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Reactor Response to Reactivity Changes During a Xenon Transient

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The response of a thermal reactor to a ramp change of reactivity is studied. When a single group of delayed neutrons is assumed, the differential equations for the time dependence of the thermal neutron flux takes the form of a linear homogeneous differential equation of the second order with linear coefficients. linearly independent solutions of the differential equation are found in the form of contour integrals. Moreover, expansions of these contour integrals into usable asymptotic developments are determined. Application of the mathematical results obtained is made to the problem of control rod calibration during a xenon transient.

INTRODUCTION

The response of a reactor to step changes in reactivity is discussed in almost every text on reactor technology (*1*, *2*). Usually this problem gives rise to a linear homogeneous ordinary differential equation with constant coefficients, and no special mathematical difficulties are encountered in its solution. However, when a *ramp* change of reactivity of the form $a + bt$, as opposed to a *step* change of reactivity, is involved in the problem, then significant mathematical difficulties arise. In the latter problem, when only one group of delayed neutrons is assumed, the differential equation for the neutron flux is of the form

$$\ddot{\phi} + (At + B)\dot{\phi} + (Ct + D)\phi = 0 \qquad (1)$$

where $\dot{\phi} = d\phi/dt$, etc. There is no serious difficulty involved in finding a particular solution of Eq. (1) in the form of a contour integral (*2*). However, it is not a straightforward mathematical problem to find linearly independent solutions so that initial conditions on the flux and its first derivative may be fulfilled. Moreover, the problem becomes still more difficult when the number of delayed neutron groups is greater than one. In this case, for each additional delayed neutron group added, the order of the differential equation corresponding to (1), and also the number of linearly independent solutions required to be

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found, increase by one. It is the mathematical problem just described which is of primary interest in this paper.

The motivation for the discussion of this paper is a problem of control rod calibration in a thermal reactor. The rod calibration technique associated with a reactor which can be maintained at criticality in a steady-state condition without substantial movement of the control rods is well known. The control rod calibration problem under study in the present discussion is concerned with a special situation where the reactor can be maintained at criticality only by a virtually continuous motion of the control rods. Such a condition arises during a xenon transient (*1*, page 335), that is, during the period of buildup and decay of xenon and samarium following operation of a reactor at power.

The conventional rod calibration technique is clearly inadequate during a xenon transient since the reactor period (*1*, page 293) cannot be measured experimentally. For example, when a rod is pulled during the rising phase of a xenon transient, a positive increment of reactivity is thus introduced and the flux starts to rise on a steep trajectory. However, before the flux has time to take off on its customary exponential ascent, the incoming xenon begins to take hold with a continuous contribution of negative reactivity. The net effect of the two opposing contributions of reactivity results in the flux attaining a peak and then falling off until the reactor becomes subcritical. Thus, the reactor period cannot be measured in the conventional manner.

Probably the simplest way to obtain an approximation to the reactivity introduced by the motion of a control rod during a xenon transient is explained in what follows. Suppose that a reactor is maintained at criticality during the rising phase of a xenon transient by the continuous motion of a control rod. Suppose further that at $t = t_0$ it is desired to measure the reactivity introduced by the motion of the control rod for the next x inches of travel of the rod. The reactor is now held critical with the control rod in question until the rod has moved the predetermined x inches and the elapsed time $t = t_1$ is then recorded. It remains to determine the change in reactivity in the interval $t_1 - t_0$. Presumably, the amount of fuel in the reactor and the power level preceding shutdown are known at the time of the experiment. Accordingly, there exists, at least in principle, a corresponding xenon buildup curve (*1*, page 337) which gives the *effective* macroscopic xenon cross section[2] in the reactor at any time. Accordingly, the change in the xenon cross section in the time interval $t - t_0$ can be found, and finally the corresponding change in reactivity can readily be computed.

The greatest weakness of the rod calibration procedure outlined above is that the correct (or most nearly correct) xenon buildup curve needed in the calculation is never available without special calculation. Evidently, the three most

[2] The reactivity equivalent of the actual inhomogeneously distributed xenon.

important parameters in the calculation of the effective xenon cross section are the *xenon weighting factor*,[3] the *xenon yield*, and the microscopic *xenon cross section*. Of these three parameters the xenon weighting factor is the only one amenable to improvement by the reactor physicist and is, moreover, an extremely difficult number to calculate to a high degree of accuracy. Thus, it is of interest to seek a more accurate procedure of control rod calibration during a xenon transient.

The present paper represents an attempt to study the problem of control rod calibration during a xenon transient from a purely analytic point of view and then correlate theory with experiment to obtain the desired results in the best possible approximation. Particulars of the problem studied are described as follows. Suppose that the reactor is critical during the rising phase of a xenon transient at $t = t_0$. Then a control rod (or set of control rods) is suddenly pulled. The approximate subsequent behavior of the flux has already been described. It is desired to find the reactivity $\delta\rho$ thus introduced by pulling the rod x inches and incidentally to find the reactor period in the conventional sense.

The analytic study involves the kinetics equations which describe the time-dependent behavior of the thermal flux. The problem can be reduced to the consideration of a linear homogeneous ordinary differential equation of the second order,[4] the coefficients of which are functions of the time. Fortunately, the reduction of the coefficients to linear functions of the time involves a physical approximation which is very reasonable. In this paper linearly independent solutions of the differential equation are found in the form of contour integrals; these latter can be expanded into usable asymptotic developments.

The analytic solution to the rod calibration problem thus exhibits the thermal flux in the form of an asymptotic series, where only a few terms are needed to give the desired degree of approximation. The parameters of particular interest which appear in the solution are the step change of reactivity $\delta\rho$, introduced by the motion of the control rod[5] and the quantity m, which is the time rate of change of the macroscopic xenon cross section at $t = t_0$. Now, the parameter m is clearly a function of the xenon weighting factor, and it would thus appear that the analytic approach to the rod calibration problem suffers from the same defect as the procedure described earlier in this section. However, it is explained in a later section how this apparent weakness in the analytic approach is overcome and in particular how the experimentally determined flux is used to determine both the reactivity $\delta\rho$ and the correct xenon weighting factor.

[3] The xenon weighting factor is a number which depends on the preshutdown control rod configuration and, most important of all, the control rod configuration at the time when the control rod calibration is initiated.

[4] On the assumption of one group of delayed neutrons.

[5] The case is also considered in which excess reactivity is introduced initially by pulling a rod or group of rods in a noninstantaneous fashion.

A true shortcoming of the theory developed in this chapter is the omission of the generalization to more than one group of delayed neutrons. Apparently, the mathematical difficulties entailed in such a generalization are substantial.

DEVELOPMENT OF BASIC KINETICS EQUATIONS

The kinetics equations needed here for a point of departure are developed in Glasstone and Edlund (*1*). The kinetics equations[6] in question are

$$T'(t) = \frac{\lambda c k_{\text{eff}}}{k\Sigma_a A l} H(t) + \frac{k_{\text{eff}}(1-\beta) - 1}{l} T(t) \tag{2}$$

$$\begin{aligned} cH'(t) &= -\lambda c H(t) + k\beta\Sigma_a A T(t) \\ \phi(\mathbf{r}, t) &= A W(\mathbf{r}) T(t) \\ C(\mathbf{r}, t) &= c W(\mathbf{r}) H(t) \\ k_{\text{eff}} &= \frac{k}{(1 + L^2B^2)(1 + \tau B^2)} \\ l &= \frac{l_0}{1 + L^2B^2}, \qquad l_0 = \frac{1}{v\Sigma_a}. \end{aligned} \tag{3}$$

In the foregoing equations ϕ is the thermal flux, C the concentration (in atomic nuclei per cm^3) of the precursors from which a single average group of delayed neutrons arises, λ the appropriate decay constant, v the average speed of thermal neutrons, β the fraction of the total number of fission neutrons in the single average group of delayed neutrons, k the multiplication constant in the infinite reactor, L the thermal diffusion length, τ the age, Σ_a the macroscopic thermal absorption cross section, B^2 the total buckling of the equivalent bare reactor, l_0 the neutron lifetime in the infinite reactor, and l the neutron lifetime in the finite reactor.

Equations (2) and (3) may also be written as

$$\phi' = \frac{\lambda k_{\text{eff}}}{k\Sigma_a l} C + \frac{k_{\text{eff}}(1-\beta) - 1}{l} \phi \tag{4}$$

$$C' = -\lambda C + k\beta\Sigma_a\phi \tag{5}$$

where it is understood that the space variables are fixed throughout the discussion. Next, it is desired to write equations (4) and (5) in a form which displays the reactivity explicitly. To this end, observe that

$$\frac{k_{\text{eff}}(1-\beta) - 1}{l} = \frac{\rho - \beta}{l} k_{\text{eff}} = \Gamma(\rho - \beta) \tag{6}$$

[6] The equation $\tilde{M}_t[\mu(t)] = \sum_{k=0}^{n} (-1)^k (d^k/dt^k)[p_k(t)\mu(t)] = 0$ is called the adjoint equation associated with $M_t[v(t)] = 0$, if $M_t[v(t)] = \sum_{k=0}^{n} p_k(t) v^{(k)}(t)$.

where

$$\Gamma = \frac{v\eta\Sigma_{25}}{1 + \tau B^2}. \tag{7}$$

Moreover,

$$\frac{\lambda k_{\mathrm{eff}}}{k\Sigma_a l} = \frac{\lambda v}{1 + \tau B^2} = \gamma\lambda \tag{8}$$

where

$$\gamma = \frac{v}{1 + \tau B^2}. \tag{9}$$

Then, with the use of (6) and (8), Eqs. (4) and (5) may be written in the form

$$\phi' = \Gamma(\rho - \beta)\phi + \lambda\gamma C \tag{10}$$

$$C' = -\lambda C + \eta\Sigma_{25}\,\beta\phi. \tag{11}$$

The form of ρ, peculiar to the discussion at hand, must be investigated in detail. The expression for ρ may be transformed as follows:

$$\begin{aligned}\rho = \frac{k_{\mathrm{eff}} - 1}{k_{\mathrm{eff}}} = 1 - \frac{(\Sigma_a + D_s B^2)(1 + \tau B^2)}{\eta\Sigma_{25}} \\ - \frac{(\Sigma_R + D_s B^2)(1 + \tau B^2)}{\eta\Sigma_{25}} - \frac{\Sigma_P(1 + \tau B^2)}{\eta\Sigma_{25}},\end{aligned} \tag{12}$$

with $\Sigma_a = \Sigma_R + \Sigma_P$, where Σ_R corresponds to reactor materials except for control rods, and where Σ_P represents the homogeneously (uniformly) distributed poison corresponding to the control rods and the transient xenon. Thus,

$$\Sigma_P = \Sigma_{\mathrm{cr}}\ (\text{control rods}) + \Sigma_{\mathrm{tr}}\ (\text{transient xenon}).$$

Suppose now a rod calibration experiment is initiated at $t = 0$ when the reactor is just critical during the rising phase of the xenon transient. The realistic approximation is then made that the xenon increases in a linear fashion for the next few minutes following the initiation of the experiment. Thus,

$$\Sigma_{\mathrm{tr}} = \Sigma_{\mathrm{tr}}]_{t=0} + mt, \qquad m = \partial\Sigma_X/\partial t]_{t=0} \tag{13}$$

where Σ_X is the macroscopic cross section of the xenon (plus samarium).

The reactivity ρ in Eq. (10) consists of two terms:

$$\rho = \delta\rho_1 + \rho_2(t) \tag{14}$$

where $\delta\rho_1$ represents a change of reactivity corresponding to a change of position of a control rod and $\rho_2(t)$ corresponds to a xenon transient. Since the reactor is critical at $t = 0$, then $\rho_2(0) = 0$. Thus, from (12) and (13),

$$\rho_2(t) = 1 - \frac{(\Sigma_R + D_s B^2)(1 + \tau B^2)}{\eta \Sigma_{25}} - \frac{(\Sigma_{\mathrm{cr}_0} + \Sigma_{\mathrm{tr}_0})(1 + \tau B^2)}{\eta \Sigma_{25}} - \frac{mt(1 + \tau B^2)}{\eta \Sigma_{25}}$$

whence, from the assumption of criticality at $t = 0$,

$$\rho_2(t) = -\frac{mt(1 + \tau B^2)}{\eta \Sigma_{25}} = -\frac{mtv}{\Gamma}. \tag{15}$$

With the use of (14) and (15), Eqs. (10) and (11) become

$$\phi' = \Gamma[\delta\rho_1 - (mtv/\Gamma) - \beta]\phi + \lambda\gamma C \tag{16}$$

$$C' = -\lambda C + \eta \Sigma_{25} \beta \phi. \tag{17}$$

The elimination of C from the pair of equations above to give the desired equation in ϕ alone is readily accomplished. Both sides of (16) are differentiated and the resulting equation solved for C'. Then, the elimination of C and C' between this equation and (16) and (17) yields

$$\phi'' + [\lambda + \Gamma\beta - \Gamma\delta\rho_1 + mvt]\phi' + [mv - \Gamma\lambda\delta\rho_1 + \lambda mvt]\phi = 0. \tag{18}$$

To obtain the initial conditions let it be assumed first that

$$\phi(0) = \phi_0 = 1. \tag{19}$$

Since $C' = 0$ when $t = 0$, then from (17) it follows that

$$C_0 = \eta \Sigma_{25} \beta \phi_0 / \lambda. \tag{20}$$

Substitution of (20) into (16), evaluated at $t = 0$, gives

$$\phi'(0) = \phi_0' = \Gamma\delta\rho_1. \tag{21}$$

The solution of (18) which fulfills the initial conditions (19) and (21) is sought. The discussion of this problem is continued in the Section "Applications to Special Cases."

DEVELOPMENT OF BASIC KINETICS EQUATIONS: NONINSTANTANEOUS ROD PULL

In the preceding section the kinetics equations associated with an initial step change in reactivity were developed. The type of problem considered corresponds to the instantaneous withdrawal of a control rod, usually the withdrawal of merely a *single* rod. If, however, the initial excess reactivity is introduced by the withdrawal of a *group* of rods, then the rod withdrawal cannot be effected rapidly and the assumption of a step change of reactivity represents a poor physical approximation. If the rod withdrawal takes place over a finite interval of time (i.e., interval large compared to the reactor period) experimentalists consider the assumption of a linear change in reactivity to be a realistic approximation. Thus, it is assumed in what follows that $\delta\rho = \mu t$, where μ is a measure of the excess reactivity inserted. If the rod withdrawal is stopped after t_0 seconds, the fixed amount of reactivity introduced by the rod withdrawal is given by $\delta\rho_0 = \mu t_0$.

On the basis of the theory developed in the preceding section the equations describing the procedure discussed above are given by

$$\left.\begin{aligned} \phi' &= \Gamma\left(\mu t - \frac{mtv}{\Gamma} - \beta\right)\phi + \lambda\gamma C \qquad (22)\\ C' &= -\lambda C + \eta\Sigma_{25}\beta\phi \qquad (23) \end{aligned}\right\}, 0 \leqq t \leqq t_0$$

$$\left.\begin{aligned} \phi' &= \Gamma\left(\mu t_0 - \frac{mtv}{\Gamma} - \beta\right)\phi + \lambda\gamma C \qquad (24)\\ C' &= -\lambda C + \eta\Sigma_{25}\beta\phi \qquad (25) \end{aligned}\right\}, t \geqq t_0.$$

Equations (24) and (25) are of the type already considered in the preceding section. Equations (22) and (23), which describe the early behavior of the reactor, pose a mathematical problem quite similar to the one presented earlier.

If C is eliminated from Eqs. (22) and (23), then

$$\phi'' + [\lambda + \Gamma\beta - \Gamma\zeta t]\phi' - [\Gamma\zeta + \lambda\Gamma\zeta t]\phi = 0 \tag{26}$$

where

$$\zeta = \mu - (mv/\Gamma). \tag{27}$$

The initial conditions associated with the solution of (26) are

$$\phi_0 = 1, \qquad \phi_0' = 0. \tag{28}$$

The procedure for finding the flux over the entire range $t \geqq 0$ is explained in what follows. Let $\phi_1(t)$ represent the flux for $0 \leqq t \leqq t_0$ and $\phi_2(t)$ the flux for $t \geqq t_0$. The arbitrary constants which appear in the general solution $\phi_1(t)$ are determined by fulfillment of the initial conditions (28). The arbitrary constants contained in the general solution $\phi_2(t)$ are determined by fulfillment of the continuity conditions $\phi_1(t_0) = \phi_2(t_0)$, $\phi_1'(t_0) = \phi_2'(t_0)$.

SOLUTION OF KINETICS EQUATIONS BY ASYMPTOTIC SERIES

The differential equations associated with the rod calibration problems under discussion here are given by Eqs. (18) and (26), both being of the form

$$W''(t) + (At + B)W'(t) + (Ct + D)W(t) = 0. \tag{29}$$

Solutions of this equation which are valid for the range of values of the parameters of interest here are discussed in detail later. The general technique was to find solutions in the form of contour integrals and to expand these in appropriate asymptotic series.

The two linearly independent solutions of Eq. (29) valid for the range of parameters of interest are given by

$$W_1(t) = e^{-Ct/A} (At + F)^{\lambda} \left\{ \sum_{j=0}^{n} \delta_j \left[\frac{2A}{(At + F)^2} \right]^j + T_n[(At + F)^2] \right\} \quad (30)$$

$$W_2(t) = \frac{e^{-Ct/A} \; e^{-(At+F)^2/2A}}{(At + F)^{\lambda+1}} \left\{ \sum_{j=0}^{n} \gamma_j \left[\frac{2A}{(At + F)^2} \right]^j + V_n[(At + F)^2] \right\} \quad (31)$$

where

$$\delta_j = \frac{\Gamma(\lambda + 1)}{4^j j! \, \Gamma(\lambda - 2j + 1)}, \qquad \gamma_j = \frac{(-1)^j \Gamma(\lambda + 2j + 1)}{4^j j! \, \Gamma(\lambda + 1)}$$

$$F = B - 2(C/A).$$

The terms T_n and V_n in Eqs. (30) and (31), respectively, are error terms which may be shown to be negligible for the range of values of the parameters under consideration and for the degree of accuracy required in the calculations.

The general solution of the differential equation (29) is a linear combination of the linearly independent solutions:

$$\phi(t) = \mu_1 W_1(t) + \mu_2 W_2(t) \quad (32)$$

where the constants are determined by the initial conditions. In the special cases considered in the next section, from three to five terms of the asymptotic expansions in (30) and (31) are needed to compute fluxes to an accuracy of four decimal places.

APPLICATIONS TO SPECIAL CASES

The theory developed in the preceding sections has to the best knowledge of the authors never been used in connection with actual experimental data. The special flux plots exhibited in this section have been calculated on the basis of a rather arbitrary selection of parameters and serve to illustrate the theory previously developed. Two cases have been considered.

Case I

With the reactor assumed to be critical at some instant during the rising phase of a xenon transient, positive excess reactivity is introduced instantaneously by withdrawal of control rods.

Case II

With the reactor assumed to be critical at some instant during the rising phase of a xenon transient, positive excess reactivity is introduced by gradual withdrawal of control rods over an interval of about 10 sec.

The differential equation associated with Case I is given by Eq. (18). The initial conditions associated with Case I are

$$\phi_0 = 1, \qquad \phi_0 = \Gamma\delta\rho_1 . \tag{33}$$

Equation (18) is readily identified with Eq. (29). With the use of the linearly independent solutions, Eqs. (30), (31), the general solution of (18) can be written as indicated in (32). Finally, the constants of combination are determined in fulfillment of the initial conditions (33). In this connection, it may be shown that the term-by-term differentiation of the asymptotic series (30) and (31) is permissible.

The final solution has not been tabulated since the writing of the complicated equation affords a negligible contribution to the understanding of the problem. The flux in Case I has been plotted for three values of $\delta\rho_1$ in Fig. 1.

The differential equation associated with the behavior of the flux in Case II is given by Eq. (26). The initial conditions for Case II are:

$$\phi_0 = 1, \qquad \phi_0' = 0 \tag{34}$$

Equation (26) is readily identified with Eq. (29). The general solution of (26) can once again be written as indicated in (32) and μ_1, μ_2 determined to satisfy the initial conditions (34). Let the solution thus determined be designated as $\phi_1(t)$.

The flux is given by $\phi_1(t)$ during the period of gradual rod pull over an interval of t_0 seconds. At the end of t_0 seconds, it has been shown earlier that the fixed amount of positive reactivity introduced by the rod withdrawal is given by

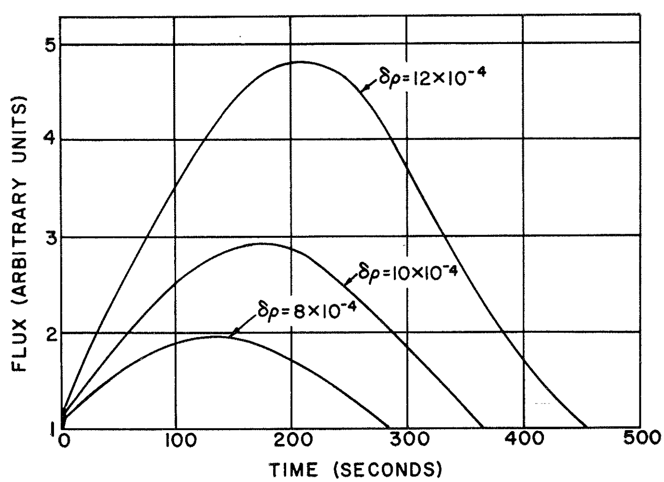

FIG. 1. Flux during a xenon transient with instantaneous rod pull.

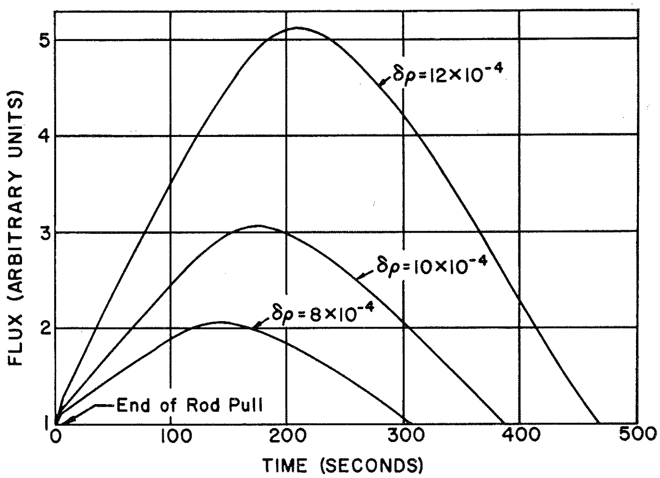

FIG. 2. Flux during a xenon transient with noninstantaneous rod pull.

$\delta\rho_0 = \mu t_0$. Thus, the behavior of the flux for $t \geqq t_0$ is dominated by Eq. (18), where $\delta\rho_1$ is replaced by μt_0. Let the general solution of the resulting equation be designated by $\phi_2(t)$. The constants of combination in $\phi_2(t)$ may be determined in fulfillment of the continuity conditions $\phi_1(t_0) = \phi_2(t_0)$ and $\phi_1'(t_0) = \phi_2'(t_0)$. Thus, the flux for Case II has been completely determined.

The flux in Case II has been plotted for three values of $\delta\rho_1$ (or μ) in Fig. 2. It is noted from Figs. 1 and 2 that the eventual behavior of the flux is not influenced greatly by whether or not the rod pull is instantaneous or gradual over a short interval of time.

IMPLICATIONS OF THE NEW TECHNIQUES

Experimentally determined flux plots would have approximately the shapes of the theoretical flux plots exhibited in Figs. 1 and 2. In principle, the initial excess reactivity introduced by the rod pull $\delta\rho$ and the xenon weighting factor ω can *both* be precisely determined from an experimental flux plot and the theoretical flux as follows. Suppose that the experimental flux has been measured at two distinct times t_1 and t_2, so that $\phi(t_1)$ and $\phi(t_2)$ have been determined experimentally. Substitution of the pairs of known values t_1, $\phi(t_1)$ and (t_2), $\phi(t_2)$ into the equation for the theoretical flux gives two equations in two unknowns, albeit the equations are extremely complicated. If it is then assumed that all of the reactor parameters save $\delta\rho$ and m (the slope of the effective macroscopic xenon cross section at $t = 0$) are known, then the two equations can be solved graphically for $\delta\rho$ and m. When m is known, the corresponding xenon weighting factor ω can be determined as follows.

It is rather commonly known that under certain circumstances which prevail in this discussion the effective xenon cross section can be written in the form

$$[\Sigma_X(t)]_{\text{eff}} = A(t) + B(t)\omega\Phi \tag{35}$$

where Φ is the average flux preceding shutdown, ω is the xenon weighting factor, and where $A(t)$ and $B(t)$ are derived in Glasstone and Edlund (*1*). Differentiation of both sides of (35) and evaluation at $t = 0$ gives

$$m = A'(0) + B'(0)\omega\Phi$$

whence ω is determined as a function of m.

The principal difficulty associated with the method outlined above, for a precise calibration of a control rod during a xenon transient, is of course the problem of the simultaneous solution of the two very complicated transcendental equations involved. However, this problem could be greatly simplified by the development of an appropriate code for the solution of the problem with the aid of a large scale digital computing device.

INTRODUCTION TO THE MATHEMATICAL PROBLEM

The remainder of this paper is devoted to a detailed study of the differential equation (29) and, in particular, the methods used in obtaining the linearly independent solutions, Eqs. (30) and (31). By substituting $W(t) = e^{-Ct/A}\eta(t)$ into (29) and then changing variables, $s = At + B - 2C/A$, the resulting equation becomes:

$$N''(s) + r_1 sN'(s) + r_2 N(s) = 0 \tag{36}$$

where $r_1 = 1/A \neq 0$, $r_2 = (C/A^3)(C/A - B) + (D/A^2)$ and $N(s) = W(t)e^{Ct/A}$. This is the standard form of the differential equations which will be considered.

Equation (36) will be solved by using asymptotic series in the following sense: a function $f(x)$ has the asymptotic representation $g(x)(a_0 + a_1 x^{-1} + a_2 x^{-2} + \cdots)$ if, for every fixed value of n,

$$\lim_{x\to\infty} x^n \left\{\frac{f(x)}{g(x)} - a_0 - a_1 x^{-1} - \cdots a_n x^{-n}\right\} = 0.$$

Any function can have but one asymptotic expansion although many different functions may have the same asymptotic expansion. If both $F(x)$ and $G(x)$ have asymptotic expansions, then $\alpha F(x) + \beta G(x)$, where α, β are constants, has the asymptotic expansion obtained by simply replacing $F(x)$ and $G(x)$ by their asymptotic expansions in $\alpha F(x) + \beta G(x)$ and adding. These are the only results from the theory of asymptotic series that will be used in this paper. For more results and a more complete theory, the books by Knopp (*4*) and Van der Corput (*5*) can be read.

If $N_1(s)$ and $N_2(s)$ are two linearly independent solutions of (36), then Eq. (36) is said to be solved by means of asymptotic series if the asymptotic expansions of $N_1(s)$ and $N_2(s)$ are used to find the general solution of (36).

METHOD I. THE USE OF DEFINITE INTEGRALS

Equation (36) can be written as

$$L_s[N] = N''(s) + r_1 s N'(s) + r_2 N(s) = 0. \tag{37}$$

A solution of the form $N(s) = \int_\alpha^\beta K(s, t)V(t)\,dt$ is assumed where α, β, $K(s, t)$ and $V(t)$ are suitably chosen in the following way. If differentiability under the integral sign is assumed, then

$$L_s[N] = \int_\alpha^\beta L_s[K(s, t)]v(t)\ dt.$$

A linear differential operator with respect to t, M_t, is then found such that $M_t[K(s, t)] \equiv L_s[K(s, t)]$. Hence

$$L_s[N] = \int_\alpha^\beta M_t[K(s, t)]\,v(t)\,dt.$$

Lagrange's identity (*3*, page 124) is now used to obtain

$$L_s[N] = \int_\alpha^\beta K(s, t)\bar{M}_t[v(t)]\,dt + P[K(s, t), v(t), t]]_{t=\alpha}^{t=\beta}$$

where $\bar{M}_t$ is the adjoint operator[6] of M_t, and $P(K, v, t)$ is the *bilinear concomitant* associated with M_t and $\bar{M}_t$. The function $v(t)$ can be found as a solution of $\bar{M}_t[v(t)] = 0$ after a kernel $K(s, t)$ has been selected. The limits α and β are chosen so that $P(K, v, t)]_{t=\alpha}^{t=\beta}$ vanishes. Then a solution of the equation $L_s[N] = 0$ has been found subject to the assumption concerning differentiability under the integral sign.

METHOD II. THE USE OF CONTOUR INTEGRALS

For the sake of completeness, an nth-order linear differential operator is considered. Let

$$L_x \equiv \sum_{r=0}^{n} (a_{ro} + a_{r1}\,x)\,D_x^r\,,\ a_{n1} \neq 0. \tag{38}$$

The linear differential operator of Eq. (37) is not of this form since in that case $a_{n1} = 0$. However, it can be changed to the desired form as will be shown later.

A solution of $L_x[y] = 0$ is assumed of the form

$$y(x) = \int_C e^{x\zeta}\,v(\zeta)\,d\zeta$$

where $v(\zeta)$ and C, a contour in the complex ζ plane, are to be determined. As in the preceding section, differentiability under the integral sign is assumed so that from (38):

$$L_x[y] = \int_C M_\zeta[e^{x\zeta}]\, v(\zeta)\, d\zeta$$

where

$$M_\zeta \equiv \sum_{r=0}^{n} \zeta^r (a_{ro} + a_{r1} D_\zeta).$$

If $\bar{M}_\zeta[v] = 0$ denotes the adjoint equation associated with $M_\zeta[v]$, then by Lagrange's identity the equation above becomes

$$L_x[y] = \int_C e^{x\zeta}\, \bar{M}_\zeta[v]\, d\zeta + [P(e^{x\zeta}, v)]_C .$$

Thus, $y(x)$ will be a solution of $L_x[y] = 0$ if $v(t)$ is a solution of $\bar{M}_\zeta[v] = 0$ and if C is chosen so that

$$[P(e^{x\zeta}, v)]_C = 0.$$

From definition,

$$\bar{M}_\zeta = P_n(\zeta) D_\zeta + Q_n(\zeta) \tag{39}$$

where $P_n(\zeta) = \sum_{r=0}^{n} a_{r1}\zeta^r$ and $Q_n(\zeta)$ is some other polynomial of degree $\leqq n$. Actually $P_n(\zeta)$ is of degree n since $a_{n1} \neq 0$. Thus

$$\bar{M}_\zeta[v] = P_n(\zeta)v'(\zeta) + Q_n(\zeta)v(\zeta) = 0 \tag{40}$$

is the equation to be solved for $v(\zeta)$. From the theory of partial fractions,

$$-\frac{Q_n(\zeta)}{P_n(\zeta)} = \mu + \frac{\lambda_1}{\zeta - \alpha_n} + \cdots + \frac{\lambda_n}{\zeta - \alpha_n}$$

where $\alpha_1, \cdots, \alpha_n$ are the n roots of $P_n(\zeta) = 0$ and are assumed to be distinct. Hence

$$v(\zeta) = e^{\mu\zeta}(\zeta - \alpha_1)^{\lambda_1} \cdots (\zeta - \alpha_n)^{\lambda_n}.$$

The corresponding bilinear concomitant is

$$P(e^{x\zeta}, v) = (\zeta - \alpha_1)^{\lambda_1+1} \cdots (\zeta - \alpha_n)^{\lambda_n+1} e^{(\mu+x)\zeta}.$$

If C_i denotes the path in the complex ζ plane starting at $-\infty$, circling α_i in a negative direction, and returning to $-\infty$, then $P(e^{x\zeta}, v)]_{C_i} = 0$ if $x + \mu > 0$. The path C_i in the neighborhood of $-\infty$ is usually taken to be parallel to, or at a small angle to, the axis of the reals in such a way that each of the roots $\alpha_1, \cdots, \alpha_{i-1}, \alpha_{i+1}, \cdots, \alpha_n$ is at a finite distance from this loop. Thus there are n solutions of $L_x[y] = 0$ corresponding to each i in

$$y_i(x) = \int_{C_i} e^{x\zeta} \phi(\zeta)\, d\zeta.$$

It is important that the polynomial $P_n(\zeta)$ defined in (39) be of degree n since only then will n solutions exist with the type of contour described above.[7] The fact that these n solutions are linearly independent will follow from subsequent work.

SELECTION OF THE KERNEL $K(s, t)$ IN METHOD I

For convenience, kernels of the type $K(s, t) = K(s \cdot t)$ are sought. Then Eq. (36) can be written as

$$L_s[N] = \{s^{-2} F(s\, d/ds) + G(s\, d/ds)\}\, N(s) = 0$$

where $F(u) = u^2 - u$, $G(u) = r_1 u + r_2$. If H is any polynomial and $K(z)$ is any solution of the ordinary differential equation

$$\{z^{-2} F(z\, d/dz) - H(z\, d/dz)\}\, K(z) = 0$$

then $K(s \cdot t)$ satisfies

$$\{s^{-2} F(s\, d/ds) + G(s\, d/ds)\}\, K(s\cdot t) = \{G(t\, d/dt) + t^2 H(t\, d/dt)\}\, K(s\cdot t) \quad (41)$$

or, equivalently, $L_s[K] = M_t[K]$. This defines the operator M_t.

If now $H(u)$ is a real polynomial of the form

$$H(u) = \alpha_1 u^2 + \alpha_2 u + \alpha_3$$

then $K(z)$ satisfies the equation

$$K''(z)(1 - \alpha_1 z^2) - zK'(z)(\alpha_1 + \alpha_2) - \alpha_3 K(z) = 0. \quad (42)$$

The real numbers $\alpha_1, \alpha_2, \alpha_3$ are arbitrary. From Eq. (41) it follows that

$$M_t[u] = u''(t)(\alpha_1 t^4) + u'(t)(r_1 t + \alpha_1 t^3 + \alpha_3 t^3) + u(t)(r_2 + \alpha_3 t^2). \quad (43)$$

In order to simplify the task of finding a solution to $\bar{M}_t[v(t)] = 0$, α_1 is set equal to 0. From (43) it is then clear that the solution of $\bar{M}_t[v(t)] = 0$ can be found by elementary separation of variables.

Solutions of (36) can now be found corresponding to certain restrictions on the ratio $\lambda = -(r_2/r_1)$.

SOLUTION FOR RESTRICTED VALUES OF r_1, r_2

By suitable choices of α_2, α_3 and the limits α, β, different solutions of (36) can be obtained for varying values of r_1, r_2. For this paper only one such case

[7] For more general techniques and contours, see reference *3*, Chapter XVIII, or reference *6*, pages 317–333.

need be considered. In this connection, let $\lambda = -(r_2/r_1)$ and consider the case $\lambda > -1$, $r_1 > 0$.

It can easily be verified that $K(z) = e^{-z^2}$ satisfies the equation $K''(z) + 2zK'(z) + 2K(z) = 0$. This corresponds to Eq. (42) with $\alpha_1 = 0$, $\alpha_2 = \alpha_3 = -2$. Thus

$$M_t[u] = u'(t)(r_1 t - 2t^3) + u(t)(r_2 - 2t^2)$$

and

$$M_t[e^{-s^2t^2}] = L_s[e^{-s^2t^2}].$$

It follows then that

$$\bar{M}_t[v] = -(d/dt)[(r_1 t - 2t^3)\, v(t)] + v(t)(r_2 - 2t^2)$$

and

$$P(e^{-s^2t^2}, v(t), t) = (r_1 t - 2t^3)\, v(t) e^{-s^2t^2}.$$

The solution of $\bar{M}_t[v] = 0$ up to a constant is

$$v(t) = t^{-1-\lambda}\left(t^2 - \frac{r_1}{2}\right)^{(\lambda-1)/2} \quad |t| \geqq \sqrt{\frac{r_i}{2}}.$$

Hence

$$[P(e^{-s^2t^2}, v(t), t)]_{t=\alpha}^{t=\beta} = \left[t^{-\lambda}\left(t^2 - \frac{r_1}{2}\right)^{(\lambda+1)/2} e^{-s^2t^2}\right]_{t=\alpha}^{t=\beta}. \tag{44}$$

For $s \neq 0$, α and β can be chosen to be $\sqrt{r_1/2}$ and ∞, respectively, to make (44) vanish provided $(\lambda + 1)/2 > 0$.

Thus, the following integral is obtained:

$$N_1(s) = \int_{\sqrt{r_1/2}}^{\infty} e^{-s^2t^2} t^{-1-\lambda}\left(t^2 - \frac{r_1}{2}\right)^{(\lambda-1)/2} dt.$$

The substitution $u = t^2$ changes this into (up to a constant)

$$N_1(s) = \int_{r_1/2}^{\infty} e^{-s^2u} u^{-(\lambda+2)/2}[u - (r_1/2)]^{(\lambda-1)/2}\, dt. \tag{45}$$

This integral exists and is an analytic function of s for $\lambda > -1$. Thus, it is one solution of $L_s[N] = 0$, since the differentiability assumption of the previous section is now valid.

By choosing different values for α_1, α_2, α_3 and α, β, other solutions valid for other ranges of r_1 and r_2 can be found. The disadvantage of this method lies in the restrictions on r_1, r_2 arising from the trial and error technique of choosing α_1, α_2, α_3 and α, β. Method II, by contour integrals, leads to a more unified theory. However, as will be seen later, the solution found above leads to a more exact remainder when asymptotic expansions are found. It might also be mentioned that only one solution was found for the set (λ, r_1) for $L_s[N] = 0$. Since $L_s[N] = 0$ is a second-order differential equation, another solution linearly independent of the first would have to be obtained in each case in order to have general solutions.

SOLUTIONS USING METHOD II

The equation

$$L_s[N] = N''(s) + r_1 s N'(s) + r_2 N(s) = 0$$

is not of the form suitable for Method II. To circumvent this difficulty, a change of the independent variable $s^2 = x$ can be made

$$L_x[y] \equiv xy''(x) + [(\tfrac{1}{2}) + (r_1 x/2)]y'(x) + (r_2/4)y(x) = 0 \tag{46}$$

where $y(x) = N(s)$. Thus

$$L_x = \sum_{r=0}^{2} \sum_{s=0}^{1} a_{rs} x^s D_x^r$$

with $a_{00} = r_2/4$, $a_{10} = \frac{1}{2}$, $a_{11} = r_1/2$, $a_{21} = 1$, $a_{01} = a_{20} = 0$. Then

$$M_\zeta = \sum_{r=0}^{2} \sum_{s=0}^{1} a_{rs} \zeta^r D_\zeta^s$$

so that

$$M_\zeta[v] = \left(\zeta^2 + \frac{r_1}{2}\zeta\right)v' + \left(\frac{1}{2}\zeta + \frac{r_2}{4}\right)v.$$

From the general theory of Method II, the adjoint is seen to be

$$-\frac{d}{d\zeta}\left[\left(\zeta^2 + \frac{r_1}{2}\zeta\right)v\right] + \left(\frac{1}{2}\zeta + \frac{r_2}{4}\right)v = 0$$

which reduces to

$$\left(\zeta^2 + \frac{r_1}{2}\zeta\right)v'(\zeta) + \left(\frac{3}{2}\zeta + \frac{r_1}{2} - \frac{r_2}{4}\right)v(\zeta) = 0. \tag{47}$$

This is an equation with *variables separable* which can easily be solved. Thus

$$v(\zeta) = \zeta^{-(\lambda+2)/2}\left(\zeta + \frac{r_1}{2}\right)^{(\lambda-1)/2}.$$

From Eq. (47), it can be seen that $P_2(\zeta) = \zeta^2 + r_1\zeta/2$, where $P_2(\zeta)$ follows the notation used in Eq. (39). The roots of $P_2(\zeta)$ are 0 and $-r_1/2$, which are distinct since $r_1 \neq 0$. Hence, from the previous analysis

$$y_i(x) = \int_{C_i} e^{x\zeta} \zeta^{-(\lambda+2)/2} \left(\zeta + \frac{r_1}{2}\right)^{(\lambda-1)/2} d\zeta, \qquad i = 1, 2 \quad (48)$$

forms two solutions of Eq. (46). The contour C_1 is composed of the following:

(i) a line making an angle $\alpha \neq 0$, $|\alpha| < \pi/2$ with the negative real axis from $-\infty$ to $|z| = r$, where r is positive,

(ii) a circle of radius r described in a negative direction about the origin,

(iii) the line of (i) from $|z| = r$ to $-\infty$.

Thus C_1 is associated with the root 0, and C_2 is a similar contour encircling $-r_1/2$. It is assumed that $r < |r_1/2|$ so that the point $z = -r_1/2$ is not included inside the circle of (ii).

If $\lambda = -2k$, with k an integer greater than 1, the integrand for $y_1(x)$ is analytic within C_1 and $y_1(x) \equiv 0$. This is a permissible solution but not a very useful one. For this case, C_1 can be chosen as the straight line R_1 starting at $-\infty$ and going to 0 at an angle α with the negative real axis. A similar case occurs for $y_2(x)$ when $\lambda = 2n + 1$, n a positive integer. This gives rise to a new path of integration R_2 similar to R_1.

ASYMPTOTIC EVALUATION OF SOLUTIONS USING METHOD I

In Eq. (45), if $u = r_1/2 + \zeta$, then

$$N_1(s) = e^{-r_1 s^2/2} (2/r_1)^{1+\lambda/2} \int_0^\infty e^{-s^2\zeta} \zeta^{(\lambda-1)/2} (1 + 2\zeta/r_1)^{-1-\lambda/2} d\zeta. \qquad (49)$$

This integral is of the form $\int_0^\infty e^{-\tau\zeta} F(\zeta)\, d\zeta$. A generalized Watson's lemma (*7*, page 218 or *8*, page 231) can be used to find an asymptotic series for this integral. Thus,

$$\int_0^\infty e^{-s^2\zeta} \frac{\zeta^{(\lambda-1)/2}}{(1 + 2\zeta/r_1)^{1+\lambda/2}} d\zeta$$

$$= \sum_{j=0}^{n} (-1)^j \frac{1}{s^{\lambda+1}} \frac{\Gamma\left(1 + \frac{\lambda}{2} + j\right) \Gamma\left(j + \frac{\lambda}{2} + \frac{1}{2}\right)}{\Gamma\left(1 + \frac{\lambda}{2}\right) j!(r_1 s^2/2)^j} + R_n(s)$$

where

$$0 \leqq |R_n(s)| < \frac{1}{s^{\lambda+1}} \frac{\Gamma\left(2 + \frac{\lambda}{2} + n\right) \Gamma\left(n + \frac{3}{2} + \frac{\lambda}{2}\right)}{\Gamma\left(1 + \frac{\lambda}{2}\right) (n + 1)! \left(\frac{r_1 s^2}{2}\right)^{n+1}}.$$

That is, for fixed n, $R_n(s) = O(|s^2|^{-n-\lambda-2})$. Hence, except for constant factors,

$$N_1(s) = e^{-s^2 r_1/2} s^{-\lambda-1} \left\{ \sum_{j=0}^{n} \alpha_j \left(\frac{2}{r_1 s^2}\right)^j + R_n'(s) \right\} \tag{50}$$

where

$$\alpha_j = (-1)^j \frac{\Gamma\left(1 + \frac{\lambda}{2} + j\right)\Gamma\left(j + \frac{\lambda}{2} + \frac{1}{2}\right)}{\Gamma\left(1 + \frac{\lambda}{2}\right)(j!)\,\Gamma\left(\frac{\lambda}{2} + \frac{1}{2}\right)} = (-1)^j \frac{\Gamma(\lambda + 2j + 1)}{4^j (j!)\ \Gamma(\lambda + 1)}$$

and

$$|R'_n(s)| < \frac{\Gamma(\lambda + 2n + 3)(r_1 s^2)^{-n-1}}{2^{n+1}(n + 1)!\,\Gamma(\lambda + 1)}$$

ASYMPTOTIC EVALUATIONS FROM CONTOUR INTEGRALS

Part I. First Solution

Except for $\lambda = -2k$, the theory developed in the section "Solutions Using Method II" leads to the integral

$$y_1(x) = \int_{C_1} e^{x\zeta} \zeta^{-(\lambda+2)/2} \left(\zeta + \frac{r_1}{2}\right)^{(\lambda-1)/2} d\zeta$$

where $x > 0$ and C_1 is a contour described previously. The quantity $(r_1/2)^{(\lambda-1)/2}$ can be factored from the integrand to give

$$y_1(x) = \int_{C_1} e^{x\zeta} \zeta^{-(\lambda+2)/2} \left(1 + \frac{2\zeta}{r_1}\right)^{(\lambda-1)/2} d\zeta$$

which is also a solution of the differential equation.

If $x\zeta = e^{-i\pi} t$, then

$$y_1(x) = x^{\lambda/2} \int_{C_1'} e^{-t} (e^{-i\pi} t)^{-(\lambda+2)/2} \left(1 + \frac{2e^{-i\pi} t}{x r_1}\right)^{(\lambda-1)/2} dt \tag{51}$$

where C_1' is the path in the complex t plane described as follows:

(i) the line at an angle of α with the axis of positive reals from $+\infty$ to $|t| = rx$;

(ii) the circle of radius rx described in a positive direction about the origin;

(iii) the line of (i) above from $+rx$ to $+\infty$.

A form of Taylor's series with remainder from MacRobert (*9*, page 296) can be used to obtain

$$\left(1 + \frac{2e^{-i\pi} t}{r_1 x}\right)^{(\lambda-1)/2} = \sum_{j=0}^{n} \frac{\Gamma\left(\frac{\lambda}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{\lambda}{2} + \frac{1}{2} - j\right)(j!)} \left(\frac{2e^{-i\pi} t}{r_1 x}\right)^j + K_n(t, x) \tag{52}$$

where

$$K_n(t, x) = \left(\frac{2e^{-i\pi}t}{r_1 x}\right)^{n+1} \frac{\Gamma\left(\frac{\lambda}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{\lambda}{2} - \frac{1}{2} - n\right)(n!)} \cdot \int_0^1 (1 - u)^n \left(1 + \frac{2e^{-i\pi}tu}{r_1 x}\right)^{(\lambda-3)/2-n} du. \tag{53}$$

The validity of this remainder formula follows from the analytic nature of $[1 + (2e^{-i\pi}t/r_1x)]^{(\lambda-1)/2}$. The function inside the parentheses does not vanish because of the construction of C_1'.

The results of (51) and (52) can be combined to give

$$y_1(x) = x^{\lambda/2}\left\{\sum_{j=0}^{n} \frac{\Gamma\left(\frac{\lambda}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{\lambda}{2} + \frac{1}{2} - j\right)(j!)} \left(\frac{2}{r_1 x}\right)^j \int_{C_1'} e^{-t}(e^{-i\pi}t)^{j-(\lambda/2)-1}\, dt \right. \\ \left. + \frac{\Gamma\left(\frac{\lambda}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{\lambda}{2} - \frac{1}{2} - n\right)(n!)} \left(\frac{2}{r_1 x}\right)^{n+1} \int_{C_1'} e^{-t}(e^{-i\pi}t)^{n-(\lambda/2)} R_n(t, x)\, dt\right\} \tag{54}$$

where

$$R_n(t, x) = \int_0^1 (1 - u)^n \left(1 + \frac{2e^{-i\pi}tu}{r_1 x}\right)^{(\lambda-3)/2-n} du.$$

Now

$$\int_{C_1'} e^{-t}(e^{-i\pi}t)^{-\mu}\, dt = \frac{-2\pi i}{\Gamma(\mu)}$$

for all μ. A proof parallel to the ones used in Whittaker and Watson (*10*, page 245) and Copson (*7*, page 226) follows, with the additional assumption that $|\alpha| < \pi/2$.

Hence the first term on the right-hand side of (54) becomes

$$x^{\lambda/2}\left\{\sum_{j=0}^{n} \frac{\Gamma\left(\frac{\lambda}{2} + \frac{1}{2}\right)(-2\pi i)}{\Gamma\left(\frac{\lambda}{2} + \frac{1}{2} - j\right) j!\, \Gamma\left(1 + \frac{\lambda}{2} - j\right)} \left(\frac{2}{r_1 x}\right)^j\right\}. \tag{55}$$

The second term of the right-hand side of (54) can be written as

$$x^{\lambda/2} \frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{\lambda}{2}-\frac{1}{2}-n\right)(n!)} \left(\frac{2}{r_1 x}\right)^{n+1} I_n, \tag{56}$$

where

$$I_n = \int_{C_1'} e^{-t}(e^{-i\pi}t)^{n-(\lambda/2)} R_n(t, x)\, dt. \tag{57}$$

By Cauchy's theorem, C_1' can be deformed into the following path:

(i) the line making an angle of α with the positive axis of reals from ∞ to $|z| = \delta$, $\delta > 0$;

(ii) the circle around the origin of radius δ, with $\delta < rx$ so that this circle is enclosed with the circle part of C_1';

(iii) the line of (i) from $|z| = \delta$ to ∞.

Thus $I_n = I_n^1 + I_n^2 + I_n^3$ where I_n^i, $i = 1, 2, 3$, corresponds respectively to the three parts of the path (i), (ii), (iii).

For I_n^1, $t = ve^{i\alpha}$, $\delta \leqq v < \infty$

$$I_n^1 = -e^{i\alpha} e^{i(\alpha-\pi)(n-\lambda/2)} \int_\delta^\infty e^{-ve^{i\alpha}} v^{n-\lambda/2} R_n(ve^{i\alpha}, x)\, dv. \tag{58}$$

For I_n^3, $t = ve^{i(\alpha+2\pi)}$, $\delta \leqq v < \infty$

$$I_n^3 = e^{i\alpha} e^{i(\alpha+\pi)(n-\lambda/2)} \int_\delta^\infty e^{-ve^{i\alpha}} v^{n-\lambda/2} R_n(ve^{i(\alpha+2\pi)}, x)\, dv. \tag{59}$$

Now, the function

$$\left| 1 + \frac{2e^{i(\alpha\pm\pi)}vu}{r_1 x} \right|,$$

considered as a function of u, has a minimum value $(1 - \cos^2 \alpha)^{1/2}$ independent of v, r_1, x. Thus, for $n > \lambda/2 - 1$, both $R_n(ve^{i\alpha}, x)$ and $R_n(ve^{i(\alpha+2\pi)}, x)$ defined in (54) are bounded in absolute value by

$$(1 - \cos^2 \alpha)^{(\lambda/4-(3/4)-n/2)} \cdot \frac{1}{(n+1)}.$$

Hence

$$|I_n^1| + |I_n^3| \leqq [2/(n+1)]\,(1 - \cos^2 \alpha)^{(\lambda/4-(3/4)-n/2)} \int_\delta^\infty e^{-v \cos \alpha} v^{n-\lambda/2}\, dv. \tag{60}$$

For I_n^2, $t = \delta e^{i\theta}$, $\alpha \leqq \theta \leqq \alpha + 2\pi$

$$I_n^2 = ie^{i\pi[n-(\lambda/2)]}\delta^{(n-(\lambda/2)+1)} \int_\alpha^{\alpha+2\pi} e^{-\delta e^{i\theta}} (e^{i\theta})^{n-(\lambda/2)+1} R_n(\delta e^{i\theta}, x)\, d\theta \tag{61}$$

which tends to zero as $\delta \to 0$ for $n > (\lambda/2) - 1$. The results of (60) and (61) combined give

$$| I_n | \leqq \frac{2}{n+1} (1 - \cos^2 \alpha)^{(\lambda/4)-(3/4)-(n/2)} \int_0^\infty e^{-v \cos \alpha} v^{n-(\lambda/2)}\, dv$$

and, consequently,

$$\begin{aligned} | I_n | &\leqq \frac{2}{n+1} [\cos^2 \alpha(1 - \cos^2 \alpha)]^{(\lambda/4)-(3/4)-(n/2)} (\cos \alpha)^{1/2}\, \Gamma\left(n - \frac{\lambda}{2} + 1\right) \\ &< \frac{2 \cdot \Gamma\left(n - \frac{\lambda}{2} + 1\right)}{n+1} \cdot 2^{n-(\lambda/2)+(3/2)} \end{aligned} \tag{62}$$

if $\alpha = \pi/4$ is chosen.

Equations (54), (55), (56), and (62) give

$$\begin{aligned} y_1(x) = x^{\lambda/2} \Bigg\{ &\sum_{j=0}^{n} \frac{\Gamma\left(\frac{\lambda}{2} + \frac{1}{2}\right)(-2\pi i)}{\Gamma\left(\frac{\lambda}{2} + \frac{1}{2} - j\right)(j!)\, \Gamma\left(1 + \frac{\lambda}{2} - j\right)} \left(\frac{2}{r_1 x}\right)^j \\ &+ \frac{\Gamma\left(\frac{\lambda}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{\lambda}{2} - \frac{1}{2} - n\right)(n!)} \left(\frac{2}{r_1 x}\right)^{n+1} I_n \Bigg\}. \end{aligned} \tag{63}$$

The identities

$$\begin{cases} \Gamma(z)\Gamma(z + \frac{1}{2}) = 2^{1-2z}\, \pi^{1/2}\, \Gamma(2z) \\ \Gamma(z)\Gamma(1 - z) = \pi/\sin \pi z \end{cases} \tag{64}$$

lead, up to a multiplicative constant, to

$$y_1(x) = x^{\lambda/2} \left\{ \sum_{j=0}^{n} \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - 2j + 1)(j!)} \frac{1}{(2r_1 x)^j} + T_n(x) \right\} \tag{65}$$

where

$$| T_n(x) | < \left| \sin \pi \left(\frac{\lambda}{2} - \frac{1}{2}\right)\right| \left(\frac{1}{2 | r_1 x |}\right)^{n+1} \frac{| \Gamma(\lambda + 1) | \Gamma(2n + 2 - \lambda)}{(n + 1)! \pi} 2^{n - \frac{\lambda}{2} + \frac{5}{2}}$$

which is valid for n $(\lambda/2) - 1$, $\lambda \neq -2k$.

Note that the minimum, $(1 - \cos^2 \alpha)$, of the expression

$$\left| 1 + \frac{2e^{i(\alpha \pm \pi)} vu}{r_1 x} \right| = 1 - 2a \cos \alpha + a^2 u^2, a = \frac{2v}{r_1 x} \tag{66}$$

occurs at $u^* = \cos \alpha/a$. This is certainly valid for all $r_1 \neq 0$, $x > 0$, and $v > 0$. However, if $r_1 < 0$, a better lower bound can be found, for in this case $u^* = \cos \alpha/a < 0$, and hence, since only those values of u for which $0 \leqq u \leqq 1$ are of interest, the value obtained from (66) when $u = 0$ can be used as a lower bound. This new value leads to the following bound for I_n defined in (56):

$$| I_n | < \frac{2\Gamma(n - \lambda/2 + 1)}{n + 1} \cdot \frac{1}{(\cos \alpha)^{n - \lambda/2 + 1}} .$$

If α is picked arbitrarily close to zero, the bound in Eq. (65) becomes

$$| T_n(x) | < | \sin \pi(\lambda/2 - 1/2) | \left(\frac{1}{2 | r_1 x |}\right)^{n+1} \frac{2 | \Gamma(\lambda + 1) | \Gamma(2n + 2 - \lambda)}{(n + 1)! \pi} \tag{67}$$

which is a slightly better bound and is true for $n > \lambda/2 - 1$.

Part II. Second Solution

The case when $\lambda = -2k$ is only slightly different from the above work and will not be done here. The asymptotic series is the same as (65); however, the bound for $T_n(x)$ changes slightly. The result is outlined in the final section.

From the section "Solutions Using Method II" the second solution is, for $\lambda \neq 2k + 1$, k an integer,

$$y_2 (x) = \int_{C_2} e^{x\zeta} \zeta^{-\lambda/2-1} (\zeta + r_1/2)^{\lambda/2 - 1/2} d\zeta$$

where C_2 has been previously described. A change of variable, $\zeta + r_1/2$ into ζ, gives rise to a new path C_2' which can be considered as the path C_1 used for $y_1(x)$. Thus the results of Part I can be used to give (up to a multiplicative constant),

$$y_2(x) = e^{(-r_1 x)/2} x^{-\lambda/2 - 1/2} \left\{ \sum_{j=0}^{n} \gamma_j \left(\frac{2}{r_1 x}\right)^j + V_n(x) \right\}. \tag{68}$$

As before, use of (64) has been made to obtain for γ_j,

$$\gamma_j = (-1)^j \frac{\Gamma(\lambda + 2j + 1)}{4^j(j!)\Gamma(\lambda + 1)},$$

and

$$|V_n(x)| < \left(\frac{1}{2|r_1 x|}\right)^{n+1} \frac{\Gamma(2n + 3 + \lambda)}{(n + 1)!\,|\Gamma(\lambda + 1)|} \frac{2^{n+\lambda/2+2}}{|\cos(\pi\lambda/2|}$$

for $n > -\lambda/2 - 3/2$ and $\lambda \neq 2k + 1$. As above, the case $\lambda = 2k + 1$ can be worked out in analogy to $\lambda = -2k$ of Part I. The result is included in the final section.

If $r_1 > 0$, the bound for $V_n(x)$ can be improved just as the bound for $T_n(x)$ was improved for Part I when $r_1 < 0$. This result is also included in the final section.

It is clear from the expressions for $y_1(x)$ and $y_2(x)$ that they are linearly independent if $r_1 \neq 0$.

SUMMARY OF SOLUTIONS AND COMPARISONS

The substitution of s^2 for x in (65) and (68) gives two solutions for $L_s[N] = 0$. These are

$$W_1(s) = s^\lambda \left\{ \sum_{j=0}^{n} \delta_j \left(\frac{2}{r_1 s^2}\right)^j + T_n(s^2) \right\} \tag{69}$$

where

$$\delta_j = \frac{\Gamma(\lambda + 1)}{4^j(j!)\,\Gamma(\lambda - 2j + 1)}$$

and $T_n(s^2)$ is the function satisfying (65) or (67);

$$W_2(s) = e^{-r_1 s^2/2} s^{-\lambda-1} \left\{ \sum_{j=0}^{n} \gamma_j \left(\frac{2}{r_1 s^2}\right)^j + V_n(s^2) \right\} \tag{70}$$

where γ_j is defined following Eq. (68) and $V_n(s^2)$ is the function satisfying (68).

If $\lambda > -1$ and $r_1 > 0$, the solution (70) corresponds to the solution (50).

To obtain solutions for the original differential equation (29), the substitution of $s = At + F$, where $F = B - 2C/A$, is used in (69) and (70), and each of these equations is multiplied by $e^{-Ct/A}$. Thus, if $w_1(t)$ and $w_2(t)$ denote the two linearly independent solutions of (29), then

$$w_1(t) = e^{-Ct/A} (At + F)^\lambda \left\{ \sum_{j=0}^{n} \delta_j \left(\frac{2}{r_1(At + F)^2}\right)^j + T_n\left[(At + F)^2\right] \right\} \tag{71}$$

where δ_j is defined following Eq. (69), and

$$|T_n[(At+F)^2]| < \begin{cases} \left|\cos\frac{\pi\lambda}{2}\right| \left(\frac{1}{2|r_1|(At+F)^2}\right)^{n+1} 2^{n-\frac{\lambda}{2}+\frac{5}{2}} \frac{|\Gamma(\lambda+1)|\Gamma(2n+2-\lambda)}{(n+1)!\,\pi} \\ \qquad \text{for } n > \frac{\lambda}{2} - 1 \text{ and } \lambda \neq -2k; \\ \left(\frac{1}{2|r_1|(At+F)^2}\right)^{n+1} 2^{n-\frac{\lambda}{2}+\frac{5}{2}} \frac{\Gamma(2n+2-\lambda)}{\Gamma(-\lambda)(n+1)!} \\ \qquad \text{for } n > \frac{\lambda}{2} - 1 \text{ and } \lambda = -2k; \\ \left|\cos\frac{\pi\lambda}{2}\right| \left(\frac{1}{2|r_1|(At+F)^2}\right)^{n+1} \frac{2\,|\Gamma(\lambda+1)|\,\Gamma(2n+2-\lambda)}{(n+1)!\,\pi} \\ \qquad \text{for } n > \frac{\lambda}{2} - 1 \text{ and } r_1 < 0; \end{cases}$$

$$w_2(t) = \frac{e^{-Ct/A}\, e^{-r_1(At+F)^2/2}}{(At+F)^{\lambda+1}} \left\{ \sum_{j=0}^{n} \gamma_j \left(\frac{2}{r_1(At+F)^2}\right)^j + V_n\left[\left(At+F\right)^2\right] \right\} \tag{72}$$

where γ_j is defined following Eq. (68), and

$$|V_n[(At+F)^2]| < \begin{cases} \left(\frac{1}{2|r_1|(At+F)^2}\right)^{n+1} \frac{2^{n+(\lambda/2)+2}}{\left|\cos\frac{\pi\lambda}{2}\right|} \frac{\Gamma(2n+3+\lambda)}{(n+1)!\,|\Gamma(\lambda+1)|} \\ \qquad \text{for } n > -\frac{\lambda}{2} - \frac{3}{2} \quad \text{and} \quad \lambda \neq 2k+1; \\ \left(\frac{1}{2|r_1|(At+F)^2}\right)^{n+1} 2^{n+(\lambda/2)+3} \frac{\Gamma(2n+3+\lambda)}{(n+1)!\,\Gamma(\lambda+1)} \\ \qquad \text{for } n > -\frac{\lambda}{2} - \frac{3}{2} \quad \text{and} \quad \lambda = 2k+1; \\ \left(\frac{1}{2|r_1|(At+F)^2}\right)^{n+1} \frac{1}{\left|\cos\frac{\pi\lambda}{2}\right|} \frac{\Gamma(2n+3+\lambda)}{(n+1)!\,|\Gamma(\lambda+1)|} \end{cases}$$

$$\text{for } n > -\frac{\lambda}{2} - \frac{3}{2} \quad \text{and} \quad r_1 > 0.$$

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