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Error Bounds for Spline Interpolation

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§1. Introduction. The object of this paper is to consider various forms of error bounds for spline interpolation in one space variable, along with some applications and extensions. Briefly, the material in §2 concerns the derivation of error bounds for spline interpolation for collections of bounded linear functionals on the Sobolev space $W_2^n[0,1]$ satisfying Property $\mathfrak{R}$, in a sense to be made precise in §2. Some special cases will be given to illustrate the results.

In §3, it is shown how the use of the theory of interpolation spaces leads to error bounds for the more general Besov spaces. In §4, application of the error bounds is made to the study of convergence of discrete variational Green's function to the continuous Green's function defined from two-point boundary value problems. Finally, in §5, Hermite spline functions are considered for the numerical approximation of ordinary differential equations, and improved error bounds are derived.

§2. Property $\mathfrak{R}$ Collections. Consider any ordinary differential operator L of order n of the form

$$(2.1) \qquad L[u] = \sum_{j=0}^{n} a_j(x)\, D^j u(x), \quad D^j \equiv \frac{d^j}{dx^j}, \quad n \geq 1 ,$$

for $u \in C^n[0,1]$, where we assume that $a_j \in C^j[0,1]$ for all $0 \leq j \leq n$, and that $a_n(x) \geq \omega > 0$ on $[0,1]$. In general, let $W_p^m[0,1]$, m a positive integer and $1 \leq p \leq +\infty$, denote the Sobolev space of all real-valued functions f defined on $[0,1]$

such that $D^{m-1}f$ is absolutely continuous with $D^m f \in L_p[0,1]$. It is well known that $W_p^m[0,1]$ is a Banach space. Fixing p = in this section, let $\Lambda = \{\lambda_i\}_{i=1}^k$ be a set of k linearly independent bounded linear functionals on $W_2^n[0,1]$. For any $r = (r_1, r_2, \ldots, r_k)$ in real Euclidean space $\mathbb{R}^k$, the minimization problem,

$$(2.2)\quad \begin{cases} \inf\{\|Lg\|_{L_2[0,1]} : g \in K_r^{\perp}\}, & \text{where} \\ K_r^{\perp} \equiv \{g \in W_2^n[0,1] : \lambda_i(g) = r_k \text{ for all } 1 \le i \le k\}, \end{cases}$$

possesses a unique solution $s(x)$ in $K_r^{\perp}$ if $\mathfrak{n}(L) \cap K^{\perp} = \{0\}$ (cf. Anselone and Laurent [2] and Jerome and Schumaker [14 here $K^{\perp}$ denotes those $g \in W_2^n[0,1]$ for which $\lambda_i(g) = 0$, $1 \le i \le k$, and $\mathfrak{n}(L)$ denotes the null-space of L. Moreove the collection of all $s(x)$ which solve the minimization prob for some $r \in \mathbb{R}^k$ is a finite-dimensional subspace of $W_2^n[0,1$ and is denoted by $Sp(L,\Lambda)$. Given any $f \in W_2^n[0,1]$, the unique element s in $Sp(L,\Lambda)$ which solves the minimizatio problem of (2.2) with $\lambda_i(s) = \lambda_i(f)$ for all $1 \le i \le n$, will be called the <u>$Sp(L,\Lambda)$-interpolate</u> of f.

In most applications, the elements λ_i of Λ are usual chosen to be point evaluations of the function or its derivativ through order $n-1$, i.e., $\lambda_i(f) = D^{j_i}f(x_i)$ where $0 \le j_i \le n-$ and $x_i \in [0,1]$. Satisfactory error bounds for $f - s$, where is the $Sp(L,\Lambda)$-interpolate of f, have been obtained for su Λ (cf. Ahlberg, Nilson, and Walsh [1], and Jerome and Varg [15]). But, as the derivations of the error estimates are bas either on Rolle's Theorem or Rayleigh-Ritz inequalities, these known error bounds can be extended to more general Λ. Thi brings us to

<u>Definition 1</u>. Consider the collection $\{\Lambda_i\}_{i=1}^{\infty}$ where each $\Lambda_i = \{\lambda_{j,i}\}_{j=1}^{k_i}$ is a set of k_i linearly independent bounded linear functionals on $W_2^n[0,1]$, $1 \le i \le \infty$. If $K^{\perp}(i)$ denot those $g \in W_2^n[0,1]$ for which $\lambda_{j,i}(g) = 0$ for all $1 \le j \le k_i$ assume that $\mathfrak{n}(L) \cap K^{\perp}(i) = \{0\}$ for each $i \ge 1$. For any $f \in W_2^n[0,1]$, let $s_i(x)$ denote the unique $Sp(L,\Lambda_i)$-interpo late of f. Then, $\{\Lambda_i\}_{i=1}^{\infty}$ satisfies <u>Property $\mathfrak{R}$</u> with respect

to $W_2^n[0,1]$ if, for each $f \in W_2^n[0,1]$, there exist distinct points $\xi_j(i)$ with $0 \le \xi_1(i) < \xi_2(i) < \ldots < \xi_{m_i}(i) \le 1$ with $m_i \ge i$, such that

$$(2.3) \qquad f(\xi_j(i)) = s_i(\xi_j(i)) \quad \text{for all } 1 \le j \le m_i \quad \text{for all } i \ge 1 ,$$

and, defining $\xi_0(i) = 0$ and $\xi_{m_i+1}(i) = 1$, there exists for each $i \ge 1$ a quantity $\bar{\Delta}_i$, independent of f, such that $\sup |\xi_{j+1}(i) - \xi_j(i)| \le \bar{\Delta}_i$ for all $i \ge 1$, and $0 \le j \le m_i$

$$(2.4) \qquad \lim_{i \to \infty} \bar{\Delta}_i = 0 .$$

With this definition, we then prove

<u>Theorem 2.1.</u> Let $\{\Lambda_i\}_{i=1}^{\infty}$ be a collection satisfying Property $\mathfrak{R}$ with respect to $W_2^n[0,1]$. Then, for any fixed $f \in W_2^n[0,1]$, there exist constants K and K', independent of i, and a positive integer i_0, such that

$$(2.5) \qquad \|D^j(f-s_i)\|_{L_\infty[0,1]} \le K(\bar{\Delta}_i)^{n-j-1/2}\|Lf\|_{L_2[0,1]} \quad \text{for all } 0 \le j \le n-1, \quad i \ge i_0 ,$$

and

$$(2.6) \qquad \|D^j(f-s_i)\|_{L_2[0,1]} \le K'(\bar{\Delta}_i)^{n-j}\|Lf\|_{L_2[0,1]} \quad \text{for all } 0 \le j \le n, \quad i \ge i_0 .$$

Similarly, if $f \in W_2^{2n}[0,1]$ and for each i, the second integral relation is valid, i.e.,

$$(2.7) \qquad \int_0^1 [L(f-s_i)]^2\,dx = \int_0^2 (f-s_i)L^* Lf\,dx, \quad i \geq 1,$$

then

$$(2.8) \quad \|D^j(f-s_i)\|_{L_\infty[0,1]} \leq K(\bar{\Delta}_i)^{2n-j-1/2}\|L^* Lf\|_{L_2[0,1]},$$

$$0 \leq j \leq n-1, \quad i \geq i_0,$$

and

$$(2.9) \quad \|D^j(f-s_i)\|_{L_2[0,1]} \leq K'(\bar{\Delta}_i)^{2n-j}\|L^* Lf\|_{L_2[0,1]},$$

$$0 \leq j \leq n, \quad i \geq i_0.$$

Proof. Since $\lim_{i\to\infty} \bar{\Delta}_i = 0$, it follows that there exists an in- teger i_1 such that for $i \geq i_1$, s_i interpolates f in the sens of (2.3) in at least $n+1$ distinct points of $[0,1]$. Using Rolle's theorem, the proof of Theorem 6 of [22] can be direct applied, giving

$$(2.10) \quad \|D^j(f-s_i)\|_{L_\infty[0,1]} \leq \frac{n!\,(\bar{\Delta}_i)^{n-j-1/2}}{\sqrt{n}\; j!}\|D^n(f-s_i)\|_{L_2[0,$$

$$0 \leq j \leq n-1, \quad i \geq i_1$$

Again because $\lim_{i\to\infty} \bar{\Delta}_i = 0$, there also exists an integer i_2 such that

$$(2.11) \quad \|D^n(f-s_i)\|_{L_2[0,1]} \leq H\|L(f-s_i)\|_{L_2[0,1]} \quad \text{for all } i \geq i$$

where H is a positive constant (cf. [22]). Next, for each $i \geq 1$, the $Sp(L, \Lambda_i)$-interpolate s_i of f satisfies $\int_0^1 Ls_i \cdot L(f-s_i)\,dx$ (cf. [2]), which yields the <u>first integral relation</u>:

$$(2.12) \quad \|Lf\|^2_{L_2[0,1]} = \|L(f-s_i)\|^2_{L_2[0,1]} + \|Ls_i\|^2_{L_2[0,1]}, \quad i \geq 1 .$$

Hence, we have from (2.12) that $\|L(f-s_i)\|_{L_2[0,1]} \leq \|Lf\|_{L_2[0,1]}$ for all $i \geq 1$. Thus, combining (2.10) and (2.11) gives the desired result of (2.5) for $i \geq \max(i_1, i_2)$. Similarly, the proof of (2.6) follows that of [22, Theorem 7], and uses Rayleigh-Ritz inequalities instead of Rolle's theorem. The inequalities of (2.7) and (2.8) are established as in the manner of Theorems 8 and 9 of [22].

As an application of the above result, suppose that $L = D^n$, and for each $i \geq n$, define $\Lambda_i = \{\lambda_{j,i}\}_{i=1}^{i}$ by means of the functionals

$$(2.13) \qquad \lambda_{j,i}(f) = \int_{(j-1)/i}^{j/i} f(t)\,dt, \quad i \leq j \leq i .$$

Clearly, the functionals $\{\lambda_{j,i}\}_{j=1}^{i}$ are linearly independent bounded linear functionals on $W_2^n[0,1]$ for all $i \geq n$. Next, $\eta(D^n)$ consists of all polynomials of degree at most $n-1$, and $K^{\perp}(i)$ consists of all function $g \in W_2^n[0,1]$ such that

$$(2.14) \qquad \int_{(j-1)/i}^{j/i} g(t)\,dt = 0, \quad 1 \leq j \leq i .$$

For each $i \geq n$, it is readily verified that $\eta(D^n) \cap K^{\perp}(i) = \{0\}$, and thus, for each $i \geq n$, there exists a unique $Sp(D^n, \Lambda_i)$-interpolate, s_i, of $f \in W_2^n[0,1]$. For $i \geq n$, it is clear that if $f(x) \not\equiv s_i(x)$ in $[\frac{j-1}{i}, \frac{j}{i}]$, then $f(x) - s_i(x)$ must change signs at least once in $[\frac{j-1}{i}, \frac{j}{i}]$; otherwise $\lambda_{j,i}(f-s_i) \neq 0$. Hence, there exist a point $\xi_j(i)$ in $(\frac{j-1}{i}, \frac{j}{i})$ such that $f(\xi_j(i)) = s_i(\xi_j(i))$ for each $1 \leq j \leq i$, $i \geq n$. While these points $\xi_j(i)$ in general depend on f, it does follow that

$$\sup_{0 \leq j \leq i+1} |\xi_{j+1}(i) - \xi_j(i)| \leq \frac{2}{i} \equiv \bar{\Delta}_i$$

for each $f \in W_2^n[0,1]$, and hence $\lim_{i\to\infty} \bar{\Delta}_i = 0$. Consequent $\{\Lambda_i\}_{i=n}^{\infty}$ as defined by (2.13) satisfies property $\mathfrak{R}$, and the error bounds of Theorem 1 are applicable.

Of course, the usual Lg-splines as considered in [14] and [15], as well as the L-splines of [22], are formulated in terms of functionals $\lambda_{i,j}$ which are point evaluations of func tions or their derivatives through order $n-1$. Hence $\{\Lambda_i\}_{i=}^{n}$ for either Lg-splines or L-splines will automatically satisfy property $\mathfrak{R}$, if the partitions π_i of $[0,1]$ defined by these point functionals, are such that $\lim_{i\to\infty} \bar{\pi}_i = 0$. In this sense, Theorem 2.1 generalizes the previously known error bounds fo Lg-splines and L-splines.

§3. Besov Spaces. Once one has the error bounds as in The 2.1, one can extend their usefulness via results from the the of interpolation spaces. The purpose of this section is to br show in a specialized way how this can be done. More deta results of this nature, as well as considerations of errors of interpolation and best approximation in higher dimensional s tings, are to be found in Hedstrom and Varga [12].

Let X_0 and X_1 be two Banach spaces with norms $\|\cdot$ and $\|\cdot\|_1$, respectively, which are contained in a linear Hausdorff space $\mathfrak{X}$, such that the identity mapping of X_i, $i = 0, 1$, in $\mathfrak{X}$ is continuous. If $X_0 + X_1 \equiv \{f \in \mathfrak{X}: f = f_0 + f_1$ where $f_i \in X_i$, $i = 0, 1\}$, then it is known (cf. Butzer and Berens [6, p. 165]) that $X_0 + X_1$ and $X_0 \cap X_1$ are Banach spaces under the norms

$$\|f\|_{X_0 \cap X_1} = \max\{\|f\|_0, \|f\|_1\} ,$$

$$\|f\|_{X_0 + X_1} = \inf\{\|f_0\|_0 + \|f_1\|_1 : f = f_0 + f_1 \text{ with } f_i \in X_i, i = 0, 1\} .$$

It is understood that the above infimum is taken over all such decompositions, $f = f_0 + f_1$ with $f_i \in X_i$, $i = 0, 1$. Moreover, it follows that

$$X_0 \cap X_1 \subset X_i \subset X_0 + X_1 \subset \mathfrak{X}, \quad i = 0, 1 , \tag{3.1}$$

where the inclusion $A \subset B$ is understood here, and in the rest of this section, to mean that the identity mapping from A into B is continuous. A Banach space $X \subset \mathfrak{X}$ is an <u>intermediate space</u> of X_0 and X_1 if it satisfies the inclusion

$$X_0 \cap X_1 \subset X \subset X_0 + X_1 \subset \mathfrak{X} . \tag{3.2}$$

Peetre (cf. [6] and [21]) has given a real-variable method for constructing intermediate spaces of X_0 and X_1, which we now describe. For each positive t and each $f \in (X_0 + X_1)$, define

$$K(t, f) = \inf\{\|f_0\|_0 + t\|f_1\|_1 : f = f_0 + f_1 \text{ with } f_i \in X_i, \ i = 0, 1\} .$$

Then, for any θ with $0 < \theta < 1$ and any extended real number q with $1 \leq q \leq +\infty$, let $(X_0, X_1)_{\theta, q}$ be the set of all elements $f \in (X_0 + X_1)$ for which the following norm is finite:

$$\|f\|_{(X_0, X_1)_{\theta, q}} \equiv \begin{cases} \left[\int_0^\infty (t^{-\theta} K(t, f))^q \frac{dt}{t}\right]^{1/q}, & 1 \leq q < +\infty , \\ \sup\limits_{t > 0} t^{-\theta} K(t, f) , & q = +\infty . \end{cases}$$

It is known [6, p. 168] that $(X_0, X_1)_{\theta, q}$ is an intermediate space of X_0 and X_1, and thus satisfies the inclusions of (3.2). In particular, $(X, X)_{\theta,q} = X$.

If Y_0 and Y_1 are two Banach spaces continuously con-tained (with respect to the identity mapping) in the linear Hausdorff space $\mathcal{Y}$, let T denote any linear transformation from $(X_0 + X_1)$ to $(Y_0 + Y_1)$ for which

$$\|Tf\|_i \le M_i \|f\|_i \quad \text{for all} \quad f \in X_i, \quad i = 0, 1 ,$$

i.e., T is a bounded linear transformation from X_i to Y_i with norm at most M_i, $i = 0, 1$. Then, the following is know (cf. [6, p. 180]).

Theorem 3.1. For $0 < \theta < 1$, $1 \le q \le +\infty$, T is a bounded linear transformation from the intermediate space $(X_0, X_1)_{\theta, q}$ whose norm $M \equiv \sup \frac{\|Tf\|_{(Y_0, Y_1)_{\theta,q}}}{\|f\|_{(X_0,X_1)_{\theta,q}}}$ satisfies

$$(3.3) \qquad M \le M_0^{1-\theta} \cdot M_1^{\theta} .$$

With the previous notation, then the Besov space $B_p^{\sigma, q}$ is defined as the intermediate space (cf. [3])

$$(3.4) \quad (L_p[0,1], W_p^m[0,1])_{\theta,q} = B_p^{\sigma,q}[0,1] \text{ where } 0 < \sigma = \theta m <$$

here, $1 \le p, q \le +\infty$. It is further known (cf. [6]) for $0 < \theta <$ that

$$(3.5) \quad (L_{p_0}[0,1], L_{p_1}[0,1])_{\theta,q} = L_p[0,1] \text{ where } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p}$$

and if $\sigma_0 \ne \sigma_1$, $0 < \theta < 1$, $1 < q_0, q_1 \le +\infty$, and $1 \le p \le \infty$, th

$$(3.6) \quad (B_p^{\sigma_0, q_0}[0,1], B_p^{\sigma_1, q_1}[0,1])_{\theta,q} = B_p^{\sigma,q} \text{ where } \sigma = \theta\sigma_0 + ($$

and for integer σ_i, either of the spaces $B_p^{\sigma_i, q_i}[0,1]$ in (3.6) can be replaced by $W_p^{\sigma_i}[0,1]$.

We now apply these results to error bounds of Theorem 2.1. Choosing $X_0 = X_1 = \mathfrak{X} = W_2^n[0,1]$, and $Y_0 = L_\infty[0,1] = \mathfrak{Y}$, $Y_1 = L_2[0,1]$, let the linear mapping T on $W_2^n[0,1]$ be defined by

$$(3.7) \qquad Tf = D^j(f - s_i) \quad ,$$

for some fixed j, $0 \le j \le n-1$, where s_i is the $Sp(L, \Lambda_i)$-interpolate of f. Because of previous assumptions on the differential operator L of (2.1), we can write (2.5) and (2.6) of the previous section as

$$\|Tf\|_{L_\infty[0,1]} \le K(\bar{\Delta}_i)^{n-j-1/2} \|f\|_{W_2^n[0,1]} \quad ,$$

and

$$\|Tf\|_{L_2[0,1]} \le K'(\bar{\Delta}_i)^{n-j} \|f\|_{W_2^n[0,1]} \quad .$$

Now, using the result of (3.3) with those of (3.5) for $p_0 = +\infty$, $p_1 = 2$, we obtain

$$(3.8) \quad \|D^j(f - s_i)\|_{L_q[0,1]} \le K'(\bar{\Delta}_i)^{n-j-\frac{1}{2}+\frac{1}{q}} \|f\|_{W_2^n[0,1]} \quad ,$$

$$0 \le j \le n-1, \quad 2 \le q \le +\infty \quad .$$

Similarly, if the error bounds of (2.8) and (2.9), depending on the second integral relation, are valid, then interpolation similarly gives

$$(3.9) \quad \|D^j(f-s_i)\|_{L_q[0,1]} \le K(\bar{\Delta}_i)^{2n-j-\frac{1}{2}+\frac{1}{q}} \|f\|_{W_2^{2n}[0,1]} \quad ,$$

$$0 \le j \le n-1, \quad 2 \le q \le +\infty \quad .$$

Next, from (3.6), we have that $(W_2^n[0,1], W_2^{2n}[0,1])_{\theta,\tau} = B_2^{\sigma,\tau}[0,1]$, where $n < \sigma = 2n - \theta n < 2n$. Thus, interpolating the results of (3.8) and (3.9) gives us

Theorem 3.2. Assuming the error bounds (2.5)-(2.9) of Theorem 2.1, let $f \in B_2^{\sigma,\tau}[0,1]$ where $n < \sigma < 2n$ and $1 \leq \tau \leq +\infty$, and let its $Sp(L,\Lambda_i)$-interpolate be s_i. Then, ther exists a constant K, independent of i, such that

$$(3.10)\quad \|D^j(f-s)\|_{L_q[0,1]} \leq K(\overline{\Delta}_i)^{\sigma-j-\frac{1}{2}+\frac{1}{q}}\|f\|_{B_2^{\sigma,\tau}[0,1]},$$

$$0 \leq j \leq n-1, \quad 2 \leq q \leq +\infty .$$

The importance of the error bounds of (3.10) lies in the fact that we now have new error bounds for functions f which are elements of $W_2^n[0,1]$, but not of $W_2^{2n}[0,1]$. In additio since the exponent of $\overline{\Delta}_i$ in (3.10) doesn't depend on τ, an since

$$(3.11)\qquad B_2^{m,1}[0,1] \subset W_2^m[0,1] \subset B_2^{m,\infty}[0,1],$$

we also have error bounds for spaces intermediate to $W_2^n[0,1]$ which can be larger than intermediate Sobolev spaces. To il-lustrate this, suppose $n = 2$ in the above discussion. Then, $W_2^4[0,1] \subset W_2^3[0,1] \subset W_2^2[0,1]$, and $W_2^3[0,1] \subset B_2^{3,\infty}[0,1] \subset W_2^2[0,1]$. This means that the error bound of (3.10), with exponent of $\overline{\Delta}_i$ equal to $3-j-\frac{1}{2}+\frac{1}{q}$, is valid not only for $W_2^3[0,1]$, but for $B_2^{3,\infty}[0,1]$ as well. Further results, simi to Theorem 3.2, can be found in [12].

§4. Discrete Variational Green's Functions. As an applicati of the splines introduced in §2, we consider the boundary val problem

$$(4.1)\qquad -L^*L[u(x)] = f(x), \quad 0 < x < 1,$$

subject to the boundary conditions of

$$(4.2) \qquad D^j u(0) = D^j u(1) = 0, \quad 0 \le j \le n-1 ,$$

where L is the n-th order differential operator of (2.1), and L^* is its formal adjoint, i.e.,

$$(4.3) \qquad L^*[v(x)] \equiv \sum_{j=0}^{n} (-1)^j D^j \{a_j(x) v(x)\} .$$

With the assumptions on L of §2, it is well known that there exists a Green's function $G(x,\xi)$ associated with the boundary value problem (4.1)-(4.2), defined on $[0,1] \times [0,1]$, such that

$$(4.4) \quad \begin{cases} G(x,\xi) = G(\xi,x) & \text{in the closed unit square,} \\ G(x,\xi) \in C^{2n-2} & \text{in the closed unit square,} \\ G(x,\xi) \in C^{2n} & \text{in the subsets } 0 \le x < \xi \le 1 \text{ and} \\ & 0 \le \xi < x \le 1 \text{ of } [0,1] \times [0,1] . \end{cases}$$

Moreover, for any $0 \le \xi \le 1$, let $G_\xi(x)$ denote the function defined on $[0,1]$ by $G_\xi(x) = G(x,\xi)$. Then, it is also well known that

$$(4.5) \quad \begin{cases} \text{i) } D^{2n-1} G_\xi \in L_\infty[0,1] \text{ and } \sup\{\|D^{2n-1}G_\xi\|_{L_\infty[0,1]} : \\ \qquad\qquad 0 \le \xi \le 1\} = \sigma < +\infty . \\ \text{ii) } D^k G_\xi(0) = D^k G_\xi(1) = 0 \text{ for all } 0 \le k \le n-1 . \\ \text{iii) } \lim_{x \to \xi+} D^{2n-1} G_\xi(x) - \lim_{x \to \xi-} D^{2n-1} G_\xi(x) = \dfrac{(-1)^n}{a_n^2(\xi)}, \text{ and} \\ \text{iv) } L^* L G_\xi(x) = 0 \text{ for all } x \ne \xi \text{ in } [0,1] . \end{cases}$$

Now, given any $f \in C^0[0,1]$, it is classic that the uniq solution ϕ of (4.1)-(4.2) can be expressed in terms of the Green's function $G(x,\xi)$ by

$$(4.6) \qquad \phi(x) = \int_0^1 G(x,\xi)\, f(\xi)\, d\xi \ .$$

In addition, if $W^n_{2,0}[0,1]$ denotes the subspace of $W^n_2[0,1]$ which satisfies the boundary conditions (4.2), then ϕ can b characterized as the unique function in $W^n_{2,0}[0,1]$ which minimizes the functional

$$(4.7) \quad F[v] = \int_0^1 \{\sum_{j=0}^{n} p_j(t)\,(D^j v(t))^2 + 2f(t)v(t)\}dt, \quad v \in W^n_{2,0}[$$

Using the Ritz-Galerkin approach, the minimization of F ove some finite-dimensional subspace S^M of $W^n_{2,0}[0,1]$ produc a unique ϕ^M in S^M which can, in analogy with (4.6), be described by

$$(4.8) \qquad \phi^M(x) = \int_0^1 G^M(x,\xi) f(\xi)\, d\xi \ .$$

Appropriately, the function $G^M(x,\xi)$ defined on $[0,1]\times[0,1$ is called the discrete variational Green's function (cf. Ciarle [7]) for the problem (4.1) - (4.2) and the subspace S^M.

Our purpose in this section is to show how the error bou for spline interpolation can be applied to the problem of estim the error in $G(x,\xi) - G^M(x,\xi)$, when S^M is a special spli subspace of $W^n_{2,0}[0,1]$.

To make matters precise, we consider the special case L-splines of Schultz and Varga [22]. If $\pi: 0 = x_0 < x_1 < x_2 < \ldots$ $< x_{N+1} = 1$ is a partition of $[0,1]$, and $\underline{Z} = (z_0, z_1, \ldots, z_{N+1}$ is an incidence vector with positive integer components satisf ing $1 \le z_i \le n$ for all $0 \le i \le N+1$, then the spline space $Sp(L, \pi, \underline{Z})$, a subspace of $W^n_2[0,1]$, is simply the particul case of $Sp(L,\Lambda)$ treated in §2 where the elements λ of Λ are all of the form

$$\lambda_{i,j}(f) = D^j f(x_i) \quad \text{for all} \quad 0 \le j \le z_i - 1, \quad 0 \le i \le N+1 .$$

Choosing $Z_0 = Z_{N+1} = n$ for convenience, the second integral relation needed in (2.7) is valid, as are the other assumptions of Theorem 2.1, and thus, we may use the error bounds of (2.5)-(2.9), with $\overline{\Delta} \equiv \max_{0 \le i \le N} |x_{i+1} - x_i|$. We further denote by $Sp_0(L, \pi, \underline{Z})$ the subspace of $Sp(L, \pi, \underline{Z})$ which satisfies the boundary conditions of (4.2). The following is a special case of Ciarlet and Varga [8].

<u>Theorem 4.1</u>. Let $G^M(x, \xi)$ be the discrete variational Green's function associated with the L-spline subspace $Sp_0(L, \pi, \underline{Z})$. Then, there exist positive constants K and K', independent of $\overline{\Delta}$, such that for all $0 \le \xi \le 1$,

$$(4.8) \qquad \|D^k(G_\xi^M - G_\xi)\|_{L_\infty[0,1]} \le K(\overline{\Delta})^{2n-k-3/2} \quad \text{for all } 0 \le k \le n-1$$

and

$$(4.9) \qquad \|D^k(G_\xi^M - G_\xi)\|_{L_2[0,1]} \le K'(\overline{\Delta})^{2n-k-1} \quad \text{for all } 0 \le k \le n .$$

In addition, if $G^M(x, \xi)$ is the discrete variational Green's function associated with the Hermite L-spline subspace $Sp_0(L, \pi, \hat{\underline{Z}})$ where $\hat{Z} \equiv (n, n, \ldots, n)$, then there exist positive constants K'', independent of $\overline{\Delta}$, such that for all $0 \le \xi \le 1$,

$$(4.10) \qquad \|D^k(G_\xi^M - G_\xi)\|_{L_\infty[0,1]} \le K''(\overline{\Delta})^{2n-k-1} \quad \text{for all } 0 \le k \le n-1 .$$

<u>Proof</u>. In [8], it is shown that $G_\xi^M(x)$, the discrete variational Green's function associated with $Sp_0(L, \pi, \underline{Z})$, is in fact the $Sp_0(L, \pi, \underline{Z})$-interpolate of $G_\xi(x)$ for each $0 \le \xi \le 1$. Since $D^{2n-1}G_\xi \in L_\infty[0,1]$ by (4.5i), then $G_\xi \in W_\infty^{2n-1}[0,1] \subset W_2^{2n-1}[0,1] \subset B_2^{2n-1,\infty}[0,1]$, using (3.11). Hence, the error

bounds of (4.8) and (4.9) follow as special cases of (3.10) with $q = \infty$ or $q = 2$, $\sigma = 2n-1$, and $\tau = +\infty$. Similarly, the assumption of Hermite L-spline subspaces $Sp_0(L, \pi, \hat{\underline{Z}})$ allow one, as in Birkhoff, Schultz, and Varga [4], to increase the exponent of $\bar{\Delta}$ in (4.8) by 1/2, which gives (4.10).

We further remark that other results, such as the positivity of the discrete variational Green's function $G^M(x, \xi)$, are also considered in [8].

§5. Improved Error Bounds for Ordinary Differential Equations
Applications of spline functions have not been made only to two-point boundary value problems and elliptic partial differential equations. Indeed, spline functions of maximum smoothness were first considered in the numerical solution of ordinary differential equations by Loscalzo and Talbot [18] and [19], and many interesting connections with standard numerical integration techniques have been proved. For example, the trapezoidal rule and the Milne-Simpson predictor-corrector method fall out as special cases of such spline functions applications. Unfortunately, higher-order smooth spline approximations turn out to be unstable, and consequently, the practical use of smooth spline functions to the numerical integration of ordinary differential equations is restricted to cases for which the resulting method turns out to be classical. For details of the above remarks, we recommend the article by F. R. Loscalzo [16] in Theory and Applications of Spline Functions, edited by T. N. E. Greville.

The main reason why the above-mentioned applications of spline functions to the numerical integration of ordinary differential equations lead to unstable methods is because the resulting numerical approximations are, in a certain sense, too smooth. Loscalzo and Schoenberg [16] and [20] have shown that the use of Hermite-splines of lower-order smoothness, to be described below, avoids completely the problem of instability. In [16] and [17], error bounds for the approximate Hermite spline solutions were obtained, but, as we shall show below in Theorem 5.1, the error bounds derived were not in all cases best possible. In keeping with the title of this paper, the basic purpose in this section is to obtain improved error bounds for such Hermite-spline applications.

Although applicable to systems of ordinary differential equations, we consider, for simplicity, the initial value problem

$$(5.1)\qquad Dy(x) = f(x, y(x)), \quad y(0) = y_0 ,$$

where y_0 is specified. We assume that f is continuous on $[0,b] \times \mathbb{R}$, subject to the usual Lipschitz condition

$$(5.2)\qquad |f(x,z) - f(x,\zeta)| \le L|z-\zeta| \quad \text{for all} \quad x \in [0,b], \quad \text{for all} \quad z, \zeta \in \mathbb{R},$$

where $L > 0$. This assures existence and uniqueness of a solution $y(x)$ of (5.1). For a positive integer n, let $h \equiv b/n$, and let $x_j \equiv jh$, $0 \le j \le n$, be the associated knots in $[0,b]$.

To explain the Hermite spline method, consider the $2q+2$ numbers

$$(5.3)\qquad D^j t(0), \quad D^j t(h), \quad 0 \le j \le q ,$$

where $t(x) \in C^q[0,h]$. It is clear that, by means of Hermite interpolation, there is a unique polynomial $s(x)$ of degree at most $2q+1$, written $s \in \pi_{2q+1}$, such that

$$(5.4)\qquad D^j t(0) = D^j s(0), \quad D^j t(h) = D^j s(h), \quad 0 \le j \le q .$$

If, however, we <u>insist</u> that $s \in \pi_{2q}$, i.e., s is of degree at most $2q$, then the following $2q+2$-nd divided difference of $s(x)$ must necessarily vanish (cf. [17, Lemma 3.1]):

$$(5.5)\qquad H_q(s;0;h) \equiv (-1)^q \frac{(q!)^2}{(2q)!} h^{2q+1} s(\overbrace{0,0,\ldots,0}^{q+1}, \overbrace{h,h,\ldots,h}^{q+1})$$

This divided difference can also be expressed as the sum

$$(5.6)\qquad H_q(s;0;h) = -\sum_{k=0}^{q} C_{k,q} \{D^k t(0) + (-1)^{k+1} D^k t(h)\} ,$$

where

$$(5.7) \qquad C_{k,q} \equiv \frac{1}{k!} \frac{q!\,(2q-k)!}{(q-k)!\,(2q)!}, \qquad 0 \le k \le q \ .$$

Later, we shall show how (5.6) and (5.7) connect the Hermite spline method with Padé approximations.

The algorithm for determining the Hermite spline function $p(x)$ which approximates the solution $y(x)$ of (5.1) can be described as follows. Given the functions

$$(5.8) \qquad g_{\ell-1}(x, u(x)) \equiv D_x^{\ell-1} f(x, u(x)), \quad 1 \le \ell \le q \ ,$$

where it is assumed for simplicity that $f \in C^{2q+1}([0,b] \times \mathbb{R})$, and $u(x)$ is any element in $C^q[0,b]$, define $p_1(x)$ in $[0,h]$ as the polynomial of degree at most $2q$ such that

$$(5.9) \qquad \begin{cases} D^k p_1(0) = g_k(0, p_1(0)), & 1 \le k \le q \ , \\ D^k p_1(h) = g_k(h, p_1(h)), & 1 \le k \le q \ , \\ p_1(0) = y_0, \quad H_q(p_1; 0; h) = 0 \ . \end{cases}$$

Using (5.6), it follows from (5.9) that $p(h)$ necessarily satisfies

$$(5.10) \qquad \begin{aligned} p_1(h) &= y_0 + \frac{h}{2}\{g_1(0, y_0) + g_1(h, p_1(h))\} \\ &\quad + \ldots + C_{qq} h^q \{g_q(0, y_0) + (-1)^{q+1} g_q(h, p_1(h))\} \end{aligned}$$

and the method of successive substitutions suggests itself for the determination of $p_1(h)$. By means of the contraction mapping theorem, it can be shown (cf. [17]) that there is an $h_0 > 0$

such that for all $0 < h \leq h_0$, there is a unique solution $p_1(h)$ of (5.10).

Now, having found $p_1(x)$ in $[0,h]$, determine similarly $p_2(x) \in \pi_{2q}$ in $[h, 2h]$ such that

$$\text{(5.11)} \quad \begin{cases} D^k p_2(h) = g_k(h, p_1(h)), & 1 \leq k \leq q, \\ D^k p_2(2h) = g_k(h, p_2(2h)), & 1 \leq k \leq q, \\ p_2(h) = p_1(h);\ H_q(p_2;0;h) = 0 . \end{cases}$$

In this way, the Hermite spline function $p(x)$, with $p(x) \equiv p_j(h)$ in $[(j-1)h, jh]$, $1 \leq j \leq [b/h]$, is a polynomial of degree at most $2q$ on each subinterval, and, by construction, $p(x) \in C^q[0,b]$. We remark that the solution of (5.10), (5.11), etc., produces only the numbers $p(jh)$, $1 \leq j < [b/h]$, and thus, the Hermite spline method can be viewed simultaneously as a single-step method (cf. [13, p. 209]). Once $p(h)$, say, is determined, the values $D^k p(h)$, $1 \leq k \leq q$, are obtained by evaluating $g_k(h, p(h))$ from (5.9), and $p(x)$ in $[0,h]$ can then be found by Hermite interpolation.

To appraise the errors in the Hermite spline method, the results of Loscalzo [17], utilizing the previous hypotheses, give us that for all $0 < h \leq h_1$, there exists a constant K_1, independent of h, such that

$$\text{(5.12)} \quad |D^k y(jh) - D^k p(jh)| \leq K_1 h^{2q}, \quad 0 \leq k \leq q, \quad 0 \leq j \leq [b/h],$$

where $y(x)$, of class $C^{2q+2}[0,b]$, is the solution of (5.1), and $p(x)$ is its Hermite spline approximation. Next, let $w(x)$ be the Hermite-interpolation of $y(x)$, i.e.,

$$\text{(5.13)} \quad D^k w(jh) = D^k y(jh), \quad 0 \leq k \leq q, \quad 0 \leq j \leq [b/h],$$

and $w(x) \in \pi_{2q+1}$ on each subinterval $[(j-1)h, jh]$, $1 \leq j \leq [b/h]$

Because $y \in C^{2q+2}[0,b]$, the interpolation error for $w(x)$ satisfies (cf. [4, Theorem 2])

$$(5.14) \qquad \|D^k(y-w)\|_{L_\infty[0,b]} \le K_2 h^{2q+2-k}, \quad 0 \le k \le q ,$$

where K_2 is independent of h. Because of (5.13), it follow that

$$(5.15) \qquad |D^k w(jh) - D^k p(jh)| \le K_1 h^{2q}, \; 0 \le k \le q, \; 0 \le j \le [b/h] .$$

To complete the picture, we now state a known result, a minor extension of a result of Swartz [23, Lemma 2] (Swartz's in- equalities):

Lemma. If there exists a constant K, independent of h, and an integer α such that

$$(5.16) \qquad \max\{|D^k s(0)|, \; |D^k s(h)|\} \le Kh^{\alpha-k} \text{ for all } 0 \le k \le q ,$$

where $s \in \pi_{2q+1}$, then there exists a K', independent of h, such that

$$(5.17) \qquad \|D^k s\|_{L_\infty[0,h]} \le K' h^{\alpha-k}, \qquad 0 \le k \le q .$$

To apply this Lemma, we see from (5.15) that $w(x) - p(x)$, an element of π_{2q+1} on each subinterval $[(j-1)h, jh]$, satisfies the inequalities of (5.16) with $\alpha = 2q$. Hence, applying (5.17) on each subinterval $[(j-1)h, jh]$ of $[0,b]$ gives

$$(5.18) \qquad \|D^k(w-p)\|_{L_\infty[0,b]} \le K' h^{2q-k}, \quad 0 \le k \le q ,$$

where K' is independent of h. Then, applying the triangle inequality to (5.14) and (5.18) gives us

Theorem 5.1. Assuming $f(x,y) \in C^{2q+1}([0,b] \times \mathbb{R})$, let $y(x)$ be the unique solution of (5.1) in $[0,b]$. Then, there exist constants $K > 0$ and $0 < h_0 \leq b$ such that for all $0 < h \leq h_0$, the Hermite spline approximation $p(x)$ of $y(x)$, defined in (5.9), satisfies

$$\|D^k(y-p)\|_{L_\infty[0,b]} \leq K\, h^{2q-k}, \quad 0 \leq k \leq q \; . \tag{5.19}$$

We remark that the special case $k = 0$ of (5.19) was obtained earlier by Loscalzo [17], but his error bounds for the higher derivatives of $p(x)$ were weaker than those of (5.19).

To conclude our discussion of the Hermite spline method for ordinary differential equations, we first remark that Loscalzo [17] showed that the Hermite spline method (of order q) is A-stable in the sense of Dahlquist for any q, [9], [10] for any q, i.e., if this method is applied to the particular ordinary differential equation

$$Dy(x) = \lambda y(x), \quad y(0) = 1, \quad \mathrm{Re}\lambda < 0 \;, \tag{5.20}$$

then its approximation $p(x)$ satisfies

$$\lim_{n\to\infty} p(nh) = 0 \quad \text{for } \underline{\text{any}} \;\; h > 0 \; . \tag{5.21}$$

It is easy to verify from (5.6) that the Hermite spline method of order q applied to (5.20) gives $p((n+1)h) = \zeta_h\, p(nh)$, where $p(0) = 1$, and

$$\zeta_h \equiv \left(\sum_{k=0}^{q} C_{q,k}\, \lambda^k\, h^k \right) \left(\sum_{k=0}^{q} (-1)^k\, C_{q,k}\, \lambda^k\, h^k \right)^{-1} . \tag{5.22}$$

However, from the definition of $C_{q,k}$ in (5.7), it turns out that ζ_h is just the q-th diagonal Padé rational approximation of $e^{\lambda h}$ (cf. [24, p. 269]). Consequently, since $\mathrm{Re}\,\lambda < 0$, then $|\zeta_h| < 1$ (cf. [5]). In this regard, it has been more generally shown in the thesis by Ehle [11] that the diagonal and the first two subdiagonals of the Padé table of the exponential function give rise to such A-stable approximations of (5.20), but the connection between Padé approximations and Hermite spline methods for ordinary differential equations appears to be new.

REFERENCES

1. J. H. Ahlberg, E. N. Nilson, and J. L. Walsh, The Theory of Splines and their Applications, Academic Press, New York, 1967.

2. P. M. Anselone and P. J. Laurent, "A general method for the construction of interpolating or smoothing spline-functions", Numer. Math. 12 (1968): 66-82.

3. O. V. Besov, "On some families of functional spaces. Imbedding and extension theorems", (in Russian), Dokl. Akad. Nauk SSSR 126 (1959): 1163-1165.

4. G. Birkhoff, M. H. Schultz, and R. S. Varga, "Piecewise Hermite interpolation in one and two variables with applications to partial differential equations", Numer. Math 11 (1968): 232-256.

5. G. Birkhoff and R. S. Varga, "Discretization errors for well-set Cauchy problems, I", J. Math. and Physics 44 (1965): 1-23.

6. P. I. Butzer and H. Berens, Semi-groups of Operators and Approximation, Springer-Verlag, New York, 1967.

7. P. G. Ciarlet, "Discrete variational Green's function I", Aequationes Mathematicae (to appear).

8. P. G. Ciarlet and R. S. Varga, "Discrete variational Green's functions. II. One dimensional problem", Numer. Math. (to appear)

9. G. Dahlquist, "A special stability problem for linear multistep methods", BIT 3 (1963): 27-43.

10. G. Dahlquist, "Stability questions for some numerical methods for ordinary differential equations", Proceedings of Symposia in Applied Mathematics, Amer. Math. Soc., Providence, Rhode Island, vol. 15 (1963), pp. 147-158.

11. B. L. Ehle, "On Padé approximations to the exponential function and A-stable methods for the numerical solution of initial value problems", (120 pp.). Doctoral Thesis, University of Waterloo 1969.

12. G. W. Hedstrom and R. S. Varga, "Applications of Besov spaces to spline approximation", J. Approx. Theory (to appear).

13. P. Henrici, Discrete Variable Methods in Ordinary Differential Equations, John Wiley and Sons, New York, 1962.

14. J. W. Jerome and L. L. Schumaker, "On Lg-splines", J. Approx. Theory 2 (1969): 29-49.

15. J. W. Jerome and R. S. Varga, "Generalizations of spline functions and applications to nonlinear boundary value and eigenvalue problems," Theory and Applications of Spline Functions (edited by T. N. E. Greville), Academic Press, New York, 1969: 103-155.

16. F. R. Loscalzo, "An introduction to the application of spline functions to initial value problems", Theory and Applications of Spline Functions, edited by T. N. E. Grevill Academic Press, New York, 1969, 37-64.

17. F. R. Loscalzo, "On the use of spline functions for the numerical solution of ordinary differential equations", Tech. Summary Report No. 869, Math. Research Center, University of Wisconsin, Madison, May 1968.

18. F. R. Loscalzo and T. D. Talbot, "Spline function approximations for solutions of ordinary differential equations", Bull. Amer. Math. Soc. 73 (1967): 438-442.

19. F. R. Loscalzo and T. D. Talbot, "Spline function approximations for solutions of ordinary differential equations", SIAM J. Numer. Anal. 4(1967): 433-445.

20. F. R. Loscalzo and I. J. Schoenberg, "On the use of spline functions for the approximation of solutions of ordinary differential equations", Tech. Summary Report No. 723, Math. Research Center, University of Wisconsin, Madison, January, 1967.

21. J. Peetre, "Espaces d'interpolation, généralisations, applications", Rend. Sem. Mat. Fis. Milano 34 (1964): 133-164.

22. M. H. Schultz and R. S. Varga, "L-splines", Numer. Math. 10 (1967): 345-369.

23. B. Swartz, "$O(h^{2n-2-\ell})$ bounds on some spline interpolation errors", Bull. Amer. Math. Soc. 74 (1968): 1072-1078.

24. R. S. Varga, Matrix Iterative Analysis, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1962.

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