Minimal G-Functions. II

DAVID H. CARLSON
Oregon State University
Corvallis, Oregon

AND

RICHARD S. VARGA*
Kent State University
Kent, Ohio

Communicated by Richard S. Varga

ABSTRACT

The concept of a G-function, introduced by Nowosad and Hoffman, gives an appropriate setting for many generalizations of the Gerschgorin Circle Theorem. In this paper, we extend our previous results for minimal G-functions to the partitioned matrix case.

1. G-FUNCTIONS

For n a fixed positive integer, $n \geqslant 2$, let $\mathbb{C}^n$ denote the n-dimensional vector space of all column vectors $x = (x_1, x_2, \ldots, x_n)^T$, and let $\mathbb{C}^{n,n}$ denote the set of all $n \times n$ complex matrices. Let $\mathscr{P}_n$ be the collection of all functions $f = (f_1, f_2, \ldots, f_n)$ such that for each i, $i = 1, 2, \ldots, n$, f_i: $\mathbb{C}^{n,n} \to \mathbb{R}_+$, i.e. $0 \leqslant f_i(A) < \infty$ for each $A \in \mathbb{C}^{n,n}$, and such that f_i depends only on the moduli of the off-diagonal entries of the matrices, i.e. if $B = (b_{i,j}) \in \mathbb{C}^{n,n}$ and $A = (a_{i,j}) \in \mathbb{C}^{n,n}$ satisfy $|b_{i,j}| = |a_{i,j}|$ for all $i, j = 1, 2, \ldots, n$, $i \neq j$, then $f_i(B) = f_i(A)$, $i = 1, 2, \ldots, n$.

As in Nowosad [8], Hoffman [4], and Carlson and Varga [1], we say that $f \in \mathscr{P}_n$ is a *G-function* if, for any $A = (a_{i,j}) \in \mathbb{C}^{n,n}$ satisfying

$$|a_{i,i}| > f_i(A), \qquad i = 1, 2, \ldots, n,$$

A is nonsingular. Let $\mathscr{G}_n$ denote the set of all G-functions f in $\mathscr{P}_n$. For example, if $x \in \mathbb{C}^n$ has positive components, written $x > 0$, then $r^x = (r_1{}^x, r_2{}^x, \ldots, r_n{}^x)$ and $c^x = (c_1{}^x, c_2{}^x, \ldots, c_n{}^x)$ are well-known G-functions,

* Research supported in part by AEC Grant (11-1)-2075.

where

$$r_i^x(A) \equiv \frac{1}{x_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}| x_j, \qquad c_i^x(A) \equiv \frac{1}{x_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{j,i}| x_j, \qquad i = 1, 2, \ldots n,$$

for any $A = (a_{i,j}) \in \mathbb{C}^{n,n}$.

If $f \in \mathscr{P}_n$ and if $A = (a_{i,j}) \in \mathbb{C}^{n,n}$, then $\mathscr{M}^f(A) = (\alpha_{i,j}) \in \mathbb{C}^{n,n}$ is defined as the matrix whose entries are

$$\alpha_{i,j} = -|a_{i,j}| \quad \text{for all} \quad i \neq j, \quad \alpha_{i,i} = f_i(A), \quad i, j = 1, 2, \ldots, n.$$

An important tool in Carlson and Varga [1] is the following result, which shows the close relationship of G-functions and M-matrices.†

PROPOSITION 1. *Let $f \in \mathscr{P}_n$. Then $f \in \mathscr{G}_n$ if and only if $\mathscr{M}^f(A)$ is a (possibly singular) M-matrix for every $A \in \mathbb{C}^{n,n}$.*

In the next section, we define the notion of a G-function for partitioned matrices, and prove a result analogous to Proposition 1. This will then be used to show that other results of [1] can be extended to the partitioned matrix case.

2. G-FUNCTIONS FOR PARTITIONED MATRICES

Let N be a fixed positive integer, $N \geqslant n$. Let π denote

(i) a set $P_1, P_2, \ldots, P_n$ of nonzero projections on $\mathbb{C}^N$ for which $P_i P_j = 0$ for all $i, j = 1, 2, \ldots, n$, $i \neq j$, and $P_1 + P_2 + \cdots + P_n = I$, the identity operator on $\mathbb{C}^N$, and

(ii) a set $\psi_1, \psi_2, \ldots, \psi_n$ of vector norms on, respectively, the subspaces $W_i \equiv P_i \mathbb{C}^N$ of $\mathbb{C}^N$, $i = 1, 2, \ldots, n$.

In the sequel, we may assume without loss of generality that

$$W_i = \text{span}\{e_{k_{i-1}+1}, e_{k_{i-1}+2}, \ldots, e_{k_i}\}, \qquad i = 1, 2, \ldots, n,$$

where $e_j = (\delta_{j,1}, \delta_{j,2}, \ldots, \delta_{j,N})^T$, and where $k_0 = 0 < k_1 < \cdots < k_n = N$.

† $A = (a_{i,j}) \in \mathbb{C}^{n,n}$ is a (possibly singular) *M-matrix* if and only if A is real with $a_{i,j} \leqslant 0$ for all $i, j = 1, 2, \ldots, n$, $i \neq j$, and $A + \text{diag}(d_1, d_2, \ldots, d_n)$ is nonsingular whenever $d_i > 0$, $i = 1, 2, \ldots, n$. If A is an irreducible M-matrix, then $A + \text{diag}(d_1, d_2, \ldots, d_n)$ is nonsingular whenever $d_i \geqslant 0$, $i = 1, 2, \ldots, n$, and $d_1 + d_2 + \cdots + d_n > 0$.

Any $B \in \mathbb{C}^{N,N}$ has a block-matrix representation

$$B = \begin{bmatrix} B_{1,1} & \cdots & B_{1,n} \\ \vdots & & \vdots \\ B_{n,1} & \cdots & B_{n,n} \end{bmatrix},$$

where, for $i, j = 1, 2, \ldots, n$, $B_{i,j}\colon W_j \to W_i$ may be thought of as $P_i B P_j$, with domain restricted to W_j and range considered as a subspace of W_i.

Given $B_{i,j}\colon W_j \to W_i$, we define, for $i, j = 1, 2, \ldots, n$,

$$M(B_{i,j}) = \sup\{\psi_i(B_{i,j}x)\colon x \in W_j \text{ with } \psi_j(x) \leqslant 1\},$$

$$m(B_{i,j}) = \inf\{\psi_i(B_{i,j}x)\colon x \in W_j \text{ with } \psi_j(x) \geqslant 1\}.$$

When $i = j$, the numbers $m(B_{i,i})$ are the so-called "reciprocal norms" (cf. Fiedler and Pták [3]) in that if $B_{i,i}$ is nonsingular, then $m(B_{i,i}) = [M(B_{i,i}^{-1})]^{-1}$ (and if $B_{i,i}$ is singular, $m(B_{i,i}) = 0$).

Let $\mathbb{C}_\pi^{N,N}$ denote the subset of $\mathbb{C}^{N,N}$ of matrices $B = (B_{i,j})$ for which each $B_{i,i}$ is nonsingular, $i = 1, 2, \ldots, n$. Let $\mathscr{P}_\pi$ then be the collection of all functions $F = (F_1, F_2, \ldots, F_n)$ for which

(i) $\quad F_i\colon \; \mathbb{C}_\pi^{N,N} \to \mathbb{R}_+, \qquad i = 1, 2, \ldots, n,$

(ii) for each $B \in \mathbb{C}_\pi^{N,N}$, $F_i(B)$ depends only on the products

$$m(B_{k,k})M(B_{k,k}^{-1}B_{k,j}), \qquad k, j = 1, 2, \ldots, n, \qquad k \neq j,$$

i.e. if $C \in \mathbb{C}_\pi^{N,N}$ with $m(C_{k,k})M(C_{k,k}^{-1}C_{k,j}) = m(B_{k,k})M(B_{k,k}^{-1}B_{k,j})$ for all $k, j = 1, 2, \ldots, n$, $k \neq j$, then $F_i(C) = F_i(B)$, $i = 1, 2, \ldots, n$.

We remark that in the case $N = n$, so that each W_i, $i = 1, 2, \ldots, n$, is, according to our previous assumption, generated by the single vector e_i, it can be verified that for any $B = (b_{i,j}) = (B_{i,j}) \in \mathbb{C}_\pi^{N,N}$,

$$m(B_{i,i})M[(B_{i,i})^{-1}B_{i,j}] = |b_{i,j}|(\psi_i(e_i)/\psi_j(e_j)), \quad i, j = 1, 2, \ldots, n, \quad i \neq j.$$

From this, it follows that any $F \in \mathscr{P}_\pi$ in this case can be trivially extended to a $f \in \mathscr{P}_n$.

This brings us to our fundamental definition.

Definition. Given $F = (F_1, F_2, \ldots, F_n) \in \mathscr{P}_\pi$, then F is a *G-function (relative to π)*, if, for each $B = (B_{i,j}) \in \mathbb{C}_\pi^{N,N}$ for which

$$m(B_{i,i}) > F_i(B), \qquad i = 1, 2, \ldots, n, \tag{2.1}$$

B is nonsingular. The collection of all G-functions relative to π is denoted by $\mathscr{G}_\pi$.

If $F \in \mathscr{P}_\pi$ and if $B = (B_{i,j}) \in \mathbb{C}_\pi^{N,N}$, then $\mathscr{M}_\pi^F(B) = (\beta_{i,j}) \in \mathbb{C}^{n,n}$ is defined as the $n \times n$ matrix whose entries are

$$\beta_{i,j} = -m(B_{i,i})M(B_{i,i}^{-1}B_{i,j}) \quad \text{for all} \quad i \neq j, \quad \beta_{i,i} = F_i(B), \quad i, j = 1, 2, \ldots, n. \tag{2.2}$$

In analogy with Proposition 1, we now establish the following theorem.

THEOREM 1. *Let $F \in \mathscr{P}_\pi$. Then $F \in \mathscr{G}_\pi$ if and only if $\mathscr{M}_\pi^F(B)$ is a (possibly singular) M-matrix for every $B \in \mathbb{C}_\pi^{N,N}$.*

Proof. First, assume that $\mathscr{M}_\pi^F(B)$ is an M-matrix for every $B \in \mathbb{C}_\pi^{N,N}$. If $B = (B_{i,j}) \in \mathbb{C}_\pi^{N,N}$ satisfies (2.1), set $0 < \hat{d}_i = m(B_{i,i}) - F_i(B)$, $i = 1, 2, \ldots, n$, so that $m(B_{i,i}) = F_i(B) + \hat{d}_i$. By our definition, $\mathscr{M}_\pi^F(B) + \operatorname{diag}(\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_n)$ is a nonsingular M-matrix, with diagonal entries $m(B_{i,i})$ and off-diagonal entries $-m(B_{i,i})M(B_{i,i}^{-1}B_{i,j})$. But then, B is nonsingular (cf. Fiedler and Pták [3, Theorem 3.3], and Robert [9, Theorem 7]), and we evidently have that $F \in \mathscr{G}_\pi$.

Conversely, assume $F \in \mathscr{G}_\pi$, and consider the matrix $\mathscr{M}_\pi^F(B)$ for any $B \in \mathbb{C}_\pi^{N,N}$. From the definition of (2.2), the entries of $\mathscr{M}_\pi^F(B)$ have the proper sign-pattern for $\mathscr{M}_\pi^F(B)$ to be an M-matrix. If, on the contrary, $\mathscr{M}_\pi^F(B)$ is not an M-matrix, there necessarily exist $d_i > 0$, $i = 1, 2, \ldots, n$, such that the matrix $T \equiv \mathscr{M}_\pi^F(B) + \operatorname{diag}(d_1, d_2, \ldots, d_n)$ is singular, and hence there exists $0 \neq y \in \mathbb{C}^n$ for which $Ty = 0$. Equivalently, $Ty = 0$ can be expressed as

$$(F_i(B) + d_i)y_i - \sum_{\substack{j=1 \\ j \neq i}}^{n} m(B_{i,i})M(B_{i,i}^{-1}B_{i,j})y_j = 0, \qquad i = 1, 2, \ldots, n. \tag{2.3}$$

Now, let $\xi_i \in W_i$ be fixed vectors with $\psi_i(\xi_i) = 1$, $i = 1, 2, \ldots, n$, and set $z = (\sum_{i=1}^{n} y_i \xi_i) \in \mathbb{C}^N$. Clearly, $z \neq 0$. We now construct a matrix $C = (C_{i,j}) \in \mathbb{C}_\pi^{N,N}$ such that $Cz = 0$, and such that

$$m(C_{i,i})M(C_{i,i}^{-1}C_{i,j}) = m(B_{i,i})M(B_{i,i}^{-1}B_{i,j}), \quad i, j = 1, 2, \ldots, n, \quad i \neq j. \tag{2.4}$$

First, if we choose

$$C_{i,i} = (F_i(B) + d_i)I_i, \qquad i = 1, 2, \ldots, n, \tag{2.5}$$

where I_i is the identity operator on W_i, then each $C_{i,i}$ is evidently nonsingular, so that $C = (C_{i,j}) \in \mathbb{C}_\pi^{N,N}$. Moreover, this choice reduces (2.4) to

$$M(C_{i,j}) = m(B_{i,i})M(B_{i,i}^{-1}B_{i,j}), \qquad i, j = 1, 2, \ldots, n, \qquad i \neq j. \tag{2.6}$$

Next, we directly verify from (2.3) and (2.5) that $Cz = 0$ if

$$C_{i,j}\xi_j = -m(B_{i,i})M(B_{i,i}^{-1}B_{i,j})\xi_i, \quad i, j = 1, 2, \ldots, n, \quad i \neq j. \tag{2.7}$$

Our problem then reduces to constructing submatrices $C_{i,j}$: $W_j \to W_i$, for all $i, j = 1, 2, \ldots, n$, $i \neq j$, which simultaneously satisfy (2.6) and (2.7). Following the construction of Johnston [7], let $\tilde{\psi}_j$ denote the conjugate norm to ψ_j in W_j. As a well-known consequence of the Hahn-Banach Theorem, there exists a vector σ_j in W_j for which both $\tilde{\psi}_j(\sigma_j) = 1$ and $\sigma_j^*\xi_j = 1$. Then, upon defining the submatrices $C_{i,j}$: $W_j \to W_i$, $i \neq j$, by

$$C_{i,j} = -m(B_{i,i})M(B_{i,i}^{-1}B_{i,j})\xi_i\sigma_j^*, \qquad i, j = 1, 2, \ldots, n, \qquad i \neq j,$$

it is readily seen that both (2.6) and (2.7) are satisfied. In summary, the above construction gives us from (2.7) that $Cz = 0$, so that C is singular. On the other hand, because of (2.4) we see that

$$F_i(C) = F_i(B), \qquad i = 1, 2, \ldots, n,$$

and from (2.5) that

$$m(C_{i,i}) = F_i(B) + d_i > F_i(B) = F_i(C), \qquad i = 1, 2, \ldots, n.$$

But, as $F \in \mathscr{G}_\pi$ by assumption, the above inequalities give us that C must be nonsingular, a contradiction. Thus, $\mathscr{M}_\pi^F(B)$ is an M-matrix for each $B \in \mathbb{C}_\pi^{N,N}$, and the proof is complete. ∎

3. THE IDENTIFICATION OF $\mathscr{P}_n$ AND $\mathscr{P}_\pi$

We now show that there is a natural identification of elements of $\mathscr{P}_n$ and $\mathscr{P}_\pi$, given n, $N(\geqslant n)$, and π. First, given $f = (f_1, f_2, \ldots, f_n) \in \mathscr{P}_n$, we define the mapping ϕ: $\mathscr{P}_n \to \mathscr{P}_\pi$ by $\phi(f) = (\phi_1(f), \phi_2(f), \ldots, \phi_n(f))$ where

$$(\phi_i(f))(B) \equiv f_i(\mathscr{M}_\pi^Z(B)), \quad i = 1, 2, \ldots, n, \quad \text{for all} \quad B \in \mathbb{C}_\pi^{N,N}, \tag{3.1}$$

where Z denotes the zero function in $\mathscr{P}_\pi$. Conversely, to define the mapping χ: $\mathscr{P}_\pi \to \mathscr{P}_n$, fix, for each ordered pair (i, j), $i, j = 1, 2, \ldots, n$, $i \neq j$, a submatrix $E_{i,j}$: $W_j \to W_i$ for which $M(E_{i,j}) = 1$. Then, given $A = (a_{i,j}) \in C^{n,n}$, define $B^A = (B_{i,j}^A) \in \mathbb{C}_\pi^{N,N}$ by

$$B^A_{i,j} = |a_{i,j}|E_{i,j}, \qquad i \neq j, \qquad B^A_{i,i} = I_i, \qquad i, j = 1, 2, \ldots, n.$$

Note that B^A depends only on the moduli of the off-diagonal entries of A, and, moreover, that $m(B^A_{i,i})M[(B^A_{i,i})^{-1}B^A_{i,j}] = |a_{i,j}|$ for all $i \neq j$. Then, given $F = (F_1, F_2, \ldots, F_n) \in \mathscr{P}_\pi$, we define the mapping $\chi\colon \mathscr{P}_\pi \to \mathscr{P}_n$ by $\chi(F) = (\chi_1(F), \chi_2(F), \ldots, \chi_n(F))$ where

$$(\chi_i(F))(A) \equiv F_i(B^A), \qquad i = 1, 2, \ldots, n, \quad \text{for all} \quad A \in \mathbb{C}^{n,n}. \quad (3.2)$$

With these definitions of ϕ and χ, we see that the composition $\chi \circ \phi$ maps $\mathscr{P}_n$ into $\mathscr{P}_n$. More precisely, given any $f = (f_1, f_2, \ldots, f_n) \in \mathscr{P}_n$, then $(\chi \circ \phi)(f) \equiv (g_1, g_2, \ldots, g_n) \in \mathscr{P}_n$ is given by

$$g_i(A) = f_i(\mathscr{M}_\pi{}^Z(B^A)), \quad i = 1, 2, \ldots, n, \quad \text{for all} \quad A \in \mathbb{C}^{n,n},$$

from which it readily follows that

$$(\chi \circ \phi)(f) = f \quad \text{for all} \quad f \in \mathscr{P}_n.$$

Similarly, $\phi \circ \chi\colon \mathscr{P}_\pi \to \mathscr{P}_\pi$, and given any $F = (F_1, F_2, \ldots, F_n) \in \mathscr{P}_\pi$, then $(\phi \circ \chi)(F) \equiv (G_1, G_2, \ldots, G_n) \in \mathscr{P}_\pi$ is given by

$$G_i(A) = F_i(B^{\mathscr{M}_\pi{}^Z(A)}), \qquad i = 1, 2, \ldots, n, \quad \text{for all} \quad A \in \mathbb{C}_\pi{}^{N,N},$$

from which analogously it follows that

$$(\phi \circ \chi)(F) = F \quad \text{for all} \quad F \in \mathscr{P}_\pi.$$

Thus, using these mappings ϕ and χ, we can identify elements of $\mathscr{P}_n$ with elements of $\mathscr{P}_\pi$, and conversely. Moreover, this identification preserves certain other properties as well in $\mathscr{P}_n$ and $\mathscr{P}_\pi$, as we show in Theorem 2 below. For added notation, we first define a partial order on $\mathscr{P}_n$. If $f = (f_1, f_2, \ldots, f_n)$ and $g = (g_1, g_2, \ldots, g_n)$ are in $\mathscr{P}_n$, we write $f \geqslant g$ if

$$f_i(A) \geqslant g_i(A) \quad \text{for all} \quad i = 1, 2, \ldots, n, \quad \text{all} \quad A \in \mathbb{C}^{n,n}.$$

The analogous partial order is then used for $\mathscr{P}_\pi$. Next, we say that $f = (f_1, f_2, \ldots, f_n)$ in $\mathscr{P}_n$ is continuous if, for each $i = 1, 2, \ldots, n$, f_i is continuous on $\mathbb{C}^{n,n}$. Similarly, $F = (F_1, F_2, \ldots, F_n)$ in $\mathscr{P}_\pi$ is continuous if, for each $i = 1, 2, \ldots, n$, F_i is continuous on $\mathbb{C}_\pi{}^{N,N}$. Because $F_i(B)$, for $B = (B_{i,j}) \in \mathbb{C}_\pi{}^{N,N}$, by definition depends only on the $n(n-1)$ products:

$$\alpha_{i,j} = m(B_{i,i})M(B_{i,i}^{-1}B_{i,j}), \qquad i, j = 1, 2, \ldots, n, \qquad i \neq j,$$

then $F \in \mathscr{P}_\pi$ is continuous if and only if each F_i is continuous with respect to the $n(n-1)$ quantities $\alpha_{i,j}$. This brings us to Theorem 2.

THEOREM 2. *Given n, N, and π, the mappings ϕ: $\mathscr{P}_n \to \mathscr{P}_\pi$ and χ: $\mathscr{P}_\pi \to \mathscr{P}_n$ are inverses of one another, i.e. given $f \in \mathscr{P}_n$ and $F \in \mathscr{P}_\pi$, then $\phi(f) = F$ if and only if $\chi(F) = f$. These mappings preserve*:

(i) nonnegative linear combinations, *i.e. for arbitrary scalars $r \geqslant 0$, $s \geqslant 0$,*

$$\phi(rf + sg) = r\phi(f) + s\phi(g), \quad \textit{for all} \quad f, g \in \mathscr{P}_n,$$

$$\chi(rF + sG) = r\chi(F) + s\chi(G), \quad \textit{for all } F, G \in \mathscr{P}_\pi;$$

(ii) partial order, *i.e. for $f = \chi(F)$, $g = \chi(G)$ in $\mathscr{P}_n$, $F = \phi(f)$, $G = \phi(g)$ in $\mathscr{P}_\pi$, then $f \leqslant g$ if and only if $F \leqslant G$;*

(iii) continuity, *i.e. $f = \chi(F) \in \mathscr{P}_n$ is continuous if and only if $F = \phi(f) \in \mathscr{P}_\pi$ is continuous;*

(iv) G-functions, *i.e. $f = \chi(F) \in \mathscr{P}_n$ is a G-function if and only if $F = \phi(f) \in \mathscr{P}_\pi$ is a G-function (relative to π).*

Proof. The proofs of (i)–(iii) are readily verified, and are omitted. To prove (iv), we make use of the identity,

$$\mathscr{M}_\pi^{\phi(f)}(B) = \mathscr{M}^f[\mathscr{M}_\pi^Z(B)] \quad \text{for all} \quad B \in \mathbb{C}_\pi^{N,N}, \quad \text{all} \quad f \in \mathscr{P}_n, \tag{3.3}$$

which follows directly from (3.1). If $f \in \mathscr{G}_n$, then from Proposition 1 and the above identity, $\mathscr{M}_\pi^{\phi(f)}(B)$ is evidently an M-matrix for every $B \in \mathbb{C}_\pi^{N,N}$. Thus, from Theorem 1, $\phi(f) \in \mathscr{G}_\pi$.

Conversely, making use of the analogous identity (cf. (3.2)),

$$\mathscr{M}^{\chi(F)}(A) = \mathscr{M}_\pi^F(B^A) \quad \text{for all} \quad A \in \mathbb{C}^{n,n}, \quad \text{all} \quad F \in \mathscr{P}_\pi, \tag{3.4}$$

assume that $F \in \mathscr{G}_\pi$. Thus, from Theorem 1 again, $\mathscr{M}^{\chi(F)}(A)$ is an M-matrix for every $A \in \mathbb{C}^{n,n}$, so that from Proposition 1, $\chi(F) \in \mathscr{G}_n$. This completes the proof. ∎

We now say that $B \in \mathbb{C}_\pi^{N,N}$ is irreducible$_\pi$ (reducible$_\pi$) if the associated matrix $\mathscr{M}_\pi^Z(B) \in \mathbb{C}^{n,n}$ is irreducible (reducible) in the usual graph-theoretic sense (cf. Varga [10, p. 19]). Note that for $B \in \mathbb{C}_\pi^{N,N}$, irreducibility$_\pi$ is equivalent to the notion of block-irreducibility of Feingold and Varga [2].

Using Theorem 2 and the above notion of irreducibility$_\pi$, many of the results of Carlson and Varga [1] directly carry over the partitioned matrix case. We give several such results now as corollaries of Theorem 2.

COROLLARY 1. *If $F \in \mathscr{G}_\pi$ and if $B \in \mathbb{C}_\pi^{N,N}$ is irreducible$_\pi$, then there is an $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$ with $x > 0$ (depending on B) such that*

$$F_i(B) \geqslant R_i^x(B) \equiv \frac{1}{x_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} m(B_{i,i}) M(B_{i,i}^{-1} B_{i,j}) x_j, \qquad i = 1, 2, \ldots, n. \tag{3.5}$$

Moreover, if $\mathcal{M}_\pi^F(B)$ *is a singular M-matrix, equality holds throughout above, i.e.* $F_i(B) = R_i^x(B)$, $i = 1, 2, \ldots, n$.

Proof. We include the proof of this corollary to show how our previous constructions and the results of Theorem 2 couple with the results of [1]. If $B \in \mathbb{C}_\pi^{N,N}$ is irreducible$_\pi$, the $n \times n$ matrix $A = (a_{i,j}) \equiv \mathcal{M}_\pi^Z(B)$ is, by definition, irreducible. Assuming $F \in \mathcal{G}_\pi$, then $f = \chi(F)$ is a G-function from (iv) of Theorem 2. Thus, from Proposition 1 of [1], there is an $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$ with $x > 0$ (depending on A, and hence depending on B) such that

$$f_i(A) \geqslant r_i^x(A) = \frac{1}{x_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}| x_j, \qquad i = 1, 2, \ldots, n.$$

But, as $\phi(f) = F$, it follows from (3.1) and the definition of the entries of the matrix A that the above inequalities turn out to be nothing more than

$$F_i(B) \geqslant \frac{1}{x_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} m(B_{i,i}) M(B_{i,i}^{-1} B_{i,j}) x_j, \qquad i = 1, 2, \ldots, n,$$

the desired result of (3.5). The case of equality in (3.5) similarly follows, which completes the proof. ∎

In analogy with [1], we denote by $\mathcal{G}_n^c$ and $\mathcal{G}_\pi^c$, respectively, the sets of continuous functions in $\mathcal{G}_n$ and $\mathcal{G}_\pi$. The next corollary follows directly from Hoffman's result [4] (cf. [1, Theorem 1]) and Theorem 2.

COROLLARY 2. *If* $F = (F_1, F_2, \ldots, F_n)$ *and* $G = (G_1, G_2, \ldots, G_n)$ *are in* $\mathcal{G}_\pi$ *and* $0 < \alpha < 1$, *then* $H = (H_1, H_2, \ldots, H_n)$ *and* $K = (K_1, K_2, \ldots, K_n)$, *defined by*

$$H_i(B) = F_i^\alpha(B) G_i^{1-\alpha}(B), \qquad i = 1, 2, \ldots, n, \quad \text{for all} \quad B \in \mathbb{C}_\pi^{N,N}, \tag{3.6}$$

$$K_i(B) = \alpha F_i(B) + (1-\alpha) G_i(B), \qquad i = 1, 2, \ldots, n, \quad \text{for all} \quad B \in \mathbb{C}_\pi^{N,N}, \tag{3.7}$$

are also in $\mathscr{G}_\pi$. *If F and G are in* $\mathscr{G}_\pi{}^c$, *so are H and K. Thus,* $\mathscr{G}_\pi$ *and* $\mathscr{G}_\pi{}^c$ *are convex sets.*

Because of the partial order and convexity that exist in $\mathscr{G}_\pi$, we say, in analogy with [1], that $F \in \mathscr{G}_\pi$ $(\mathscr{G}_\pi{}^c)$ is *minimal in* $\mathscr{G}_\pi$ $(\mathscr{G}_\pi{}^c)$ if, for every $G \in \mathscr{G}_\pi$ $(\mathscr{G}_\pi{}^c)$ for which $G \leqslant F$, we have $G = F$. Similarly, we say that $F \in \mathscr{G}_\pi$ $(\mathscr{G}_\pi{}^c)$ is an *extreme point* of the convex set $\mathscr{G}_\pi$ $(\mathscr{G}_\pi{}^c)$ if $F = \alpha G + (1 - \alpha)H$, where $0 < \alpha < 1$ and where G and H are in $\mathscr{G}_\pi$ $(\mathscr{G}_\pi{}^c)$, implies that $F = G = H$. The proof given in [1] then shows that the minimal elements of $\mathscr{G}_\pi$ $(\mathscr{G}_\pi{}^c)$ are precisely the extreme points of $\mathscr{G}_\pi$ $(\mathscr{G}_\pi{}^c)$. Other characterizations of minimal elements in $\mathscr{G}_\pi{}^c$ follow directly from Corollary 1 and from the analogous results of [1, Theorem 2], which we state as Theorem 3 below.

THEOREM 3. *Let* $F \in \mathscr{G}_\pi{}^c$. *Then the following are equivalent*:

(i) *F is minimal in* $\mathscr{G}_\pi{}^c$;

(ii) *F is an extreme point in* $\mathscr{G}_\pi{}^c$;

(iii) *for every* $B \in \mathbb{C}_\pi^{N,N}$, *the matrix* $\mathscr{M}_\pi{}^F(B)$ *is singular*;

(iv) *for every irreducible*$_\pi$ $B \in \mathbb{C}_\pi^{N,N}$, *there exists an* $x \in \mathbb{C}^n$ *with* $x > 0$ (*depending on B*) *for which* (cf. (3.5))

$$F_i(B) = R_i^x(B), \qquad i = 1, 2, \ldots, n.$$

In a similar way, it is easily verified from the proof of [1, Theorem 4] that the following result is also valid.

THEOREM 4. *For* $n > 2$, *let F and G be two distinct minimal elements in* $\mathscr{G}_\pi{}^c$, *and, for* $0 < \alpha < 1$, *let* $H \in \mathscr{G}_\pi{}^c$ *be defined by* (3.6) *of Corollary* 2. *Then, H is not minimal in* $\mathscr{G}_\pi{}^c$.

The reader will readily see that other results from [1], such as (α, β)-convolutions of elements in $\mathscr{G}_n$, the characterization [1, Theorem 6] of the minimal elements in $\mathscr{G}_n$, which allows both for discontinuous elements in $\mathscr{G}_n$ and reducible matrices in $\mathbb{C}^{n,n}$, and results on domains of dependence (cf. [5]), carry over directly to the partitioned case. For brevity, we have omitted these extensions.

As a final note, we remark that the above analysis could just as well have been carried out by using $M(B_{i,j})$ throughout in place of $m(B_{i,i}) \cdot M(B_{i,i}^{-1}B_{i,j})$, $i, j = 1, 2, \ldots, n$, $i \neq j$, and by defining the elements of $\mathscr{P}_\pi$ as functions from $\mathbb{C}^{N,N}$ (instead of $\mathbb{C}_\pi^{N,N}$) to $\mathbb{R}_+^n$. While this would have given a more direct "off-diagonal" analogue of the results of [1], it is however known from the results of Fiedler and Pták [3] and Robert [9] that, for fixed vector norms ψ_i on W_i, $i = 1, 2, \ldots, n$, and for an arbitrary $B_{i,j}$: $W_j \to W_i$ and for an arbitrary nonsingular $B_{i,i}$: $W_i \to W_i$,

$$m(B_{i,i})M(B_{i,i}^{-1}B_{i,j}) \leqslant M(B_{i,j}), \qquad i \neq j.$$

This means that the Gerschgorin sets defined by

$$\{z \in \mathbb{C}\colon m(B_{i,i} - zI) \leqslant F_i(B)\}, \qquad i = 1, 2, \ldots, n, \tag{3.8}$$

where $F = (F_1, F_2, \ldots, F_n) \in \mathscr{G}_\pi$ depends on the products $m(B_{i,i}) \cdot M(B_{i,i}^{-1}B_{i,j})$, $i \neq j$, will be *smaller* than the analogous sets with $F_i(B)$ depending on the numbers $M(B_{i,j})$, $i \neq j$, if each F_i is a *monotone non-decreasing* function of its argument. (Note that the row sums R_i^x of (3.5) have this property.) Consequently, the union of the sets of (3.5) will determine a *smaller* region in the complex plane which contains all the eigenvalues of B which are not also eigenvalues of some $B_{i,i}$ (cf. [6]), than that produced analogously by the $F_i(B)$ depending on the numbers $M(B_{i,j})$.

REFERENCES

1 David H. Carlson and Richard S. Varga, Minimal G-functions, *Lin. Alg. Appl.* **6**(1973), 97–117.

2 David G. Feingold and Richard S. Varga, Block diagonally dominant matrices and generalizations of the Gerschgorin Circle Theorem, *Pacific J. Math.* **12**(1962), 1241–1250.

3 Miroslav Fiedler and Vlastimil Pták, Generalized norms of matrices and the location of the spectrum, *Czech. Math. J.* **12**(1962), 558–571.

4 A. J. Hoffman, *Generalizations of Geršgorin's Theorem: G-Generating Families*, lecture notes, University of California at Santa Barbara, August, 1969, 46 pp.

5 Alan J. Hoffman and Richard S. Varga, Patterns of dependence in generalizations of Gerschgorin's Theorem, *SIAM J. Numer. Anal.* **7**(1970), 571–574.

6 A. S. Householder, R. S. Varga, and J. H. Wilkinson, A note on Gerschgorin's inclusion theorem for eigenvalues of matrices, *Numerische Math.* **16**(1970), 141–144.

7 R. L. Johnston, Gerschgorin theorems for partitioned matrices, *Lin. Alg. Appl.* **4**(1971), 205–220.

8 P. Nowosad, On the functional (x^{-1}, Ax) and some of its applications, *An. Acad. Brasil Ci.* **37**(1965), 163–165.

9 F. Robert, Recherche d'une M-matrice parmi les minorantes d'un opérateur linéaire, *Numerische Math.* **9**(1966), 189–199.

10 Richard S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, N.J. (1962).

Received October, 1972