uniform mesh on the unit square with mesh size $h_N = 1/N+1$. The dimension of $H_0^1(\pi_N)$ is N^2. Denoting the basis functions of $H_0^1(\pi_N)$ as $\{w_i(x,y)\}_{i=1}^{N^2}$, we see from Eq. 9 that our solution is obtained by solving the system

$$0 = \sum_{j=1}^{N^2} \alpha_j \int_0^1\int_0^1 w_{x,i}w_{x,j} + w_{y,i}w_{y,j}\,dxdy$$

$$+\int_0^1\int_0^1 6xye^x e^y(xy+x+y-3)w_i(x,y)\,dxdy,$$

$$1 \le i \le N^2, \quad \ldots\ldots\ldots\ldots (51)$$

for the unknowns $\{a_j\}_{j=1}^{N^2}$. If $\{\hat{a}_j\}_{j=1}^{N^2}$ is the unique solution to this system, then letting

$$\hat{w}_N(x,y) = \sum_{j=1}^{N^2} \hat{\alpha}_j w_j(x,y)$$

we have from Eq. 35 that

$$\|u(x,y)-\hat{w}_N(x,y)\|_{1,2} \le Mh_N, \quad N > 1,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (52)$$

where M is a constant, independent of h.

If the integral on the right-hand side of Eq. 51 is evaluated using a nine-point quadrature scheme (see Herbold,[5] Page 175), then this would be consistent with Eq. 52 and we should expect $0(h)$ convergence. The numerical results for this problem are given in Table 1. In this table we include the quantity ν defined as

$$\nu \equiv \log\left(\frac{\|u-\hat{w}_{h_{N_1}}\|_{L^\infty}}{\|u-\hat{w}_{h_{N_2}}\|_{L^\infty}}\right) \Big/ \log\left(\frac{h_{N_1}}{h_{N_2}}\right),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (53)$$

where h_{N_1} and h_{N_2} are two successive values of h. The motivation for Eq. 53 is the assumption that

$$\|u(x) - \hat{w}_{h_N}(x)\|_{L^\infty} \approx C(h_N)^\nu$$

for some ν and C constant. Then, for two successive values of h, h_{N_1} and h_{N_2}, we have

$$\frac{\|u-\hat{w}_{h_{N_1}}\|_{L^\infty}}{\|u-\hat{w}_{h_{N_2}}\|_{L^\infty}} \approx \left(\frac{h_{N_1}}{h_{N_2}}\right)^\nu \quad \ldots\ldots (54)$$

and Eq. 53 follows from Eq. 54. In our table, we try to take enough values of h to anticipate the value of ν.

We see from Table 1, therefore, that convergence seems to be $0(h_N)$ in the norm $\|\cdot\|_{L^\infty}$, although we could only prove $0(h_N^2)$ in the $\|\cdot\|_{1,2}$ norm. This was the case for one-dimensional problems until Ciarlet[2] and later Perrin, Price and Varga[11] improved the theoretical results. These improvements are a strong indication of things to come for two-dimensional problems.

Table 2 shows the results of solving Eqs. 48 and 49 by the usual five-point difference method. In comparing Tables 1 and 2, we see that the usual five-point difference method looks somewhat superior; this is due, however, to the fact that Table 1 displays the maximum error of a continuous function over the region, while in Table 2 the error is computed only at the mesh point.

The solution of Eqs. 48 and 49 was also approximated by using Galerkin's technique for functions of the smooth Hermite subspace $H_0^2(\pi_N)$ where π_N is the same as above. If $\hat{w}_N(x,y) \in H_0^2(\pi_N)$ is the approximate solution, then, using Theorem 16 of Birkhoff, Schultz and Varga,[1] we can deduce

$$\|u-\hat{w}_N\|_{1,2} \le M_2(h_N^3), \quad N > 1,$$

for some constant M_2. In this case a two-dimensional, 25-point quadrature scheme is consistent.[5] The numerical results are given in Table 3. The quantities in this table indicate that for the subspace $H_0^2(\pi_N)$, the accuracy in the norm

TABLE 1 — SMOOTH HERMITE SUBSPACE $H_0^1(\pi_N)$

h	Unknowns	$\|u-\hat{w}_N\|_{L^\infty}$	ν	Computer Time (seconds)
1/7	36	$3.10 \cdot 10^{-2}$	—	<.6
1/8	49	$2.43 \cdot 10^{-2}$	1.84	<.6
1/9	64	$1.96 \cdot 10^{-2}$	1.85	1.2
1/10	81	$1.60 \cdot 10^{-2}$	1.94	1.8
1/11	100	$1.33 \cdot 10^{-2}$	1.97	2.4

TABLE 2 — FIVE-POINT FINITE DIFFERENCE METHOD

h	Unknowns	$\max_i \|u(ih)-v(ih)\|$	Computer Time (seconds)
1/5	16	$1.11 \cdot 10^{-2}$	<.6
1/6	25	$7.33 \cdot 10^{-3}$	<.6
1/7	36	$5.68 \cdot 10^{-3}$	<.6
1/8	49	$4.27 \cdot 10^{-3}$	<.6
1/9	64	$3.42 \cdot 10^{-3}$	.6
1/10	81	$2.77 \cdot 10^{-3}$	1.2
1/11	100	$2.27 \cdot 10^{-3}$	1.8

$\|\cdot\|_{L^\infty}$ is probably $O(h_N^4)$. Notice also that the first partial derivatives seems to converge in the norm $\|\cdot\|_{L^\infty}$ like $O(h_N^3)$. Moreover, if we compare the computer times for a given accuracy for the Galerkin method applied to the space $H_0^2(\pi_N)$ to the times for the usual finite differences of Table 2, we see that a significant time reduction results.

The final subspace considered was the nonsmooth space $H_0^1(\pi,\underline{z})$, where $z_{i,j} = 1$, $1 \le i,j \le N$. That is, we add one element to each mesh block $R_{i,j}$. The element added to each block, in this case, was just the bilinear basis element given by $t_i(x)t_j(y)$ where

$$t_i(x) = \begin{cases} \dfrac{2(x-ih)}{h}, & ih \le x \le (i+\frac{1}{2})h, \\ \dfrac{2((i+1)h-x)}{h}, & (i+\frac{1}{2})h \le x \le (i+1)h, \\ 0, & \text{otherwise.} \end{cases}$$

The results are given in Table 4. It can be seen that these nonsmooth spaces greatly reduce the error displayed in Table 1 while increasing this running time an immeasurable amount.

We next considered the problem

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = Q\delta(x-x_0)(y-y_0),$$

$$(x,y) \in G, \quad \ldots\ldots\ldots\ldots\ldots (55)$$

$$p(x,y) = 0, \quad (x,y) \in \partial G \quad \ldots\ldots (56)$$

where now G is the open square $(0,2000) \times (0,2000)$ with boundary ∂G and $x_0 = y_0 = 1{,}000$. We choose $Q = 886.905 B_o \mu / K$, which is equivalent to a reservoir problem with the following properties:

μ = .33 cp,
k = 1.5 md,
Q = 1.0 B/D-ft,
B_o = 1.5 res bbl/STB.

Choosing the space $H_0^1(\pi\underline{z})$, where $z_j = \delta_{j,t}$, $1 \le j \le N^2$ and in the t mesh block, which is the one containing (x_0,y_0), we add the single element $\psi(x,y)$ given by Eq. 44 as was done in Eq. 47. The results of this problem are presented in Figs. 7 and 8 and are compared with the solution using central differences and the exact solution. Notice that the Galerkin solution is very good near the source for a very coarse grid (Fig. 7), while the finite difference solution gives relatively low accuracy there. These results say that the partition chosen need only be as fine as is necessary to coincide with the permeability changes, and excellent accuracy wili be obtained. This problem was also solved using other subspaces. The experimentation here was not exhaustive, but it can be said that other numerical experiments did not give results comparable to those of the nonsmooth space selected.

CONCLUSIONS

A method of high-order accuracy, based on Galerkin's technique, is presented for solving two-dimensional elliptic differential equations. The procedure is described in detail and is shown to have tremendous computational advantages over finite difference techniques, especially for obtaining pressures and pressure gradients around wells. Although no results are given here for typical problems of reservoir mechanics, methods are nonetheless quite general and should extend to the coupled systems of nonlinear elliptic differential equations of reservoir mechanics.

NOMENCLATURE

μ = viscosity
k = permeability
Q = specific production or injection rate
B_o = oil formation volume factor
h = mesh spacing
x = horizontal coordinate
y = vertical coordinate
$p(x,y)$ = pressure

TABLE 3 — SMOOTH HERMITE SUBSPACE $H_0^2(\pi_N)$

h	Unknowns	$\|u-\hat{w}_N\|_{L^\infty}$	ν	$\|u_x-(\hat{w}_N)_x\|_{L^\infty}$	ν_x	Computer Time (seconds)
1/3	36	$8.87 \cdot 10^{-4}$	—	$2.08 \cdot 10^{-2}$	—	1.2
1/4	64	$2.99 \cdot 10^{-4}$	3.78	$8.75 \cdot 10^{-3}$	3.00	2.4
1/5	100	$1.33 \cdot 10^{-4}$	3.63	$4.86 \cdot 10^{-3}$	2.64	4.2
1/6	144	$6.97 \cdot 10^{-5}$	3.53	$2.84 \cdot 10^{-3}$	2.94	9.0
1/7	196	$3.89 \cdot 10^{-5}$	3.79	$1.80 \cdot 10^{-3}$	2.96	16.8
1/8	256	$2.29 \cdot 10^{-5}$	3.94	$1.23 \cdot 10^{-3}$	2.86	28.2
1/9	324	$1.43 \cdot 10^{-5}$	4.02	$8.63 \cdot 10^{-4}$	3.02	44.4

TABLE 4 — NONSMOOTH HERMITE SUBSPACE $H_0^1(\pi,\underline{z})$

h	Unknowns	$\|u-\hat{w}_N\|_{L^\infty}$	ν	Computer Time (seconds)
1/6	71	$1.56 \cdot 10^{-2}$	—	<.6
1/7	85	$1.16 \cdot 10^{-2}$	1.92	<.6
1/8	113	$8.63 \cdot 10^{-3}$	2.22	<.6
1/9	145	$6.78 \cdot 10^{-3}$	2.02	1.2
1/10	181	$5.56 \cdot 10^{-3}$	1.88	1.8
1/11	221	$4.47 \cdot 10^{-3}$	2.08	2.4

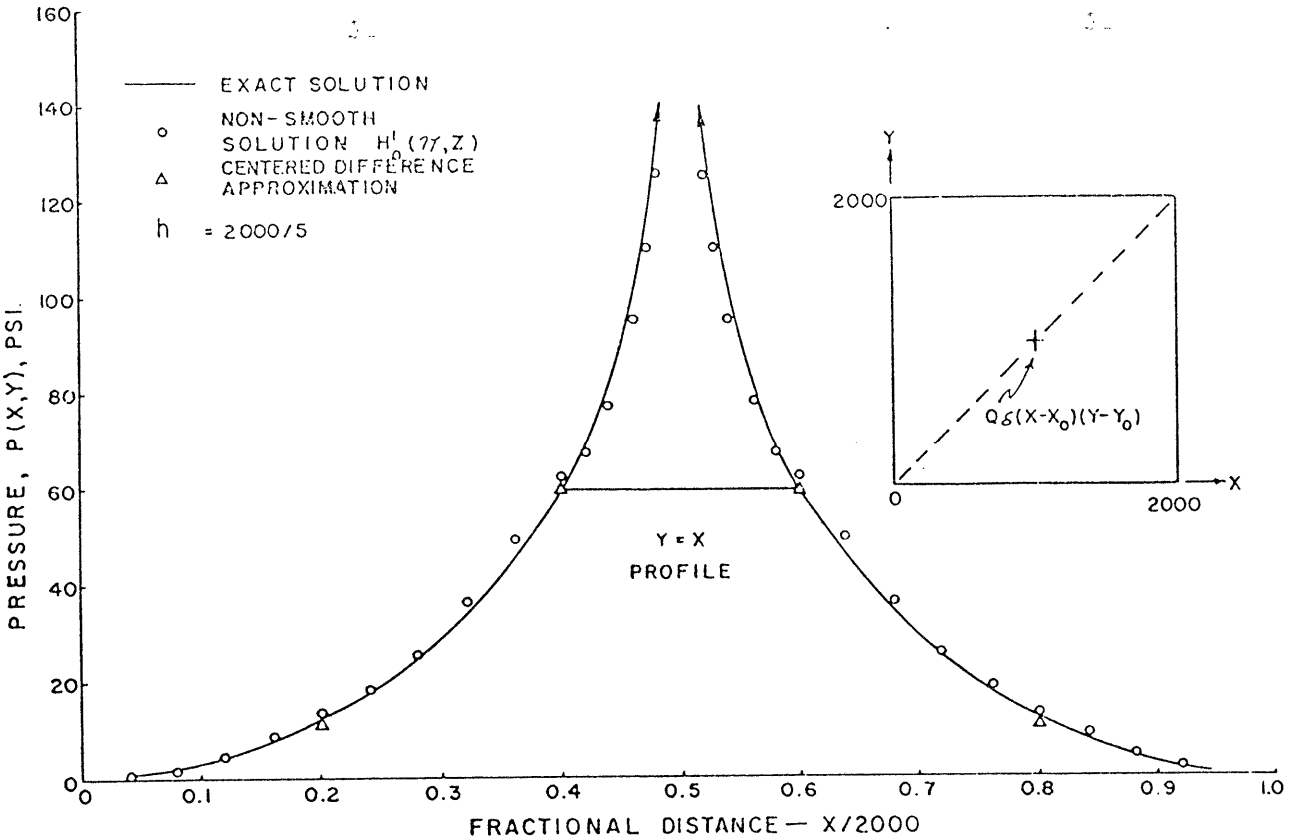


FIG. 7 — COMPARISON OF NONSMOOTH APPROXIMATION WITH CENTRAL DIFFERENCE APPROXIMATION.

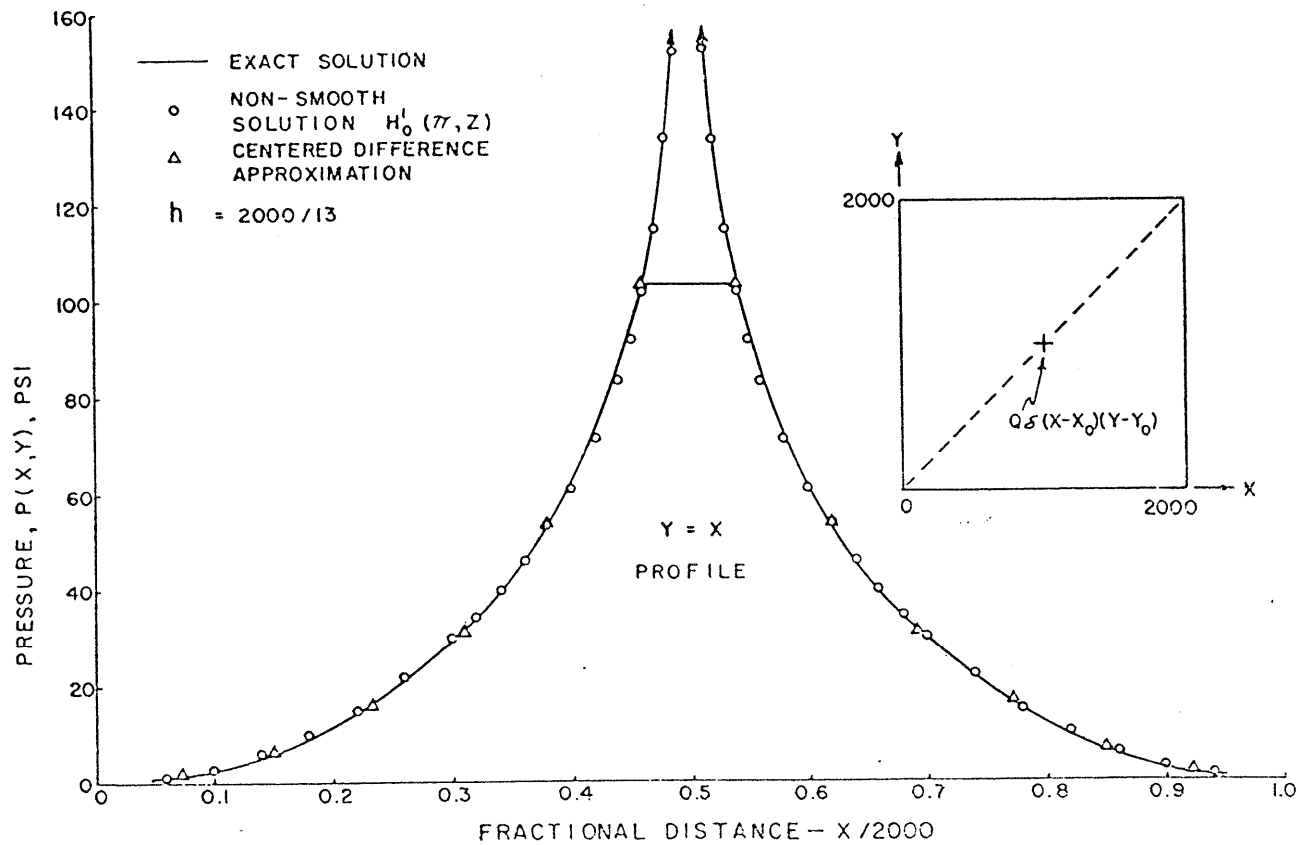


FIG. 8 — COMPARISON OF NONSMOOTH APPROXIMATION WITH CENTRAL DIFFERENCE APPROXIMATION.

SUBSCRIPTS AND SUPERSCRIPTS

k = summation index

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APPENDIX A

Let π_x and π_y be partitions of the coordinate axis and let

$$\{w_t(x,y)\}_{t=1}^{N_x N_y}$$

be a basis for $H_0^1(\pi)$. Then any element $w(x,y) \in H_0^1(\pi)$ can be written as

$$w(x,y) = \sum_{t=1}^{N_x N_y} \alpha_t w_t(x,y).$$

Let us now consider the case of adding a single element $w_{i,j}(x,y)$, which satisfies Eqs. 36a and 36b in the mesh block $R_{i,j} = [x_i, x_{i+1}] \times [y_i, y_{i+1}]$. Now, any element $g(x,y) \in H_0^1(\pi, \underline{z})$, where $z_{\ell,k} = \delta_{\ell,i}\delta_{k,j}$, $1 \le i, \ell \le N_x$, $1 \le j, k \le N_y$, can be written as

$$g(x,y) = \sum_{t=1}^{N_x N_y} \alpha_t w_t(x,y) + C w_{i,j}(x,y).$$

We now have $N_x N_y + 1$ equations and $N_x N_y + 1$ unknowns a_t, $1 \le t \le N_x N_y$ and C. Since the element $w_{i,j}(x,y)$ is zero outside the mesh block $R_{i,j}$, the only couplings possible between C and the a_t's in the system of equations would be to a_k, a_{k+1}, a_{k+N_x} and a_{k+N_x+1}, where the subscript $k = (j-1)N_x + i$. (See Fig. 9.) We need only consider one such coupling, since the argument for the other three is the same. Let us choose a_k. From Eq. 13 the coupling from a to C, $a_{k,N_x N_y+1}$, is given by

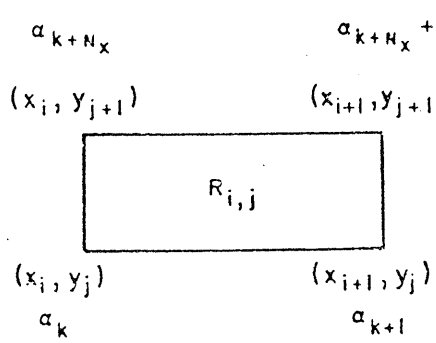

FIG. 9

$$a_{k,N_xN_y+1} = \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} (w_{k,x}w_{x,i,j}+w_{k,y}w_{y,i,j})dxdy. \quad (A-1)$$

Since the basis elements for $H_0^1(\pi)$ are of the form $t_i(x)t_j(y)$, it follows from Eq. 15a that $w_{k,x}$ is a function only of y, and $w_{k,y}$ is a function only of x; i.e.,

$$w_{k,x} = g_1(y), \quad (A-2)$$

$$w_{k,y} = g_2(x). \quad (A-3)$$

Using Eqs. A-2 and A-3, Eq. A-1 becomes

$$a_{k,N_xN_y+1} = \int_{y_j}^{y_{j+1}} g_1(y) \int_{x_i}^{x_{i+1}} w_{x,i,j}dxdy + \int_{x_i}^{x_{i+1}} g_2(x) \int_{y_j}^{y_{j+1}} w_{y,i,j}dydx$$

but

$$\int_{x_i}^{x_{i+1}} w_{x,i,j}(x,y)dx = w_{i,j}(x,y)\Big|_{x_i}^{x_{i+1}} = 0 \quad (A-4)$$

because $w_{i,j}(x,y)$ satisfies Eq. 36b. Similarly,

$$\int_{y_j}^{y_{j+1}} w_{y,i,j}dy = 0 \quad (A-5)$$

giving

$$a_{k,N_xN_y+1} = 0.$$

By symmetry,

$$a_{N_xN_y+1,k} = 0,$$

and so we can solve for the a_k separately from the single equation for C. Since the argument is the same for any basis element $w_{i,j}$ in $R_{i,j}$, if we had r such elements we would have a system of N_xN_y equations for the a_t's

$$A\underline{a} = \underline{t}, \quad (A-6)$$

and a system of r equations for the C_k's

$$B\underline{C} = \underline{g} \quad (A-7)$$

both of which can be solved separately. If we had r-basis elements $w_{i,j}$ in each block the same argument applies and we would again have Eq. A-6 to solve for the a_t and $(N_x+1)(N_y+1)$ systems like Eq. A-7 to solve. For a system with $M = N_xN_y + r(N_x+1)(N_y+1)$ unknowns we do not have to solve an $M \times M$ system of equations to obtain our coefficient. Instead, we must solve many smaller systems, which is computationally much faster.

It should be noted that Eqs. A-4 and A-5 imply that $w_{x,i,j}$ and $w_{y,i,j}$ are continuous in $R_{i,j}$. However, all that is necessary is that $w_{x,i,j}$ and and $w_{y,i,j}$ be piecewise continuous in $R_{i,j}$ and satisfy Eqs. 36a and 36b.

APPENDIX B

This appendix shows that the $M \times M$ matrix $A = (a_{i,j})$ of Eq. 11 is *positive definite*. With the inner product of Eq. 14, we define the norm $\|\cdot\|_D$ as

$$\|w\|_D^* = <w,w>^{1/2} = \{\int_0^1\int_0^1 (w_x^2 + w_y^2)dx\,dy\}^{1/2}. \quad (B-1)$$

It can be verified that the norm of Eq. B-1 is *equivalent* to the Sobolev norm[19]

$$\|w\|_{1,2} \equiv \left\{\int_0^1\int_0^1 \left[(w)^2 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]dxdy\right\}^{1/2}; \quad (B-2)$$

that is, there exists constants C_1 and $C_2 = 1$ such that

$$C_1\|w\|_{1,2} \le \|w\|_D \le C_2\|w\|_{1,2}. \quad (B-3)$$

If we now define

$$\|w\|_0 = \{\int_0^1\int_0^1 (w)^2dxdy\}^{1/2}, \quad (B-4)$$

it is clear from Eq. B-2 that

$$\|w\|_0 \le \|w\|_{1,2}. \quad (B-5)$$

With these definitions it is now easily seen that the matrix A is positive definite since, if $\underline{z} = (z_1, \ldots, z_M)^T$ is any non-zero vector, then

With the inequalities of Eq. C-8, we have from Eq. C-9

$$\sum_{i=1}^{N}\sum_{j=1}^{N}(\tilde{\alpha}_{i,j}-\hat{\alpha}_{i,j})b_{i,j}$$

$$\leq K_1 Mh\{\|\tilde{w}_x-\hat{w}_x\|_0+\|\tilde{w}_y-\hat{w}_y\|_0\}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(C-10)}$$

But since $\|\tilde{w}_x-\hat{w}_x\|_0$ and $\|\tilde{w}_y-\hat{w}_y\|_0$ are, from Eq. B-1 each bounded above by $\|\tilde{w}-\hat{w}\|_D$, which in turn from Eq. B-3 is bounded above by $C_2\|w\|_{1,2}$, then

$$\sum_{i=1}^{N}\sum_{j=1}^{N}(\tilde{a}_{i,j}-\hat{a}_{i,j})\,b_{i,j} \leq 2C_2k_1Mb\|\tilde{w}-\hat{w}\|_{1,2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(C-11)}$$

Thus, combined with the inequality of Eq. C-7, we can deduce the desired result (compare Eq. 33) that

$$\|\tilde{w}-\hat{w}\|_{1,2} \leq C_3h, \quad \ldots\ldots\ldots \text{(C-12)}$$

for some constant C_3, independent of h. ***

$$\underline{z}^T A \underline{z} = \int_0^1\int_0^1 [(Z_x(x,y))^2 + (Z_y(x,y))^2]dx\ dy$$

$$= \|Z\|_D^2 \geq C_1^2 \|Z\|_{1,2}^2$$

$$\geq C_1^2 \int_0^1\int_0^1 (Z(x,y))^2 dx\ dy > 0$$

where

$$Z(x,y) = \sum_{i=1}^{M} z_i w_i(x,y).$$

Because of this, the system of equations (Eq. 11) has a unique solution $\hat{\underline{a}}$.

APPENDIX C

In this appendix, we derive the inequality of Eq. 33. Recalling from Eqs. 35 and 36 that

$$\tilde{w}(x,y) \equiv \sum_{i=1}^{N}\sum_{j=1}^{N} \tilde{\alpha}_{i,j} w_{i,j}(x), \quad \text{(C-1)}$$

where $\tilde{a}_{i,j} = \phi(x_i, y_j)$, we define the numbers $b_{i,j}$ through

$$\int_0^1\int_0^1 k(x,y)[w_{x,i,j}(x,y)\tilde{w}_x(x,y) + w_{y,i,j}(x,y)\tilde{w}_y(x,y)]\,dx\,dy + \int_0^1\int_0^1 f(x,y)\,w_{i,j}(x,y)\,dx\,dy = b_{i,j},$$

$$1 \leq i,j \leq N. \quad \text{(C-2)}$$

Subtracting Eqs. 29 and 28 from Eqs. C-2 gives, respectively,

$$\int_0^1\int_0^1 k(x,y)[w_{x,i,j}(\tilde{w}_x - \phi_x) + w_{x,i,j}(\tilde{w}_y - \phi_y)]\,dx\,dy = b_{i,j}, \quad 1 \leq i,j \leq N, \quad \text{(C-3)}$$

and

$$\int_0^1\int_0^1 k(x,y)[w_{x,i,j}(\tilde{w}_x - \hat{w}_x) + w_{y,i,j}(\tilde{w}_y - \hat{w}_y)]\,dx\,dy = b_{i,j}, \quad 1 \leq i,j \leq N. \quad \text{(C-4)}$$

Multiplying Eqs. C-3 and C-4 by $\tilde{a}_{i,j} - \hat{a}_{i,j}$ and summing over i and j gives

$$\int_0^1\int_0^1 k(x,y)[(\tilde{w}_x - \phi_x)(\tilde{w}_x - \hat{w}_x) + (\tilde{w}_y - \phi_y)(\tilde{w}_y - \hat{w}_y)]\,dx\,dy = \sum_{i=1}^{N}\sum_{j=1}^{N} (\tilde{a}_{i,j} - \hat{a}_{i,j})\,b_{i,j}, \quad \text{(C-5)}$$

and

$$\int_0^1\int_0^1 k(x,y)[(\tilde{w}_x - \hat{w}_x) + (\tilde{w}_y - \hat{w}_y)^2]\,dx\,dy = \sum_{i=1}^{N}\sum_{j=1}^{N} (\tilde{a}_{i,j} - \hat{a}_{i,j})\,b_{i,j}. \quad \text{(C-6)}$$

Since $k(x,y) \geq k_2 > 0$ in G from Eq. 27, it follows from Eq. C-6 that

$$\sum_{i=1}^{N}\sum_{j=1}^{N} (\tilde{a}_{i,j} - \hat{a}_{i,j})b_{i,j} \geq k_2 \int_0^1\int_0^1 \{(\tilde{w}_x - \hat{w}_x)^2 + (\tilde{w}_y - \hat{w}_y)^2\}\,dx\,dy,$$

and with the definition of the norm $\|\cdot\|_{1,2}$ in Eq. B-2, we can express this inequality from Eq. B-3 as

$$\sum_{i=1}^{N}\sum_{j=1}^{N} (\tilde{a}_{i,j} - \hat{a}_{i,j})\,b_{i,j} \geq k_2 C_1^2 \|\tilde{w} - \hat{w}\|_{1,2}^2. \quad \text{(C-7)}$$

We now bound the double sum

$$\sum_{i=1}^{N}\sum_{j=1}^{N} (\tilde{\alpha}_{i,j} - \hat{\alpha}_{i,j})b_{i,j}$$

from above. Assuming that ϕ_{xx}, ϕ_{xy} and ϕ_{yy} are square-integrable on G; i.e., the integrals

$$\int_0^1\int_0^1 \varphi_{xx}^2\,dxdy, \quad \int_0^1\int_0^1 \varphi_{xy}^2\,dxdy, \quad \int_0^1\int_0^1 \varphi_{yy}^2\,dxdy$$

are all bounded; then since $\tilde{w}(x,y)$ is the $H_0^1(\pi)$-interpolation of $\phi(x)$, it follows (Birkhoff, Schultz and Varga,[1] Theorem 6) that there is a constant M, independent of k, such that

$$\|\tilde{w}_x - \varphi_x\|_0 \leq Mh \text{ and } \|\tilde{w}_y - \varphi_y\|_0 \leq Mh, \quad \text{(C-8)}$$

where $\|\cdot\|_2$ is defined in Eq. B-4. With the expression of Eq. C-5, we can bound $k(x,y)$ above by $k_1 > 0$ from Eq. 27, and then by applying Schwarz's inequality Eq. C-5 yields

$$k_1\{\|\tilde{w}_x - \phi_x\|_0 \cdot \|\tilde{w}_x - \hat{w}_x\| + \|\tilde{w}_y - \phi_y\|_2 \cdot \|\tilde{w}_y - \hat{w}_y\|\} \geq \sum_{i=1}^{N}\sum_{j=1}^{N} (\tilde{a}_{i,j} - \hat{a}_{i,j})b_{i,j}. \quad \text{(C-9)}$$