Numer. Math. 19, 155—169 (1972)

Higher Order Convergence Results for the Rayleigh-Ritz Method Applied to Eigenvalue Problems: 2. Improved Error Bounds for Eigenfunctions*

J. G. Pierce and R. S. Varga

Received August 15, 1971

Abstract. The application of the Rayleigh-Ritz method for approximating the eigenvalues and eigenfunctions of linear eigenvalue problems in several dimensions is investigated. The object is to improve upon known error estimates for the approximate eigenfunctions. Results for the Galerkin approximation of the eigenfunctions are developed under varying assumptions on the boundary conditions and domain of definition of the eigenvalue problem. These results, coupled with a previous result relating Galerkin and Rayleigh-Ritz approximation of the eigenfunctions, are then used to obtain improved error estimates for the approximate eigenfunctions in the L^2 and uniform norms.

1. Introduction

Our concern here will be to develop high-order error estimates for the Rayleigh-Ritz approximation of the eigenvalues and eigenfunctions of the linear eigenvalue problem (cf. (2.1))

$$\mathcal{N}u(x)=\lambda\mathcal{M}u(x), \qquad x\in\Omega, \tag{1.1}$$

subject to the homogeneous boundary conditions ((cf. 2.2))

$$\mathcal{B}u(x)=0, \qquad x\in\partial\Omega. \tag{1.2}$$

For a discussion of some previous work on this problem, see Pierce and Varga [31]. The error estimates developed here for the approximate *eigenvalues* are generally not new. However, the corresponding estimates for the approximate *eigenfunctions* in the L^2 and uniform norms constitute a significant improvement over previous results. In [31], we obtained a general theorem (Theorem 4.2) which shows that the Rayleigh-Ritz and Galerkin approximations to the eigenfunctions of (1.1)–(1.2) are generally close in the relevant energy norm. The approach here is to use the results of Nitsche [27, 28], and Strang and Fix [34], relating to Galerkin approximation, to obtain the corresponding results for the Rayleigh-Ritz approximation of eigenfunctions of (1.1)–(1.2).

In Section 2, we summarize the basic theory for the problem (1.1)–(1.2), the Rayleigh-Ritz method, and the results of [31]. Section 3 deals with the one-dimensional eigenvalue problem. Error estimates are developed for the Rayleigh-Ritz approximation of the eigenvalues and eigenfunctions on subspaces $S_{n,p}^h$ satisfying a certain approximation theoretic property (cf. § 3). For example the Lg-splines of Jerome and Varga [22] and the Λ-splines of Jerome and Pierce [20] can be used to generate such $S_{n,p}^h$ spaces.

* This research was supported in part by AEC Grant (11-1)-2075.

In Section 4, the error estimates for the Rayleigh-Ritz eigenvalues and eigenfunctions of Section 3 are generalized for a number of different regions and boundary conditions in R^N and for particular finite-dimensional subspaces.

We shall throughout use the letter K to denote a generic constant. The parameters upon which K depends will be clear in each case.

2. Basic Results

Let Ω be an open bounded subset of R^N, $N \geqq 1$, with boundary $\partial\Omega$. We consider the eigenvalue problem (1.1)–(1.2), where $\mathcal{N}$ and $\mathcal{M}$ are the formally self-adjoint differential operators

$$\begin{aligned} \mathcal{N}u(x) &\equiv \sum_{|\alpha|\leqq n} (-1)^{|\alpha|} D^\alpha[p_\alpha(x) D^\alpha u(x)], \quad x \in \Omega, \\ \mathcal{M}u(x) &\equiv \sum_{|\alpha|\leqq r} (-1)^{|\alpha|} D^\alpha[q_\alpha(x) D^\alpha u(x)], \quad x \in \Omega, \end{aligned} \tag{2.1}$$

where $0 \leqq r < n$, and where we use the usual multi-index notation. The coefficient functions $p_\alpha(x)$ and $q_\alpha(x)$ are assumed to be real-valued functions of class $C^{|\alpha|}(\bar{\Omega})$. The homogeneous boundary conditions $\mathcal{B} \equiv \{B_j\}_{j=1}^n$ of (1.2) will consist of n linearly independent conditions of the form

$$B_j u(x) \equiv \sum_{|\alpha|\leqq 2n-1} m_{j,\alpha}(x) D^\alpha u(x) = 0, \quad 1 \leqq j \leqq n, \ x \in \partial\Omega. \tag{2.2}$$

In the case $N = 1$ with $\Omega = (a, b)$, we shall also allow the more general boundary conditions of the form

$$B_j u(x) \equiv \sum_{k=1}^{2n} \{m_{j,k} D^{k-1} u(a) + n_{j,k} D^{k-1} u(b)\} = 0, \quad 1 \leqq j \leqq 2n. \tag{2.2'}$$

In addition, we assume in this case that

$$p_n(x) \text{ and } q_r(x) \text{ do not vanish on } [a, b]. \tag{2.3}$$

For any nonnegative integer s, let $W_s^2(\Omega)$ denote the usual Sobolev space, with inner product

$$(u, v)_s \equiv \int_\Omega \Big\{ \sum_{|\alpha|\leqq s} D^\alpha u(x) D^\alpha v(x) \Big\} dx \quad \text{for all } u, v \in W_2^s(\Omega),$$

and with norm $\|\cdot\|_s \equiv (\cdot, \cdot)_s^{\frac{1}{2}}$, and let $\|u\|_\infty \equiv \sup_{x\in\Omega} |u(x)|$ for all $u(x)$ defined on Ω.

Let $\mathcal{D}$ be the linear space of all real-valued functions $u(x) \in C^{2n}(\bar{\Omega})$ satisfying the boundary conditions of (2.2). We make the following assumptions:

$$(\mathcal{N}u, v)_0 = (u, \mathcal{N}v)_0 \equiv (u, v)_N, \quad \text{for all } u, v \in \mathcal{D}, \tag{2.4}$$

$$(\mathcal{M}u, v)_0 = (u, \mathcal{M}v)_0 \equiv (u, v)_D, \quad \text{for all } u, v \in \mathcal{D}, \tag{2.5}$$

$$\|u\|_N^2 \geqq K \|u\|_D^2, \quad \text{and} \quad \|u\|_D^2 \geqq K \|u\|_0^2, \quad \text{for all } u \in \mathcal{D}. \tag{2.6}$$

Let H_N and H_D denote, respectively, the Hilbert space completions of $\mathcal{D}$ with respect to $\|\cdot\|_N$ and $\|\cdot\|_D$. Then, from (2.6), $H_N \subset H_D$, and we assume throughout that

$$\text{bounded sets in } H_N \text{ are precompact in } H_D. \tag{2.7}$$

An *eigenvalue* of (1.1)–(1.2) is a value of λ for which there exists a non-trivial solution, or *eigenfunction*, $u(x)$, of (1.1)–(1.2). It is known (Gould [13]) that, with the assumptions of (2.4)–(2.7), the eigenvalue problem (1.1)–(1.2) has countably many real eigenvalues $0<\lambda_1\leqq\lambda_2\leqq\cdots\leqq\lambda_k\leqq\lambda_{k+1}\leqq\cdots$, having no finite limit point, and a corresponding sequence of eigenfunctions $\{f_j(x)\}_{j=1}^{\infty}$, with $f_j\in H_N$, such that

$$\mathcal{N} f_j(x)=\lambda_j \mathcal{M} f_j(x), \quad j\geqq 1. \tag{2.8}$$

The eigenfunctions can be chosen to be orthonormal in H_D, i.e.,

$$(f_i, f_j)_D=\delta_{i,j}, \quad i,j\geqq 1, \tag{2.9}$$

and $\{f_j\}_{j=1}^{\infty}$ is complete in H_D.

In addition, when $N=1$, and $\Omega=(a,b)$, the assumptions of (2.3)–(2.6) are sufficient to guarantee that (2.7) is satisfied, and that $f_j(x)\in C^{2n}[a,b]$ for all $j\geqq 1$, for the more general boundary conditions of (2.2′) (cf. Brauer [8] and Kamke [24, 25]).

The eigenvalues and eigenfunctions of (1.1)–(1.2) can then be characterized as follows (cf. Collatz [10], Gould [13] and Mikhlin [26]):

$$\lambda_k=\min\left\{\frac{\|w\|_N^2}{\|w\|_D^2}: w\in H_N,\ w\not\equiv 0,\ (w,f_l)_D=0,\ 1\leqq l\leqq k-1\right\}=\|f_k\|_N^2. \tag{2.10}$$

The Rayleigh-Ritz method for obtaining approximate eigenvalues $\hat{\lambda}_k$ and approximate eigenfunctions $\hat{f}_k$ is then defined as follows. Given a finite dimensional subspace S_M of H_N, of dimension $M\geqq k$,

$$\hat{\lambda}_k\equiv\min\left\{\frac{\|w\|_N^2}{\|w\|_D^2}: w\in S_M,\ w\not\equiv 0,\ (w,\hat{f}_l)_D=0,\ 1\leqq l\leqq k-1\right\}=\|\hat{f}_k\|_N^2. \tag{2.11}$$

Now, let j be a fixed positive integer, and $\{S_{M_t}\}_{t=1}^{\infty}$ be a sequence of finite dimensional subspaces of H_N, with $\dim S_{M_t}\equiv M_t\geqq j$ for all $t\geqq 1$, such that

$$\lim_{t\to\infty}\ \inf_{w\in S_{M_t}}\|f_k-w\|_N=0, \quad 1\leqq k\leqq j. \tag{2.12}$$

Let $\hat{\lambda}_{j,t}$ and $\hat{f}_{j,t}$ be, respectively, the j-th approximate eigenvalue and eigenfunction of (1.1)–(1.2), obtained by applying the Rayleigh-Ritz method to S_{M_t}, $t\geqq 1$. Then we have the following convergence result (cf. Ciarlet, Schultz, and Varga [9], and Schultz [32]).

Theorem 2.1. If assumption (2.12) is satisfied, then $\hat{\lambda}_{j,t}$ converges to λ_j from above. Moreover, for all t sufficiently large (say $t\geqq t_j$), there exist j functions $\{\tilde{f}_{k,t}\}_{k=1}^{j}$ in S_{M_t} for which $\sum_{k=1}^{j}\|\tilde{f}_{k,t}-f_k\|_D^2<1$, and

$$\lambda_j\leqq\hat{\lambda}_{j,t}\leqq\lambda_j+\frac{\sum_{k=1}^{j}\|\tilde{f}_{k,t}-f_k\|_N^2}{\left(1-\left(\sum_{k=1}^{j}\|\tilde{f}_{k,t}-f_k\|_D^2\right)^{\frac{1}{2}}\right)^2} \quad \text{for all } t\geqq t_j. \tag{2.13}$$

We now state the two basic results of Pierce and Varga [31]. For a given function $g \in H_N$, let $\bar{g}_t$ denote the N-norm projection of g on the subspace S_{M_t}, $t \geqq 1$, i.e.,

$$(g - \bar{g}_t, w)_N = 0 \quad \text{for all } w \in S_{M_t}. \tag{2.14}$$

Such an element $\bar{g}_t$ always exists and is unique, since H_N is a Hilbert space. Equivalently, we have that

$$\|\bar{g}_t - g\|_N = \inf_{w \in S_{M_t}} \|w - g\|_N. \tag{2.15}$$

We remark that $\bar{g}_t$ can be alternatively viewed as the Galerkin approximation on S_{M_t} to the solution of the boundary value problem $\mathcal{N} u = \mathcal{N} g$, with the boundary conditions of (1.2). If $\bar{f}_{j,t}$ denotes the N-norm projection of the eigenfunction f_j on S_{M_t}, we then have the following result of [31].

Theorem 2.2. Let $\{S_{M_t}\}_{t=1}^{\infty}$ satisfy the assumption of (2.12). If λ_j is a simple eigenvalue of (1.1)–(1.2), then there exists a positive integer t_j and a positive constant K depending only on j such that

$$\|\bar{f}_{j,t} - \hat{f}_{j,t}\|_N \leqq K \|f_j - \bar{f}_{j,t}\|_D \quad \text{for all } t \geqq t_j. \tag{2.16}$$

If λ_j is an eigenvalue of multiplicity $\nu + 1$ of (1.1)–(1.2), let f be any element of the $(\nu+1)$-dimensional subspace of H_D spanned by $\{f_k\}_{k=j}^{j+\nu}$. Let $\{\hat{f}_{k,t}\}_{k=j}^{j+\nu}$ be a corresponding set of approximate eigenfunctions chosen so that $(\hat{f}_{m,t}, \hat{f}_{n,t})_D = \delta_{m,n}$, $j \leqq m$, $n \leqq j + \nu$, and let $\hat{f}_t \equiv \sum_{k=j}^{j+\nu} (f, \hat{f}_{k,t})_D \hat{f}_{k,t}$ be the D-norm projection of f onto the $(\nu+1)$-dimensional subspace of S_{M_t} spanned by the $\{\hat{f}_{k,t}\}_{k=j}^{j+\nu}$. Then, there exists a positive integer t_j and a positive constant K, depending only on j and ν, such that

$$\|\bar{f}_t - \hat{f}_t\|_N \leqq K \|f - \bar{f}_t\|_D, \quad \text{for all } t \geqq t_j. \tag{2.17}$$

For a proof of Theorem 2.2, and a more detailed discussion of the results of this section, see Pierce and Varga [31].

Continuing our discussion of basic results, we now specialize to the one-dimensional eigenvalue problem (1.1)–(1.2), with $\Omega = (a, b)$, and to the more general boundary conditions of (2.2'). We first make several remarks. In treating the general boundary conditions of (2.2'), we must distinguish between the so-called *essential* and *suppressible* (or natural) boundary conditions (cf. Collatz [10], p. 4). We write the boundary conditions of (2.2') in the following form:

$$\text{(2.18)} \quad \begin{aligned} &\text{(i)} \quad B_j^E(u) = \sum_{l=0}^{n-1} \{\alpha_{j,l} D^l u(a) + \beta_{j,l} D^l u(b)\} = 0, \quad 1 \leqq j \leqq k \quad (\text{if } k \neq 0), \\ &\text{(ii)} \quad B_j^S(u) = \sum_{l=0}^{2n-1} \{\gamma_{j,l} D^l u(a) + \delta_{j,l} D^l u(b)\} = 0, \quad 1 \leqq j \leqq 2n - k \quad (\text{if } k \neq 2n). \end{aligned}$$

The essential boundary conditions $B_j^E(u)$ consist of the maximum set of linearly independent boundary conditions involving only derivatives of $u(x)$ of order at most $n - 1$, evaluated at a and b, which can be obtained by linear combinations of the boundary conditions of (2.2′). Let $W_E^n[a, b]$ denote the subspace of all functions u in $W_2^n[a, b]$, satisfying the essential boundary conditions of (2.18 (i)). From the classical result of Kamke (cf. [25]), the inner products $(u, v)_N$ and $(u, v)_D$ can, from the assumptions of (2.4) and (2.5), be written in the form

$$(2.19)\qquad (u, v)_N = \int_b^a \left\{ \sum_{j=0}^{n} p_j(x) D^j u(x) D^j v(x) \right\} dx + N_0(u, v), \qquad u, v \in \mathscr{D},$$

and

$$(2.20)\qquad (u, v)_D = \int_a^b \left\{ \sum_{j=0}^{r} q_j(x) D^j u(x) D^j v(x) \right\} dx + D_0(u, v) \qquad u, v \in \mathscr{D},$$

where $N_0(u, v)$ and $D_0(u, v)$ are bilinear forms in the derivatives of u and v of order at most $n-1$, evaluated at $x=a$ and $x=b$. It then follows from the smoothness of the coefficients in (2.1) and Sobolev's Imbedding Theorem in one-dimension (cf. Yosida [37], p. 174), that there exists a positive constant K for which

$$(2.21)\qquad \|u\|_N \leqq K\|u\|_n \quad \text{for all } u \in \mathscr{D}.$$

Consequently, it follows that (2.19), (2.20), and (2.21) hold also for all $u, v \in W_E^n[a, b]$, and thus,

$$(2.22)\qquad \mathscr{D} \subset W_E^n[a, b] \subset H_N.$$

Next, as a consequence of the assumption (2.3), the eigenfunctions f_j of (2.8) satisfy $f_j \in \mathscr{D} \subset W_E^n[a, b]$, $j \geqq 1$, and we can rewrite (2.10) as follows:

$$(2.23)\quad \lambda_k = \min\left\{ \frac{\|w\|_N^2}{\|w\|_D^2} : w \in W_E^n[a, b],\ w \not\equiv 0,\ (w, f_l)_D = 0,\ 1 \leqq l \leqq k-1 \right\} = \|f_k\|_N^2.$$

We shall thus apply the Rayleigh-Ritz method to subspaces $\{S_{M_i}\}_{i=1}^{\infty}$ of $W_E^n[a, b]$ which satisfy the assumption of (2.12).

We make use later of one or both of the following assumptions: There exist positive constants K such that

$$(2.24)\qquad \|u\|_n \leqq K\|u\|_N \quad \text{for all } u \in \mathscr{D}$$

(i.e., $\mathscr{N}$ is *elliptic*, and with (2.21), $W_E^n[a, b] = H_N$ and $\|\cdot\|_n$ and $\|\cdot\|_N$ are equivalent norms on H_N), and/or

$$(2.25)\qquad \|u\|_D \leqq K\|u\|_r \quad \text{for all } u \in \mathscr{D}.$$

As examples of eigenvalue problems for which (2.24)–(2.25) are valid, consider first:

$$(2.26)\qquad D^4 u(x) = -\lambda D^2 u(x), \qquad a < x < b,$$

with the boundary conditions

$$(2.27)\qquad u(a) = Du(a) = u(b) = Du(b) = 0.$$

The assumptions of (2.4) and (2.5) are readily verified in this case. The conditions of (2.6) follows from the Rayleigh-Ritz inequality (cf. Hardy, Littlewood and Pólya [15], p. 184), and the boundary conditions of (2.27). In this case, all boundary conditions of (2.27) are essential, and $H = W_E^2[a, b]$. Moreover, it then follows that there exists a positive constant K such that

$$\|u\|_2 \leqq K\|u\|_N \quad \text{for all } u \in W_E^2[a, b]. \tag{2.28}$$

As another example, take the second order problem

$$-D\{p_1(x) D u(x)\} + p_0(x) u(x) = \lambda q_0(x) u(x), \quad a < x < b, \tag{2.29}$$

where $p_1 \in C^1[a, b]$, $p_0, q_0 \in C^0[a, b]$, and $p_1(x)$ and $q_0(x)$ are positive on $[a, b]$, with the boundary conditions

$$\begin{aligned} &\text{(i)} \quad \alpha_1 u(a) - \alpha_2 D u(a) = 0, \\ &\text{(ii)} \quad \beta_1 u(b) + \beta_2 D u(b) = 0, \end{aligned} \tag{2.30}$$

where the constants α_1, α_2, β_1, and β_2 are non-negative, with $\alpha_1 + \alpha_2 > 0$ and $\beta_1 + \beta_2 > 0$. This problem has been considered in detail in Birkhoff, de Boor, Swartz and Wendroff [4]. The assumptions of (2.4) and (2.5) are easily verified. By a suitable modification of (2.29), we may assume without essential loss of generality that $p_0(x) > 0$ on $[a, b]$, and the assumption of (2.6) then follows directly. Moreover, it is easily seen that there exists a positive constant K such that

$$\|u\|_1 \leqq K\|u\|_N \quad \text{for all } u \in W_E^1[a, b]. \tag{2.31}$$

The eigenvalues of (2.29)–(2.30) are all simple, and under the stronger assumption that $p_0 \in C^3[a, b]$ and $p_0, q_0 \in C^2[a, b]$, then all eigenfunctions f_j are in $C^4[a, b]$, $j \geqq 1$, a fact which will be used later.

3. Error Estimates—One Dimensional Case

We wish to obtain error estimates for the Rayleigh-Ritz approximations to the eigenvalues and eigenfunctions of the linear eigenvalue problem (1.1)–(1.2) in one dimension on subspaces of functions satisfying a rather general approximation-theoretic property. Let $S_{n,p}^h$ be a finite-dimensional subspace of $W_E^n[a, b]$, depending on a parameter h, with $0 < h \leqq 1$, and positive integers p and n, with $p \geqq 2n$, such that, for any $f \in W_2^p[a, b] \cap W_E^n[a, b]$, there exists $\tilde{f} \in S_{n,p}^h$, such that

$$\|f - \tilde{f}\|_k \leqq K h^{p-k} \|f\|_p, \quad \text{for all } 0 \leqq k \leqq n, \tag{3.1}$$

where K is independent of f and h. We restrict attention to the case $p \geqq 2n$ because the eigenfunctions f_j are all elements of $C^{2n}[a, b]$. Later in (4.1) for higher-dimensional problems, we assume only $p > n$ because of lack of regularity in the eigenfunctions.

We remark that such subspaces $S_{n,p}^h$ can be easily generated using spline functions of various types (cf. Lg-splines of Jerome and Varga [22], and (non-singular) Λ splines of Jerome and Pierce [20], for example). In fact, let S be any subspace of interpolating spline functions in $W_2^n[a, b]$ such that, for any $f \in W_2^n[a, b]$, the interpolation $\tilde{f}$, of f in S satisfies $D^j(f - \tilde{f})(a) = 0$ and $D^j(f - \tilde{f})(b) = 0$, for all $0 \leqq j \leqq n-1$. Then let $S_{n,p}^h$ be the subspace of S satisfying the essential boundary conditions of (2.18(i)). It then follows that for every $f \in W_E^n[a, b]$,

$\tilde{f}\in S^h_{n,p}$, and the estimate of (3.1) follows directly from estimates of $\|f-\tilde{f}\|_k$ obtained in [20] and [22], and extended in Hedstrom and Varga [17] to larger classes of integers p, where h is the maximum mesh size of the partition associated with the spline space. The above conditions on S can be weakened to the requirement that, for all $f\in W_2^n[a,b]$, $\tilde{f}$ interpolate f in the essential boundary conditions of (2.18(i)), provided $f-\tilde{f}$ satisfies a "second integral relation" (cf. [20], [22]).

Let $\bar{f}^h\in S^h_{n,p}$ be such that

$$\|f-\bar{f}^h\|_N=\inf\{\|f-g\|_N: g\in S^h_{n,p}\}. \tag{3.2}$$

In order to obtain error estimates for the Rayleigh-Ritz approximation of the eigenvalues and eigenfunctions of (1.1) on $S^h_{n,p}$, it follows from Theorem 2.2 that we first need estimates of $f-\bar{f}^h$. To obtain such estimates, we employ the methods of Nitsche [27, 28]. The following theorem is a slight generalization of the work of Nitsche to general boundary conditions. The proof is given partially for this reason and partially for the sake of completeness. In § 4, we shall use extensions of these results to problems in higher dimensions.

Theorem 3.1. Let $f\in W_2^p[a,b]\cap W_E^n[a,b]$, where $p\geqq 2n$. With the assumptions of (2.3)–(2.6) and (3.1), then

$$\|f-\bar{f}^h\|_0\leqq K h^p\|f\|_p, \tag{3.3}$$

and

$$\|f-\bar{f}^h\|_N\leqq K h^{p-n}\|f\|_p, \tag{3.4}$$

where K is independent of f and h. If, in addition (2.24) is satisfied, then

$$\|f-\bar{f}^h\|_l\leqq K h^{p-l}\|f\|_p,\qquad 0\leqq l\leqq n, \tag{3.5}$$

and

$$\|D^l(f-\bar{f}^h)\|_{L_\infty[a,b]}\leqq K h^{p-l-\frac{1}{2}}\|f\|_p,\qquad 0\leqq l\leqq n-1. \tag{3.6}$$

Proof. We first prove (3.4). Let $\tilde{f}^h\in S^h_{n,p}$ satisfy (3.1). Since, by hypothesis, $f\in W_2^p[a,b]\cap W_E^n[a,b]$, we have from (3.2), (2.21), and (3.1) that

$$\|f-\bar{f}^h\|_N\leqq\|f-\tilde{f}^h\|_N\leqq K\|f-\tilde{f}^h\|_n\leqq K h^{p-n}\|f\|_p,$$

which proves (3.4). Next, by definition,

$$(f-\bar{f}^h, w)_N=0\qquad\text{for all } w\in S^h_{n,p}.$$

Let g then be the solution of the boundary value problem

$$\mathcal{N}g=f-\bar{f}^h,\qquad \mathcal{B}g=0, \tag{3.7}$$

with B_j defined as in (2.2′). Because of the assumed smoothness of the coefficients in (2.1) and the assumption of (2.6), the boundary value problem (3.7) has an associated Green's function (cf. Ince [19], p. 254), and g, the solution of (3.7), is consequently an element of $W_2^{2n}[a,b]$. Now, let $\bar{g}^h$ be the projection of g onto the space $S^h_{n,p}$ in the norm $\|\cdot\|_N$. Thus,

$$(f-\bar{f}^h, g)_N=(f-\bar{f}^h, g-\bar{g}^h)_N,$$

and since $(f-\bar{f}^h, g)_N = (f-\bar{f}^h, \mathcal{N}g)_0 = \|f-\bar{f}^h\|_0^2$, we have that

$$\|f-\bar{f}^h\|_0^2 = (f-\bar{f}^h, g-\bar{g}^h)_N \leqq \|f-\bar{f}^h\|_N \cdot \|g-\bar{g}^h\|_N. \tag{3.8}$$

Since $g \in W_2^{2n}[a,b] \cap W_E^n[a,b]$, it follows from (3.4) with $p=2n$ that

$$\|g-\bar{g}^h\|_N \leqq K h^n \|g\|_{2n}. \tag{3.9}$$

But, because the problem of (3.7) has an associated Green's function and because of the assumption of (2.3), it can easily be shown that

$$\|g\|_{2n} \leqq K \|\mathcal{N}g\|_0 = K\|f-\bar{f}^h\|_0. \tag{3.10}$$

Combining (3.8)–(3.10) and (3.1), we have that

$$\|f-\bar{f}^h\|_0^2 \leqq K h^p \|f\|_p \cdot \|f-\bar{f}^h\|_0, \tag{3.11}$$

and the result of (3.3) follows.

To obtain the results of (3.5) and (3.6), we use the theory of interpolation spaces (cf. [17]). For, from the assumption of (2.24), (3.3), and (3.4) give us that

$$\|f-\bar{f}^h\|_0 \leqq K h^p \|f\|_p \quad \text{and} \quad \|f-\bar{f}^h\|_n \leqq K h^{p-n} \|f\|_p. \tag{3.12}$$

Hence, the first inequality of (3.12) states that the mapping T, defined by $Tf \equiv f-\bar{f}^h$, is a bounded linear transformation from $W_2^p[a,b]$ to $L_2[a,b]$, with norm at most $M_0 = K h^p$, while the second inequality gives T as a bounded linear transformation from $W_2^p[a,b]$ to $W_2^n[a,b]$, with norm at most $M_1 = K h^{p-n}$. Thus, from the theory of interpolation spaces (cf. [17], Theorem 2.2), T is a bounded linear transformation from $W_2^p[a,b]$ to $W_2^k[a,b]$, $0 \leqq k \leqq n$, with norm at most $M_0^{1-k/n} M_1^{k/n} = K h^{p-k}$, so that

$$\|Tf\|_k \equiv \|f-\bar{f}^h\|_k \leqq K h^{p-k} \|f\|_p, \tag{3.13}$$

which is the desired result of (3.5). The final result of (3.6) similarly follows from the known continuous embedding (cf. [17], Theorem 2.3, Eq. (2.20)) of the Besov space $B_2^{\frac{1}{2},1}[a,b] \equiv (L_2[a,b], W_2^1[a,b])_{\frac{1}{2},1}$ in $L_\infty[a,b]$, i.e.,

$$\|w\|_{L_\infty[a,b]} \leqq K(\|w\|_0 \cdot \|w\|_1)^{\frac{1}{2}} \quad \text{for all } w \in W_2^1[a,b]. \qquad \text{Q.E.D.}$$

We now consider the application of the Rayleigh-Ritz method to subspaces $S_{n,p}^h$ of $W_2^n[a,b]$ satisfying (3.1). Let $\{h_t\}_{t=1}^\infty$ be a sequence of real numbers, with $0 < h_t \leqq 1$ for all $t \geqq 1$, and such that $\lim_{t\to\infty} h_t = 0$, and, for each $t \geqq 1$, let $S_{n,p}^{h_t}$ satisfy (3.1). We let $\{S_{M_t}\}_{t=1}^\infty \equiv \{S_{n,p}^{h_t}\}_{t=1}^\infty$, and for a fixed integer $j>0$, we assume $\dim(S_{n,p}^{h_t}) \geqq j$. In our previous notation, we let $\bar{f}_{j,t} \equiv \bar{f}_j^{h_t}$, $\hat{f}_{j,t} \equiv f_j^{h_t}$, and $\hat{\lambda}_{j,t} \equiv \lambda_j^{h_t}$. Since $f_j \in C^{2n}[a,b]$, it follows from (3.4) that condition (2.12) is satisfied for the sequence of subspaces $\{S_{n,p}^{h_t}\}_{t=1}^\infty$, and the results of Theorems 2.1 and 2.2 are therefore applicable. To avoid undue notation, we eliminate the dependence upon t, and state the following convergence theorems for subspaces $S_{n,p}^h$ of $W_E^n[a,b]$ satisfying (3.1) with h sufficiently small.

Theorem 3.2. With the assumptions of (2.3)–(2.6), let λ_j and $f_j(x)$ be, respectively, the j-th eigenvalue and eigenfunction of (1.1)–(1.2), and assume that λ_j is a simple eigenvalue, and that $f_j \in W_2^p[a,b]$ with $p \geqq 2n$. If $S_{n,p}^h$ is a subspace of

$W_E^n[a, b]$ of dimension at least j, which satisfies (3.1), let λ_j^h and $f_j^h(x)$ denote, respectively, the j-th approximate eigenvalue and eigenfunction of (1.1)–(1.2) obtained from the Rayleigh-Ritz method applied to $S_{n,p}^h$. Then, for h sufficiently small, there exist constants K, depending only on j, such that

$$0 \leqq \lambda_j^h - \lambda_j \leqq K h^{2(p-n)}, \tag{3.14}$$

and

$$\|f_j - f_j^h\|_N \leqq K h^{p-n} \|f_j\|_p. \tag{3.15}$$

If $r=0$ in (2.1), then

$$\|f_j - f_j^h\|_0 \leqq K h^p \|f_j\|_p. \tag{3.16}$$

Finally, if both (2.24) and (2.25) are satisfied, then there exist constants K, depending only on j and l, such that

$$\|f_j - f_j^h\|_l \leqq K h^{p_l} \|f_j\|_p, \quad p_l = \min(p-r, p-l), \quad 0 \leqq l \leqq n, \tag{3.17}$$

and

$$\|D^l(f_j - f_j^h)\|_{L_\infty[a,b]} \leqq K h^{q_l} \|f_j\|_p, \quad q_l = \min(p-r, p-l-\tfrac{1}{2}), \quad 0 \leqq l \leqq n-1. \tag{3.18}$$

Proof. The result of (3.14) follows directly from Theorem 2.1, (3.4), and the inequality of (2.6). To prove (3.15), we have from the triangle inequality that

$$\|f_j - f_j^h\|_N \leqq \|f_j - \bar{f}_j^h\|_N + \|\bar{f}_j^h - f_j^h\|_N.$$

But, from Theorem 2.2, and (2.6),

$$\|\bar{f}_j^h - f_j^h\|_N \leqq K\|f_j - \bar{f}_j^h\|_D \leqq K\|f_j - \bar{f}_j^h\|_N,$$

and applying (3.4) we obtain (3.15). Now, from the triangle inequality,

$$\|f_j - f_j^h\|_0 \leqq \|f_j - \bar{f}_j^h\|_0 + \|\bar{f}_j^h - f_j^h\|_0.$$

From (2.6) and (2.16) of Theorem 2.2 with $r=0$,

$$\|\bar{f}_j^h - f_j^h\|_0 \leqq K\|\bar{f}_j^h - f_j^h\|_N \leqq K\|f_j - \bar{f}_j^h\|_0.$$

Thus, (3.16) follows directly from (3.3) of Theorem 3.1. To prove (3.17), we have from the triangle inequality that

$$\|f_j - f_j^h\|_l \leqq \|f_j - \bar{f}_j^h\|_l + \|\bar{f}_j^h - f_j^h\|_l.$$

By assumptions (2.24) and (2.25), and (2.16) of Theorem 2.2,

$$\|\bar{f}_j^h - f_j^h\|_l \leqq \|\bar{f}_j^h - f_j^h\|_n \leqq K\|\bar{f}_j^h - f_j^h\|_N \leqq K\|f_j - \bar{f}_j^h\|_D \leqq K\|f_j - \bar{f}_j^h\|_r.$$

The result of (3.17) then follows from (3.5) of Theorem 3.1. The result of (3.18) follows similarly, using Sobolev's inequality in one dimension, and (3.6) of Theorem 3.1. Q.E.D.

The convergence rates obtained in Theorem 3.2 are essentially unchanged if λ_j is a multiple eigenvalue of (1.1)–(1.2). We state the following without proof.

Theorem 3.3. With the assumptions of (2.3)–(2.6), suppose that λ_j is an eigenvalue of (1.1)–(1.2) of multiplicity $\nu+1$, $\nu\geqq 1$. Let $\{f_k(x)\}_{k=j}^{j+\nu}$ be a corresponding set of eigenfunctions, chosen orthonormal in the D-norm, and assume that $f_k\in W_2^p[a,b]$, $j\leqq k\leqq j+\nu$, with $p\geqq 2n$. Let $S_{n,p}^h$ be a finite dimensional subspace of $W_E^n[a,b]$ of dimension at least $j+\nu$, satisfying (3.1). Let $\{\lambda_k^h\}_{k=j}^{j+\nu}$ be the corresponding set of approximate eigenvalues and $\{f_k^h\}_{k=j}^{j+\nu}$ a corresponding set of approximate eigenfunctions, orthonormal in the D-norm, obtained by applying the Rayleigh-Ritz method to $S_{n,p}^h$. Let $f(x)$ be any element of the $(\nu+1)$-dimensional subspace of $\mathscr{D}$ spanned by the $\{f_k(x)\}_{k=j}^{j+\nu}$, with $\|f\|_D=1$ and let $\hat{f}^h(x)\equiv\sum_{k=j}^{j+\nu}(f,f_k^h)_D f_k^h$. Then, for all h sufficiently small, there exists a positive constant K such that

$$(3.19)\qquad 0\leqq\lambda_k^h-\lambda_k\leqq Kh^{2(p-n)},\qquad j\leqq k\leqq j+\nu.$$

Moreover, the error estimates (3.15)–(3.18) of Theorem 3.2 hold analogously for $f-\hat{f}^h$.

The error estimates developed in Theorems 3.2 and 3.3 for the approximate *eigenvalues* are generally not new. For particular subspaces $S_{n,p}^h$, such results have previously been obtained by Wendroff [36], Birkhoff, de Boor, Swartz and Wendroff [4], and Ciarlet, Schultz and Varga [9]. As a consequence of the explicit calculations of Birkhoff and de Boor [3] for the example of (2.29)–(2.30), for the case of cubic spline functions, these results appear to be best possible with respect to the exponent of h.

The corresponding estimates for the approximate eigenfunctions, however, give a rather significant improvement of the results of [4] and [9]. For example, if $r=0$, the estimates of [9] for $\|f_j-f_j^h\|_0$, with $f_j\in W_2^{2m}[a,b]$, and subspaces of type $S_{n,2m}^h$, are improved here from $\mathcal{O}(h^{2m-n})$ to $\mathcal{O}(h^{2m})$. These results are moreover essentially independent of the assumption made in both [4] and [9] that λ_j is a simple eigenvalue, and that $\lambda_1<\lambda_2<\cdots<\lambda_j$. Finally, it is clear from approximation-theoretic arguments that these estimates are best possible with respect to the exponent of h.

In case $r>0$, the L^2-norm estimates of Theorems 3.2 and 3.3 appear to be best possible, but we have not been able to prove this. The uniform norm estimates of Theorems 3.2 and 3.3 are also best possible for $f_j(x)$ in class $W_2^p[a,b]$. With the stronger assumption that $f_j\in C^p[a,b]$, one would expect to be able to prove that the exponent q_l of (3.18) could be *increased* to $q_l\equiv\min(p-r,p-l)$, $0\leqq l\leqq n-1$. Such a result follows immediately from the proof of Theorem 3.2, provided one has an inequality of the following form:

$$(3.20)\qquad \|D^l(f-\bar{f}^h)\|_{L^\infty[a,b]}\leqq Kh^{p-l},\quad 0\leqq l\leqq n-1,\quad \text{for all}\quad f\in C^p[a,b],$$

where K depends on l and f, but is independent of h, and $\bar{f}^h$ is the N-norm projection of f onto $S_{n,p}^h$. While such results, as in (3.20), are not available in general, the inequality is known to be valid in certain cases. If the spaces $S_{n,p}^h$ are chosen to be Λ-spline spaces, with Λ a differential operator chosen in relation to the operator $\mathscr{N}$ of (1.1), and provided that, an inequality of the form

$$(3.21)\qquad \|D^l(f-\tilde{f})\|_{L^\infty[a,b]}\leqq Kh^{p-l},\quad 0\leqq l\leqq n-1,\quad \text{for all } f\in C^p[a,b]$$

is valid, for an interpolation $\tilde{f} \in S^h_{n,p}$ of f, then (3.20) follows. We remark that the inequality (3.21) has recently been shown to hold for subspaces of polynomial splines with uniform mesh spacing, and Hermite interpolating L-splines (cf. Swartz and Varga [35]).

To be more precise, suppose the operator $\mathcal{N}$ is of the following form,

$$\mathcal{N} u(x) = \Lambda u(x) + R u(x), \qquad \text{for all } u \in C^{2n}[a, b], \tag{3.22}$$

with Λ a formally self-adjoint, non-singular differential operator, and R a differential operator of form

$$R u(x) = \sum_{0 \leqq i+j \leqq n} (-1)^i D^i (\sigma_{i,j}(x) D^j u(x)), \qquad \text{for all } u(x) \in C^{2n}[a, b], \tag{3.23}$$

with $\sigma_{i,j}(x) = \sigma_{j,i}(x)$ and $\sigma_{i,j} \in C^{\max(i,j)}[a, b]$, $0 \leqq i+j \leqq n$. Then, letting $S^h_{n,2n}$ be a subspace of Λ-spines, the validity of inequality (3.21) implies that of (3.20), for $p = 2n$. This results from the fact that the interpolation of f and the N-norm projection of f in $S^h_{n,2n}$ are very close in the N-norm. The proof is a straightforward generalization of the results of Perrin, Price, and Varga [29].

If the eigenfunctions $f_j(x)$ of (1.1)–(1.2) are smoother, i.e., $f_j \in C^p[a, b]$, with $p = 2n + 2q$, q a positive integer, the inequality of (3.20) can be again shown to be valid in certain cases, when $S^h_{n,p}$ is chosen properly. The proof is based on a construction of Hulme [18], and Perrin, Price and Varga [29]. One defines an "interpolation" of f_j which again can be shown to be extremely close to the N-norm projection of f_j. Important special cases in which (3.20) is valid are when $\Lambda \equiv D^{2n}$, in (3.22), and the $S^h_{n,p}$ are chosen to be natural polynomial splines or piecewise-polynomial Hermite spaces of order $2n + 2q$, on a uniform mesh.

As an application of these results, consider the example of (2.29)–(2.30). Here $n = 1$, and $r = 0$, and the assumption of (2.24) is satisfied. If the eigenfunctions of (2.29)–(2.30) are of class $C^4[a, b]$, then taking $\Lambda = D^2$ and $S^h_{1,4}$ to be subspaces of cubic polynomial splines, or piecewise-cubic Hermite polynomials on a uniform mesh, we obtain (cf. Pierce [30]) that

$$\|f_j - f_j^h\|_{L^\infty[a,b]} = \mathcal{O}(h^{4-l}), \qquad l = 0,$$

as conjectured in [4]. If the eigenfunctions, f_j, of (2.29)–(2.30) are smoother, e.g., if $f_j \in C^{2m}[a, b]$, then using subspaces of polynomial splines or piecewise Hermite polynomials of degree $2m - 1$, on a uniform mesh, we have that

$$\|f_j - f_j^h\|_{L^\infty[a,b]} = \mathcal{O}(h^{2m}).$$

For the example of (2.26)–(2.27), $n = 2$ and $r = 1$, and the associated eigenfunctions f_j are in $C^\infty[a, b]$. Therefore, using subspaces of polynomial splines or piecewise Hermite polynomials of degree $2m - 1$ with $m \geqq 2$ on a uniform mesh, we have that

$$\|D^l(f_j - f_j^h)\|_{L^\infty[a,b]} = \mathcal{O}(h^{2m-1}), \qquad l = 0, 1.$$

4. Error Estimates — Multidimensional Case

The arguments made in Theorems 3.2 and 3.3 carry over easily to multidimensional problems. That is, based upon the results of Theorem 2.2, known error estimates for Galerkin approximation can be used directly to obtain improved error estimates for the Rayleigh-Ritz approximation of the eigenfunctions of (1.1)–(1.2).

We shall consider two classes of problems (1.1)–(1.2) with varying assumptions on the region Ω, and the boundary conditions of (2.2):

I. Ω is an arbitrary bounded open subset in R^N, and the boundary conditions of (2.2) are all *natural* or suppressible, i.e.,

$$D^\alpha u(x) = 0, \quad x \in \partial\Omega, \quad \text{for all } n \leqq |\alpha| \leqq 2n-1.$$

II. Ω is a rectangular parallelopiped, and the boundary conditions of (2.2) are all *essential*, i.e.,

$$D^\alpha u(x) = 0, \quad x \in \partial\Omega, \quad \text{for all } 0 \leqq |\alpha| \leqq n-1.$$

We shall make use of the error estimates of Schultz [32], and Strang and Fix [34] for Galerkin approximations in both cases.

We assume in both cases above that the conditions of (2.4)–(2.6) and (2.24)–(2.25) are all satisfied. Since in both cases,

$$(u, v)_N = \sum_{|\alpha| \leqq n} \Big\{ \int_\Omega p_\alpha(x) D^\alpha u(x) D^\alpha v(x)\, dx \Big\}, \quad u \in \mathscr{D},$$

it follows from (2.24) and the assumption that $p_\alpha \in C^{|\alpha|}(\bar{\Omega})$, that $\|\cdot\|_N$ and $\|\cdot\|_n$ are equivalent norms on H_N.

I. In this case, $H_N = W_2^n(\Omega)$, i.e., the elements of H_N need satisfy no boundary conditions. In analogy with § 3, let $S_{n,p}^h$ be a finite-dimensional subspace of $W_2^n(\Omega)$, depending on a parameter h, with $0 < h \leqq 1$, and positive integers p and n, with $p > n$, such that for any $f \in W_2^p(\Omega)$, there exists an $\tilde{f} \in S_{n,p}^h$ for which

$$\|f - \tilde{f}\|_k \leqq K h^{p-k} \|f\|_p, \quad \text{for all } 0 \leqq k \leqq n, \tag{4.1}$$

where K is independent of f and h. Based upon an argument of Schultz, Strang and Fix [34] have shown that the N-norm or Galerkin projection $\bar{f}^h$ of f on $S_{n,p}^h$, defined by

$$(f - \bar{f}^h, w)_N = 0, \quad \text{for all } w \in S_{n,p}^h, \tag{4.2}$$

then satisfies the following inequalities:

$$\|f - \bar{f}^h\|_k \leqq K h^\sigma \|f\|_p, \quad \text{where } \sigma \equiv \min(p-k, 2(p-n)), \quad 0 \leqq k \leqq n. \tag{4.3}$$

Using (4.3) as in the proof of Theorem 3.2, it follows easily that the Rayleigh-Ritz approximate eigenfunction f_j^h in $S_{n,p}^h$ of the j-th eigenfunction, f_j, satisfies

$$\|f_j - f_j^h\|_l \leqq K h^{\tau_l} \|f\|_p, \quad \tau_l \equiv \min(p-r, p-l, 2(p-n)), \quad \text{for all } 0 \leqq l \leqq n, \tag{4.4}$$

for h sufficiently small. Similarly, using (4.1), it follows that the eigenvalue error estimates for $S_{n,p}^h$ are still of the form (cf. (3.14))

$$0 \leqq \lambda_j^h - \lambda_j \leqq K h^{2(p-n)}. \tag{4.5}$$

We remark that subspaces $S^h_{n,p}$ of $W^n_2(\Omega)$ satisfying the approximation-theoretic estimates of (4.1) can be explicitly constructed from the methods described in Strang and Fix [34], and Bramble and Hilbert [7]. In addition, as mentioned in [34], the estimates of (4.4)–(4.5) are also valid for eigenvalue problems defined in the unit cube of R^N, which can be extended periodically over all of R^N by means of periodic boundary conditions in (2.2).

In this periodic case, when $\mathcal{N}$ and $\mathcal{M}$ are differential operators with *constant coefficients*, Strang and Fix [34], using methods of Fourier analysis, have proved the stronger result:

$$\|f_j - f_j^h\|_l \leqq K h^{p-l} \|f\|_p, \quad 0 \leqq l \leqq n. \tag{4.6}$$

From approximation-theoretic arguments, this result is best possible with respect to the exponent of h. However, the result of (4.6) is not valid in the nonconstant coefficient case. More precisely, Strang and Fix have shown in this case that the result of (4.3) for Galerkin approximation is best possible with respect to the exponent of h. It then follows directly from Theorem 2.2, that the estimate of (4.4) is best possible in the case $r=0$. For $r>0$, it appears that the estimate of (4.4) is best possible with respect to the exponent of h (cf. Hald and Widlund [14]).

II. In case II, $H_N \supset \mathring{W}^n_2(\Omega)$, where $\mathring{W}^n_2(\Omega)$ is defined to be the closure in $\|\cdot\|_n$ of all infinitely differentiable functions with compact support in Ω. We require that any eigenfunction of (1.1)–(1.2) be an element of $\mathring{W}^n_2(\Omega)$. The major practical difficulty is in finding finite-dimensional subspaces of $\mathring{W}^n_2(\Omega)$ for arbitrary Ω satisfying an approximation-theoretic property similar to (4.1).

With Ω a rectangular parallelopiped in R^N, finite dimensional subspaces $S^h_{n,p}$ of $\mathring{W}^n_2(\Omega)$ satisfying (4.1) have been constructed by Schultz [32] by means of tensor products of one-dimensional spline functions. An extension of Schultz's argument (as presented in Strang and Fix [34]) can then be used to show that the error estimates of (4.3), and thus of (4.4) and (4.5) are again valid.

We remark that, when Ω is a rectangular parallelopiped, the results for cases I and II can be combined so that boundary conditions of essential type can be specified on certain faces of Ω, while boundary conditions of natural type are specified on the remaining faces of Ω, i.e., the boundary conditions of I and II can be mixed on $\partial\Omega$.

Finally, when Ω is a bounded open subset of R^N which is contained in a rectangular parallelopiped, i.e., $\Omega \supset \mathop{\times}\limits_{i=1}^{N} (a_i, b_i)$, one can generate finite-dimensional subspaces of $\mathring{W}^n_2(\Omega)$ in the following way due to Harrick [16] and Schultz [32]. Assume that there exists a function $\theta \in C^1(\bar{\Omega})$, with $\theta(x)>0$ for all $x\in\Omega$ and $\sum_{i=1}^{N} |D_i\theta(x)| \neq 0$ for all $x\in\Omega$ (where $D_i \equiv \partial/\partial x_i$), and such that $\partial\Omega = \{x\in R^N : \theta(x)=0\}$. Then, the restriction to Ω of $(\theta(x))^n g(x)$ for any $g\in W^n_2(\mathop{\times}\limits_{i=1}^{N}(a_i, b_i))$ is an element of $\mathring{W}^n_2(\Omega)$. In particular, if $g\in W^n_2(\mathop{\times}\limits_{i=1}^{N}(a_i, b_i))$ is a tensor product of one-dimensional spline functions, the resulting collection, say $S^h_{n,p}$, is a finite-dimensional subspace of $\mathring{W}_2^n(\Omega)$. In this spline case, Schultz [32] has shown that the Galerkin estimate $\bar{f}^h$ of f in $S_{n,p}^h$ satisfies, for $p \geqq 2n$,

$$\|f - \bar{f}^h\|_n \leqq K h^{p-2n} \|f\|_p, \tag{4.7}$$

provided that $f \in W_2^p(\Omega) \cap \mathring{W}_2^n(\Omega)$. The above Galerkin estimate, however, does not lead to improved eigenfunction error bounds, in contrast with cases I and II of this section, and the one-dimensional theory of § 3. It remains an open question if the Galerkin error estimates of (4.7) are best possible.

References

1. Ahlberg, J. H., Nilson, E. N., Walsh, J. L.: The theory of splines and their applications. New York: Academic Press 1967.
2. Birkhoff, G., Boor, C. de: Error bounds for spline interpolation. J. Math. Mech. **13**, 827–836 (1964).
3. Birkhoff, G., Boor, C. de: Piecewise polynomial interpolation and approximation. Approximation of functions (H. L. Garabedian, ed.), pp. 164–190. Amsterdam: Elsevier Pub. Co. 1965.
4. Birkhoff, G., Boor, C. de, Swartz, B., Wendroff, B.: Rayleigh-Ritz approximation by piecewise cubic polynomials. SIAM J. Numer. Anal. **3**, 188–203 (1966).
5. Birkhoff, G., Schultz, M. H., Varga, R. S.: Piecewise Hermite interpolation in one and two variables with applications to partial differential equations. Numer. Math, **11**, 232–256 (1968).
6. Boor, C. de: Uniform approximation by splines. J. Approx. Theory **1**, 219–235 (1968).
7. Bramble, J. H., Hilbert, S. R.: Bounds for a class of linear functionals with applications to Hermite interpolation. Numer. Math. **16**, 362–369 (1971).
8. Brauer, F.: Singular self-adjoint boundary value problems for the differential equation $Lx = \lambda Mx$. Trans. Amer. Math. Soc. **88**, 331–345 (1958).
9. Ciarlet, P. G., Schultz, M. H., Varga, R. S.: Numerical methods of high-order accuracy for nonlinear boundary value problems. III. Eigenvalue problems. Numer. Math. **12**, 120–133 (1968).
10. Collatz, L.: The numerical treatment of differential equations, 3rd. ed. Berlin-Göttingen-Heidelberg: Springer 1960.
11. Courant, R., Hilbert, D.: Methods of mathematical physics. I. New York: Interscience 1953.
12. Dailey, J. W., Pierce, J. G.: Error bounds for the Galerkin method applied to singular and nonsingular boundary value problems (to appear).
13. Gould, S. H.: Variational methods for eigenvalue problems. Toronto: University of Toronto Press 1966.
14. Hald, O., Widlund, P.: On the eigenvalue problems for the finite element variational method (to appear).
15. Hardy, G. H., Littlewood, J. E., Pólya, G.: Inequalities, 2d. ed. Cambridge: Cambridge University Press 1952.
16. Harrick, I. I.: Approximation of functions which vanish on the boundary of a region, together with their partial derivatives, by functions of special type. Akad. Nauk. SSSR Izv. Sibirsk, Otd. **4**, 408–425 (1963).
17. Hedstrom, G. W., Varga, R. S.: Application of Besov spaces to spline approximation. J. Approx. Theory **4**, 295–327 (1971).
18. Hulme, B. L.: Interpolation by Ritz approximation. J. Math. Mech. **18**, 337–342 (1968).
19. Ince, E. L.: Ordinary differential equations. New York: Dover Publications 1948.
20. Jerome, J. W., Pierce, J. G.: On splines associated with singular self-adjoint differential operators. J. Approx. Theory (to appear).

21. Jerome, J. W., Schumaker, L. L.: On Lg-splines. J. Approx. Theory **2**, 29–49 (1969).
22. Jerome, J. W., Varga, R. S.: Generalizations of spline functions and applications to nonlinear boundary value and eigenvalue problems. Theory and applications of spline functions (T. N. E. Greville, ed.), pp. 103–155. New York: Academic Press 1969.
23. Johnson, O.: Error bounds for Sturm-Liouville eigenvalue approximations by several piecewise cubic Rayleigh-Ritz methods. SIAM J. Numer. Anal. **6**, 317–333 (1969).
24. Kamke, E. A.: Über die definiten selbstadjungierten Eigenwertaufgaben bei gewöhnlichen linearen Differentialgleichungen. II, III. Math. A. **46**, 231–286 (1940).
25. Kamke, E. A.: Über die definiten selbstadjungierten Eigenwertaufgaben bei gewöhnlichen linearen Differentialgleichungen. IV. Math. Z. **48**, 67–100 (1942).
26. Mikhlin, S. G.: Variational methods in mathematical physics. New York: Macmillan and Co. 1964.
27. Nitsche, J.: Ein Kriterium für die Quasi-Optimalität des Ritzschen Verfahrens. Numer. Math. **11**, 346–348 (1968).
28. Nitsche, J.: Verfahren von Ritz und Spline-Interpolation bei Sturm-Liouville-Randwertproblemen. Numer. Math. **13**, 260–265 (1969).
29. Perrin, F. M., Price, H. S., Varga, R. S.: On higher-order numerical methods for nonlinear two-point boundary value problems. Numer. Math. **13**, 180–198 (1969).
30. Pierce, J. G.: Higher order convergence results for the Rayleigh-Ritz method applied to a special class of eigenvalue problems. Thesis, Case Western Reserve University (1969).
31. Pierce, J. G., Varga, R. S.: Higher order convergence results for the Rayleigh-Ritz method applied to eigenvalue problem. I. Estimates relating Rayleigh-Ritz and Galerkin approximations to eigenfunctions. SIAM J. Numer. Anal. (to appear, V. 9, March 1972).
32. Schultz, M. H.: Multivariate spline functions and elliptic problems, Approximation with special emphasis on spline functions (ed. I. J. Schoenberg), pp. 279–347. New York: Academic Press 1969.
33. Schultz, M. H., Varga, R. S.: L-splines. Numer. Math. **10**, 345–369 (1967).
34. Strang, Gilbert, Fix, George: An analysis of the finite element method. Prentice-Hall, Inc. (to appear).
35. Swartz, B. K., Varga, R. S.: Error bounds for spline and L-spline interpolation. J. Approx. Theory (to appear).
36. Wendroff, B.: Bounds for eigenvalues of some differential operators by the Rayleigh-Ritz method. Math. Comp. **19**, 218–224 (1965).
37. Yosida, K.: Functional analysis. New York: Academic Press 1965.

John G. Pierce
Department of Mathematics
University of Southern California
Los Angeles, California
U.S.A.

R. S. Varga
Department of Mathematics
Kent State University
Kent, Ohio
U.S.A.

Universitätsdruckerei H. Stürtz AG Würzburg
Printed in Germany