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On Collections of G-Functions*

DAVID H. CARLSON** AND RICHARD S. VARGA†
Kent State University
Kent, Ohio

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1. INTRODUCTION

Weighted row and column sums of an $n \times n$ complex matrix have always played a central role in the Gerschgorin circle of ideas for obtaining regions of eigenvalue inclusion for matrices. The object of this paper is to generalize previous results [4], dealing with the sharpness of the boundary of an eigenvalue inclusion region for a collection of weighted row sums, to general collections of the more recently introduced G-functions (see [3]).

2. NOTATION

Following Hoffman [3] and Carlson and Varga [1], let $\mathscr{P}_n$, $n \geqslant 2$, be the collection of all functions $f = (f_1, \ldots, f_n)$ for which $f\colon \mathbb{C}^{n,n} \to \mathbb{R}_+^n$, that is, $+\infty > f_i(A) \geqslant 0$ for all $i = 1, 2, \ldots, n$, and all $A \in \mathbb{C}^{n,n}$, and for which f depends only on the moduli of the off-diagonal entries of any $A = (a_{i,j}) \in \mathbb{C}^{n,n}$, that is, if $B = (b_{i,j})$ and $A = (a_{i,j})$ are in $\mathbb{C}^{n,n}$ with $|b_{i,j}| = |a_{i,j}|$ for all $i \neq j$, $i, j = 1, 2, \ldots, n$, then $f_i(B) = f_i(A)$ for $i = 1, 2, \ldots, n$. An $f \in \mathscr{P}_n$ is said to be a *G-function* if, for every $A = (a_{i,j}) \in \mathbb{C}^{n,n}$ with

$$|a_{i,i}| > f_i(A), \qquad i = 1, 2, \ldots, n, \tag{2.1}$$

then A is nonsingular. Equivalently, if λ is any eigenvalue of A, then λ

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** On leave from Oregon State University, Corvallis, Oregon.

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is contained in at least one of the n disks $G_i^f(A)$ in the complex plane, where

$$G_i^f(A) \equiv \{z \in \mathbb{C}\colon |a_{i,i} - z| \leqslant f_i(A)\}. \tag{2.2}$$

Thus, if $S(A)$ denotes the collection of all eigenvalues of A, then for any $A \in \mathbb{C}^{n,n}$,

$$S(A) \subset \bigcup_{i=1}^{n} G_i^f(A) \equiv G^f(A). \tag{2.3}$$

Next, for any $A = (a_{i,j}) \in \mathbb{C}^{n,n}$, let

$$\Omega_A \equiv \{B = (b_{i,j}) \in \mathbb{C}^{n,n}\colon b_{i,i} = a_{i,i},\ |b_{i,j}| = |a_{i,j}|,\ i, j = 1, 2, \ldots, n\}. \tag{2.4}$$

Because any G-function depends only on the moduli of the off-diagonal entries of a matrix, the inclusion of (2.3) can be strengthened to $S(B) \subset G^f(A)$ for any $B \in \Omega_A$. Thus, if

$$S(\Omega_A) \equiv \{\lambda \in \mathbb{C}\colon \det(\lambda I - B) = 0 \text{ for some } B \in \Omega_A\}, \tag{2.5}$$

then for any $A \in \mathbb{C}^{n,n}$ and any G-function f in $\mathscr{P}_n$,

$$S(\Omega_A) \subset G^f(A). \tag{2.6}$$

Consider now any non-empty collection $\mathfrak{F}$ of G-functions in $\mathscr{P}_n$. As (2.6) is then valid for *each* $f \in \mathfrak{F}$, it necessarily follows for any $A \in \mathbb{C}^{n,n}$ that

$$S(\Omega_A) \subset \bigcap_{f \in \mathfrak{F}} G^f(A) \equiv G^{\mathfrak{F}}(A). \tag{2.7}$$

As in [4], we are interested in whether the inclusion of (2.7) is *sharp* for all $A \in \mathbb{C}^{n,n}$, that is, if each boundary point of the set $G^{\mathfrak{F}}(A)$ in the complex plane is an eigenvalue of *some* $B \in \Omega_A$ for *every* $A \in \mathbb{C}^{n,n}$, which we would write as

$$\partial G^{\mathfrak{F}}(A) \subset S(\Omega_A) \quad \text{for any} \quad A \in \mathbb{C}^{n,n}. \tag{2.8}$$

Next, for any $A \in \mathbb{C}^{n,n}$, we define

$$\hat{\Omega}_A \equiv \{B = (b_{i,j}) \in \mathbb{C}^{n,n}\colon b_{i,i} = a_{i,i},\ |b_{i,j}| \leqslant |a_{i,j}|,\ i, j = 1, 2, \ldots, n\}, \tag{2.9}$$

and analogously set

$$S(\hat{\Omega}_A) \equiv \{\lambda \in \mathbb{C}: \det(\lambda I - B) = 0 \text{ for some } B \in \hat{\Omega}_A\}. \tag{2.10}$$

It is clear that

$$S(\Omega_A) \subset S(\hat{\Omega}_A) \subset \bigcup_{B \in \hat{\Omega}_A} G^{\mathfrak{F}}(B), \tag{2.11}$$

the second inclusion following from (2.7). However, we are interested in when the more precise formula

$$S(\hat{\Omega}_A) = G^{\mathfrak{F}}(A) \tag{2.12}$$

is valid. We shall consider the questions of (2.8) and (2.12) for general collections of G-functions in $\mathscr{P}_n$.

For $f, g \in \mathscr{P}_n$, we say that $f \geqslant g$ if

$$f_i(A) \geqslant g_i(A) \quad \text{for all} \quad i = 1, 2, \ldots, n, \quad \text{and all} \quad A \in \mathbb{C}^{n,n}. \tag{2.13}$$

A G-function f is *minimal* (see [1]) if it is minimal with respect to the partial order determined by (2.13), that is, if $g \in \mathscr{P}_n$ is a G-function with $g \leqslant f$, then $g = f$.

3. SOME LEMMAS

We begin with

LEMMA 1. *For $\mathfrak{F}$ any collection of functions in $\mathscr{P}_n$ and for any $A \in \mathbb{C}^{n,n}$, define, for any $z \in \mathbb{C}$,*

$$\nu(z) = \nu_{\mathfrak{F}}(z; A) \equiv \inf_{f \in \mathfrak{F}}\{\max_{1 \leqslant i \leqslant n}(f_i(A) - |a_{i,i} - z|)\}. \tag{3.1}$$

Then, $z \in G^{\mathfrak{F}}(A)$ if and only if $\nu(z) \geqslant 0$.

Proof. If $z \in G^{\mathfrak{F}}(A)$, then it follows from (2.7) that for each $f \in \mathfrak{F}$, there is an integer i, $1 \leqslant i \leqslant n$, such that $|a_{i,i} - z| \leqslant f_i(A)$, that is, $\max_{1 \leqslant i \leqslant n}(f_i(A) - |a_{i,i} - z|) \geqslant 0$. As this is true for all $f \in \mathfrak{F}$, then $\nu(z) \geqslant 0$. Conversely, if $\nu(z) \geqslant 0$, then $\max_{1 \leqslant i \leqslant n}(f_i(A) - |a_{i,i} - z|) \geqslant 0$ for any $f \in \mathfrak{F}$. Thus, for each $f \in \mathfrak{F}$, there is an i, $1 \leqslant i \leqslant n$, for which $|a_{i,i} - z| \leqslant f_i(A)$, that is, $z \in G_i^f(A) \subset G^f(A)$. As this is true for each $f \in \mathfrak{F}$, then $z \in G^{\mathfrak{F}}(A)$. Q.E.D.

For any collection $\mathfrak{F}$ of functions in $\mathscr{P}_n$ and for any (fixed) $A \in \mathbb{C}^{n,n}$, it is easy to see that $\nu(z) = \nu_{\mathfrak{F}}(z; A)$ is a (uniformly) *continuous* function of z. This is useful in proving

LEMMA 2. *For $\mathfrak{F}$ any collection of functions in $\mathscr{P}_n$ and for any $A \in \mathbb{C}^{n,n}$, then $z \in \partial G^{\mathfrak{F}}(A)$ if and only if $\nu(z) = 0$ and there exists a sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ with $z_n \to z$ for which $\nu(z_n) < 0$ for all $n = 1, 2, \ldots$.*

Proof. Since $G^f(A)$, from (2.2) and (2.3), is the union of closed bounded disks in $\mathbb{C}$, then $G^{\mathfrak{F}}(A)$ is a *closed* bounded set in $\mathbb{C}$. Thus, if $z \in \partial G^{\mathfrak{F}}(A) \equiv \overline{G^{\mathfrak{F}}(A)} \cap \overline{(G^{\mathfrak{F}}(A))'}$, then $z \in G^{\mathfrak{F}}(A)$, which implies from Lemma 1 that $\nu(z) \geqslant 0$. On the other hand, if $z \in \overline{(G^{\mathfrak{F}}(A))'}$, there is a sequence $\{z_n\}_{n=1}^{\infty}$ in $(G^{\mathfrak{F}}(A))'$ for which $z_n \to z$. By Lemma 1, $\nu(z_n) < 0$ for each $n = 1, 2, \ldots$. Thus, from the continuity of ν, it follows that $\nu(z) = 0$, which establishes one part of Lemma 2. Conversely, if $\nu(z) = 0$ and $z_n \to z$ with $\nu(z_n) < 0$, then $z \in G^{\mathfrak{F}}(A)$ from Lemma 1, and $z \in \overline{(G^{\mathfrak{F}}(A))'}$, that is, $z \in \partial G^{\mathfrak{F}}(A)$. Q.E.D.

LEMMA 3. *For $\mathfrak{F}$ any collection of G-functions in $\mathscr{P}_n$, and for any $A \in \mathbb{C}^{n,n}$, let $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{C}^n$ with $\tau \geqslant 0$ be such that the $n \times n$ matrix*

$$\mathfrak{M}^{\tau}(A) \equiv \begin{bmatrix} \tau_1 & -|a_{1,2}| & \cdots & -|a_{1,n}| \\ -|a_{2,1}| & \tau_2 & \cdots & -|a_{2,n}| \\ \vdots & & & \vdots \\ -|a_{n,1}| & -|a_{n,2}| & \cdots & \tau_n \end{bmatrix} \tag{3.2}$$

is a singular M-matrix. Then,

$$\inf_{f \in \mathfrak{F}}\{\max_{1 \leqslant i \leqslant n}(f_i(A) - \tau_i)\} \geqslant 0. \tag{3.3}$$

Proof. For f any G-function, it is known [1, Proposition 1] that the matrix $\mathfrak{M}^f(A)$, determined from (3.2) with $\tau_i \equiv f_i(A)$, $i = 1, 2, \ldots, n$, is an M-matrix. Consequently, if $\mathfrak{M}^{\tau}(A)$ of (3.2) is a *singular* M-matrix, then $f_i(A)$ cannot be less than τ_i for all $i = 1, 2, \ldots, n$, that is, $\max_{1 \leqslant i \leqslant n}(f_i(A) - \tau_i) \geqslant 0$, from which (3.3) follows. Q.E.D.

Lemma 3 then serves to motivate our next definition.

DEFINITION 1. *Let $\mathfrak{F}$ be a collection of G-functions in $\mathscr{P}_n$. Then, for $A \in \mathbb{C}^{n,n}$, $\mathfrak{F}$ is full at A if, for each $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{C}^n$ with $\tau \geqslant 0$ for* *which the* $n \times n$ *matrix* $\mathfrak{M}^{\tau}(A)$ *of* (3.2) *is a singular M-matrix, then*

$$\inf_{f \in \mathfrak{F}}\{\max_{1 \leqslant i \leqslant n}(f_i(A) - \tau_i)\} = 0. \tag{3.4}$$

If $\mathfrak{F}$ *is full at each* $A \in \mathbb{C}^{n,n}$, *then* $\mathfrak{F}$ *is said to be full.*

Before proving Lemma 4, it is necessary to introduce some additional notation. Given any reducible $A \in \mathbb{C}^{n,n}$, it is well-known (see [5, p. 46]) that there is a permutation matrix $P \in \mathbb{C}^{n,n}$ and a positive integer m with $2 \leqslant m \leqslant n$, such that

$$PAP^T = \begin{bmatrix} \tilde{A}_{1,1} & \tilde{A}_{1,2} & \cdots & \tilde{A}_{1,m} \\ 0 & \tilde{A}_{2,2} & \cdots & \tilde{A}_{2,m} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{A}_{m,m} \end{bmatrix}, \tag{3.5}$$

where each square submatrix $\tilde{A}_{k,k}$, $k = 1, 2, \ldots, m$, is either irreducible or a 1×1 null matrix. The form (3.5), called the *reduced normal form of* A, gives rise to a partitioning of $\{1, 2, \ldots, n\}$ into m disjoint non-empty sets $S_k = S_k(A)$, corresponding to the distinct connected components of the directed graph for A. The subsets S_k do not depend on the choice of the permutation matrix P. For each $i = 1, 2, \ldots, n$, let $\langle i \rangle$ denote the unique subset S_k containing i, and, for each $x \in \mathbb{C}^n$ with $x > 0$, define the G-function $\hat{r}^x = (\hat{r}_1^x, \ldots, \hat{r}_n^x)$ in $\mathscr{P}_n$ (see [1]) by

$$\hat{r}_i^x(A) \equiv \frac{1}{x_i} \sum_{\substack{j \in \langle i \rangle \\ j \neq i}} |a_{i,j}| x_j, \quad i = 1, 2, \ldots, n, \tag{3.6}$$

where we take $\hat{r}_i^x(A)$ to be zero if $\langle i \rangle = \{i\}$. If A is irreducible, then we define $\langle i \rangle = \{1, 2, \ldots, n\}$ for each $i = 1, 2, \ldots, n$. In this case, the sum of (3.6) becomes the familiar row sum

$$r_i^x(A) = \frac{1}{x_i} \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}| x_j, \qquad i = 1, 2, \ldots, n, \tag{3.7}$$

and as is well-known, $r^x = (r_1^x, \ldots, r_n^x)$ is a G-function in $\mathscr{P}_n$ for each $x \in \mathbb{C}^n$ with $x > 0$.

As a result of Theorem 6 of [1], we can establish

LEMMA 4. *Given* $A \in \mathbb{C}^{n,n}$, *a collection* $\mathfrak{F}$ *of G-functions in* $\mathscr{P}_n$ *is full at* A *if and only if, for every* $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ *with* $x > 0$, *we have*

$$\inf_{\mathfrak{F}}\{\max_{1\leqslant i\leqslant n}(f_i(A) - \hat{r}_i^x(A))\} = 0. \tag{3.8}$$

Proof. Suppose that $\mathfrak{F}$ is full at A. Then, for any $x \in \mathbb{C}^n$ with $x > 0$, let $\tau_i \equiv \hat{r}_i^x(A)$, $i = 1, 2, \ldots, n$. From [1, Theorem 6], the matrix $\mathfrak{M}^\tau(A)$ of (3.2) is a singular M-matrix. Since $\mathfrak{F}$ is full at A, then (3.4) is valid with $\tau_i = \hat{r}_i^x(A)$, that is, (3.8) holds. Conversely, suppose that $\mathfrak{F}$ is a collection of G-functions in $\mathscr{P}_n$ which satisfies (3.8) for any $x \in \mathbb{C}^n$ with $x > 0$. For any $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{C}^n$ with $\tau \geqslant 0$ such that $\mathfrak{M}^\tau(A)$ of (3.2) is a singular M-matrix, it follows, whether A is irreducible or reducible, that there is a vector $y \in \mathbb{C}^n$ with $y > 0$ such that (see [1, Theorems 2 and 6])

$$\tau_j \geqslant \hat{r}_j^y(A) \quad \text{for all} \quad j = 1, 2, \ldots, n. \tag{3.9}$$

Now, writing $f_j(A) - \tau_j$ as the sum

$$(f_j(A) - \tau_j) = (f_j(A) - \hat{r}_j^y(A)) + (\hat{r}_j^y(A) - \tau_j),$$

it follows that

$$\max_{1\leqslant j\leqslant n}(f_j(A) - \tau_j) \leqslant \max_{1\leqslant j\leqslant n}(f_j(A) - \hat{r}_j^y(A)) + \max_{1\leqslant j\leqslant n}(\hat{r}_j^y(A) - \tau_j),$$

and taking infimums over $\mathfrak{F}$ and invoking (3.8) gives

$$\inf_{\mathfrak{F}}\{\max_{1\leqslant j\leqslant n}(f_j(A) - \tau_j)\} \leqslant \max_{1\leqslant j\leqslant n}(\hat{r}_j^y(A) - \tau_j) \leqslant 0,$$

the last inequality following from (3.9). But as the reverse inequality necessarily also holds (see (3.3) of Lemma 3), then $\inf_{\mathfrak{F}}\{\max_{1\leqslant j\leqslant n}(f_j(A) - \tau_j)\} = 0$, that is, $\mathfrak{F}$ is full at A. Q.E.D.

In what follows, we will consistently use the notation $\mathfrak{F}'$ for the following collection of G-functions in $\mathscr{P}_n$,

$$\mathfrak{F}' = \{\hat{r}^x = (\hat{r}_1^x, \ldots, \hat{r}_n^x) : x \in \mathbb{C}^n \text{ with } x > 0\}, \tag{3.10}$$

where $\hat{r}_i^x$ is defined in (3.6). It is clear from Lemmas 3 and 4 that $\mathfrak{F}'$ is *full*, that is, full at each $A \in \mathbb{C}^{n,n}$. It is also convenient to define the collection $\tilde{\mathfrak{F}}$ of G-functions in $\mathscr{P}_n$ as

$$\tilde{\mathfrak{F}} = \{r^x = (r_1^x, \ldots, r_n^x) : x \in \mathbb{C}^n \text{ with } x > 0\}, \tag{3.11}$$

where r_i^x is defined in (3.7). It is clear that $\tilde{\mathfrak{F}}$ is full at each irreducible matrix in $\mathbb{C}^{n,n}$, for if A is irreducible, then $r^x(A) = \hat{r}^x(A)$. We shall later show that $\tilde{\mathfrak{F}}$ is also full.

4. MAIN RESULT

With the Lemmas of Sec. 3, we can prove our main result, which generalizes Theorems 4 and 6 of [4].

THEOREM 1. *Let $\mathfrak{F}$ be a collection of G-functions in $\mathscr{P}_n$. The following conditions are equivalent:*

(i) *$\mathfrak{F}$ is full at A;*
(ii) $G^{\mathfrak{F}}(A) = G^{\mathfrak{F}'}(A)(\equiv \bigcap_{x>0} G^{\hat{r}^x}(A))$;
(iii) $\partial G^{\mathfrak{F}}(A) \subset S(\Omega_A)$;
(iv) $S(\hat{\Omega}_A) = G^{\mathfrak{F}}(A)$.

Proof. First, assume that $\mathfrak{F}$ is full at A. Since $\mathfrak{F}$ is composed of G-functions in $\mathscr{P}_n$, then, for any $f \in \mathfrak{F}$, it is known (see Fan [2] for the irreducible case and [1, Theorem 6] for the general case) that

$$f_i(A) \geqslant \hat{r}_i^x(A), \qquad i = 1, 2, \ldots, n, \tag{4.1}$$

for some $x \in \mathbb{C}^n$ with $x > 0$. Thus it follows from (4.1) and the definitions of $\nu_{\mathfrak{F}}(z; A)$ and $\nu_{\mathfrak{F}'}(z; A)$ in (3.1) that

$$\nu_{\mathfrak{F}}(z; A) \geqslant \nu_{\mathfrak{F}'}(z; A) \tag{4.2}$$

for all $z \in \mathbb{C}$. On the other hand, for any fixed $x \in \mathbb{C}^n$ with $x > 0$, we can write

$$f_i(A) - |a_{i,i} - z| = (f_i(A) - \hat{r}_i^x(A)) + (\hat{r}_i^x(A) - |a_{i,i} - z|),$$

so that

$$\max_{1 \leqslant i \leqslant n}(f_i(A) - |a_{i,i} - z|) \leqslant \max_{1 \leqslant i \leqslant n}(f_i(A) - \hat{r}_i^x(A)) + \max_{1 \leqslant i \leqslant n}(\hat{r}_i^x(A) - |a_{i,i} - z|).$$

Taking infimums over $\mathfrak{F}$ and applying (3.8) of Lemma 4, since $\mathfrak{F}$ is assumed full at A, we have

$$\nu_{\mathfrak{F}}(z; A) \leqslant \max_{1 \leqslant i \leqslant n}(\hat{r}_i^x(A) - |a_{i,i} - z|),$$

and since this is true for any $x \in \mathbb{C}^n$ with $x > 0$, then $\nu_{\mathfrak{F}}(z; A) \leqslant \nu_{\mathfrak{F}'}(z; A)$, which, when coupled with the inequality of (4.2), gives us that

$$\nu_{\mathfrak{F}}(z; A) = \nu_{\mathfrak{F}'}(z; A) \tag{4.3}$$

for any $z \in \mathbb{C}$. Thus, from Lemmas 1 and 2, we necessarily have from (4.3) that (ii) is valid. Thus, if A is irreducible, then $\hat{r}^x(A) = r^x(A)$ for every $x \in \mathbb{C}^n$ with $x > 0$, and it follows directly from [4, Theorems 4 and 6] that (iii) and (iv) are also valid. If, however, A is reducible, one can apply the results of [4, Theorems 4 and 6] to each of the diagonal submatrices of the reduced normal form for A (see (3.5)) to again conclude that (iii) and (iv) are valid.

Conversely, assume that $\mathfrak{F}$ is not full at A. Thus, there is a $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{C}^n$ with $\tau \geqslant 0$ for which the matrix $\mathfrak{M}^\tau(A)$ of (3.2) is a singular M-matrix, such that

$$\inf_{\mathfrak{F}}\{\max_{1 \leqslant i \leqslant n}(f_i(A) - \tau_i)\} = \alpha \neq 0. \tag{4.4}$$

From Lemma 3, we know that α must then be positive. Defining $\sigma = \max_{1 \leqslant i \leqslant n} \tau_i$, set $a_{i,i} = \sigma - \tau_i$, $i = 1, 2, \ldots, n$, so that each $a_{i,i}$ is non-negative. Since $f(A)$, for each $f \in \mathfrak{F}$, depends only on the moduli of the off-diagonal entries of A, we may assume these entries to be non-negative; this then fully defines a non-negative matrix $A \in \mathbb{C}^{n,n}$.

It is easily seen that σ is $\rho(A)$, the Perron-Frobenius eigenvalue of the non-negative matrix A. Consequently, $\rho(A) \in S(\Omega_A)$, and thus from (2.7),

$$\rho(A) \in G^{\mathfrak{F}}(A). \tag{4.5}$$

Consider now $\nu(\rho(A)) = \nu_{\mathfrak{F}}(\rho(A); A)$. From (3.1) and (4.4), we have

$$\nu(\rho(A)) = \inf_{\mathfrak{F}}\{\max_{1 \leqslant i \leqslant n}(f_i(A) - |a_{i,i} - \rho(A)|)\} = \inf_{\mathfrak{F}}\{\max_{1 \leqslant i \leqslant n}(f_i(A) - \tau_i)\} = \alpha > 0.$$

Next, from Lemma 2, the fact that $\nu(\rho(A)) = \alpha > 0$ gives us that $\rho(A) \notin \partial G^{\mathfrak{F}}(A)$. Thus, if

$$\omega \equiv \max\{\mu \geqslant 0 : \mu \in G^{\mathfrak{F}}(A)\}, \tag{4.6}$$

then $\omega \in \partial G^{\mathfrak{F}}(A)$ and $\omega > \rho(A)$, using the continuity of ν. But again, from the Perron-Frobenius theory of non-negative matrices, it follows for any $\lambda \in S(\hat{\Omega}_A)$ that $|\lambda| \leqslant \rho(A)$. Thus, we see that ω, as defined in (4.6), cannot be an eigenvalue of any $B \in \hat{\Omega}_A$, that is, $\omega \notin S(\hat{\Omega}_A)$. We have $\omega \in \partial G^{\mathfrak{F}}(A) \subset G^{\mathfrak{F}}(A)$, yet $\omega \notin S(\hat{\Omega}_A) = G^{\mathfrak{F}'}(A)$, the last equality following again from [4]. Hence, $\omega \notin S(\Omega_A)$, and none of (ii), (iii) and (iv) hold. Q.E.D.

COROLLARY. The collection $\tilde{\mathfrak{F}}$ of (3.11) is full.

Proof. From Theorem 1, it suffices to show that $S(\hat{\Omega}_A) = G^{\tilde{\mathfrak{F}}}(A)$ for all $A \in \mathbb{C}^{n,n}$. But this is precisely Theorem 6 of [4]. Q.E.D.

We remark that if all elements $f = (f_1, \ldots, f_n)$ of a collection $\mathfrak{F}$ of G-functions in $\mathscr{P}_n$ are *continuous*, that is, each f_i is continuous on $\mathbb{C}^{n,n}$, $i = 1, 2, \ldots, n$, then to show that $\mathfrak{F}$ is full, it suffices to show that $\mathfrak{F}$ is full at each irreducible $A \in \mathbb{C}^{n,n}$.

5. EXAMPLES

The collections $\mathfrak{F}'$ and $\tilde{\mathfrak{F}}$ of (3.10) and (3.11) are of course examples of full collections. (In fact, if D is any dense subset of $\{x \in \mathbb{C}^n : x > 0\}$, the collections $\{r^x : x \in D\}$ and $\{\hat{r}^x : x \in D\}$ are still full.) However, the collection $\tilde{\mathfrak{F}}$ can be viewed as being generated by the single G-function $r = r^e$ (where $e = (1, \ldots, 1)^T$) of unweighted row sums, in the sense that for every $x \in \mathbb{C}^n$ with $x > 0$,

$$r^x(A) = r(X^{-1}AX), \quad \text{where} \quad X = \operatorname{diag}(x_1, \ldots, x_n).$$

We shall see in Theorem 2 that every element of $\mathscr{P}_n$ generates, in a different way, a full collection of minimal G-functions.

Let g be any element in $\mathscr{P}_n$. For any $A = (a_{i,j}) \in \mathbb{C}^{n,n}$, let

$$\mathscr{P}^g(A) \equiv \begin{bmatrix} g_1(A) & |a_{1,2}| & \cdots & |a_{1,n}| \\ |a_{2,1}| & g_2(A) & & |a_{2,n}| \\ \vdots & \vdots & & \vdots \\ |a_{n,1}| & |a_{n,2}| & \cdots & g_n(A) \end{bmatrix}. \tag{5.1}$$

Clearly, $\mathscr{P}^g(A)$ is a non-negative matrix, and if $\mathscr{P}^g(A)$ is reducible, the normal reduced form for $\mathscr{P}^g(A)$ (see (3.5)) gives us a partitioning of $\mathscr{P}^g(A)$ whose diagonal submatrices $\mathscr{P}^g_{k,k}(A)$ are either irreducible non-negative matrices, or 1×1 null matrices. Let $\lambda_k{}^g(A)$ denote the Perron-Frobenius eigenvalue of the diagonal submatrix $\mathscr{P}^g_{k,k}(A)$. Note that if A is irreducible, then $k = 1 = m$, and $\lambda_1{}^g(A)$ is the spectral radius of $\mathscr{P}^g(A)$. With this, define $f^g = (f_1{}^g, \ldots, f_n{}^g) \in \mathscr{P}_n$ by

$$f_i{}^g(A) = \lambda_k{}^g(A) - g_i(A), \quad \text{where} \quad i \in S_k, \quad i = 1, 2, \ldots, n. \tag{5.2}$$

If $\mathfrak{M}^{f^g}(A) \equiv \mathfrak{M}^{f^g(A)}(A)$ is reducible, the normal reduced form for $\mathfrak{M}^{f^g}(A)$ gives us a partitioning of $\mathfrak{M}^{f^g}(A)$ whose diagonal submatrices are $\mathfrak{M}^{f^g}_{k,k}(A)$. We have by (5.2) that

$$\mathfrak{M}^{f^g}_{k,k}(A) = \lambda_k{}^g(A)I - \mathscr{P}^g_{k,k}(A), \qquad k = 1, 2, \ldots, m; \tag{5.3}$$

by the definition of $\lambda_k{}^g(A)$, each $\mathfrak{M}^{f^g}(A)$ is a singular M-matrix, hence by [1, Theorem 6], f^g is a *minimal* G-function.

On the other hand, consider any minimal G-function f in $\mathscr{P}_n$; for each $A \in \mathbb{C}^{n,n}$, define $\tau_k(A) \equiv \max_{i \in S_k} f_i(A)$, $k = 1, 2, \ldots, m$; and set

$$g_i(A) \equiv \tau_k(A) - f_i(A), \quad \text{where} \quad i \in S_k, \quad i = 1, 2, \ldots, n. \tag{5.4}$$

Because each $g_i(A)$ in (5.3) is non-negative and depends only on the moduli of the off-diagonal entries of A, $g \in \mathscr{P}_n$. Now, we have for $\mathfrak{M}^f(A) \equiv \mathfrak{M}^{f(A)}(A)$,

$$\mathscr{P}^g_{k,k}(A) = \tau_k(A)I - \mathfrak{M}^f_{k,k}(A), \qquad k = 1, 2, \ldots, m. \tag{5.5}$$

Since f is a minimal G-function, each $\mathfrak{M}^f_{k,k}(A)$ is a singular M-matrix, and hence $\tau_k(A) = \lambda_k{}^g(A)$, the Perron-Frobenius eigenvalue of $\mathscr{P}^g_{k,k}(A)$. Hence by (5.2),

$$f_i{}^g(A) = \lambda_k{}^g(A) - g_i(A) = \tau_k(A) - g_i(A) = f_i(A), \tag{5.6}$$

where $i \in S_k$, $i = 1, 2, \ldots, n$, that is, $f^g = f$. We have thus shown

LEMMA 5. *Given any $g \in \mathscr{P}_n$, define $f^g \in \mathscr{P}_n$ by* (5.2). *Then, f^g is a minimal G-function. Conversely, every minimal G-function in $\mathscr{P}_n$ has the form f^g for some $g \in \mathscr{P}_n$.*

Next, for any fixed $g \in \mathscr{P}_n$, let $\mathfrak{F}^g$ be the collection of all minimal G-functions of the form f^h (see Lemma 5), where each $h \in \mathscr{P}_n$ satisfies, for some $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$, with $\gamma \geqslant 0$,

$$h_i(A) = g_i(A) + \gamma_i, \qquad i = 1, 2, \ldots, n, \quad \text{all} \quad A \in \mathbb{C}^{n,n}. \tag{5.7}$$

Thus, $\mathfrak{F}^g$ is generated by a single fixed $g \in \mathscr{P}_n$. We now show that such collections $\mathfrak{F}^g$ are full. Pick any $A \in \mathbb{C}^{n,n}$, and any $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{C}^n$ with $\tau \geqslant 0$ for which $\mathfrak{M}^\tau(A)$ of (3.2) is a singular M-matrix. Defining

$$\lambda_k(A) = \max_{i \in S_k}(g_i(A) + \tau_i), \qquad k = 1, 2, \ldots, m, \tag{5.8}$$

set

$$\gamma_i \equiv \lambda_k(A) - g_i(A) - \tau_i, \quad \text{where} \quad i \in S_k, \quad i = 1, 2, \ldots, n. \tag{5.9}$$

By definition, $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$ with $\gamma \geqslant 0$. For this γ, let $h \in \mathscr{P}_n$ be given by (5.7). We have

$$h_i(A) = g_i(A) + \gamma_i = \lambda_k(A) - \tau_i, \quad \text{where} \quad i \in S_k, \quad i = 1, 2, \ldots, n, \tag{5.10}$$

and, by direct calculation,

$$\mathfrak{M}^{f^h}_{k,k}(A) = \lambda_k{}^h(A)I - \mathscr{P}^h_{k,k}(A) = (\lambda_k{}^h(A) - \lambda_k(A))I + \mathfrak{M}^{\tau}_{k,k}(A), \quad k = 1, 2, \ldots, m. \tag{5.11}$$

Since $\mathfrak{M}^{f^h}_{k,k}(A)$ and $\mathfrak{M}^{\tau}_{k,k}(A)$ are M-matrices, the first of which is singular, we must have $\lambda_k{}^h(A) - \lambda_k(A) \leqslant 0$ for all $k = 1, 2, \ldots, m$. Now, looking at the diagonal entries of the matrices in (5.11), we have, for all $i = 1, 2, \ldots, n$, that $f_i{}^h(A) \leqslant \tau_i$, and hence

$$\inf_{f \in \mathfrak{F}^g} (\max_{1 \leqslant i \leqslant n} (f_i(A) - \tau_i)) \leqslant 0.$$

But by Lemma 3, this quantity is non-negative; hence it is zero. It follows that $\mathfrak{F}^g$ is full at A. But as A is arbitrary in $\mathbb{C}^{n,n}$, we have proved

THEOREM 2. *For any $g \in \mathscr{P}_n$, the collection $\mathfrak{F}^g$ is full.*

As a final remark, it is clear that if $\mathfrak{F}$ is a full collection, then so is the collection

$$\mathfrak{F}^+ = \{f + \varepsilon e : f \in \mathfrak{F}, \varepsilon > 0\},$$

where $e \in \mathscr{P}_n$ is defined by

$$e_i(A) = 1, \quad i = 1, 2, \ldots, n, \quad \text{all} \quad A \in \mathbb{C}^{n,n}.$$

This shows that a full collection of G-functions need *not* contain any *minimal* G-functions.

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Extended L_p-Error Bounds for Spline and L-Spline Interpolation*

STEPHEN DEMKO AND RICHARD S. VARGA

Kent State University, Kent, Ohio 44242

1. INTRODUCTION

Our basic aim here is to extend and improve the error bounds for spline and L-spline interpolation recently given by Swartz and Varga [11]. In so doing, we also extend some recent results of Scherer [9]. To illustrate one such improvement, consider the interpolation of a given function $f \in C^k[a, b]$, with $0 \leqslant k < 2m$, by a smooth polynomial spline $s \in C^{2m-2}[a, b]$, of local degree $2m - 1$ on each segment of a uniform partition Δ of $[a, b]$, where s is uniquely determined from f by means of

$$\begin{aligned} (f - s)(x_i) &= 0, \qquad 1 \leqslant i \leqslant N - 1, \\ D^j(f - s)(a) &= D^j(f - s)(b) = 0 \quad \text{for} \quad 0 \leqslant j \leqslant \min(k, m - 1), \\ D^j s(a) &= D^j s(b) = 0 \quad \text{if} \quad k < j \leqslant m - 1, \end{aligned} \tag{1.1}$$

with $x_i \equiv a + ih$, $h = (b - a)/N$, $0 \leqslant i \leqslant N$. It is known from [11, Theorem 7.4] that there exists a constant K, independent of f and h, such that

$$Kh^{k-j}\omega_\infty(D^k f, h) \geqslant \begin{cases} \| D^j(f - s)\|_{L_\infty[a,b]}, & 0 \leqslant j \leqslant k, \\ \| D^j s \|_{L_\infty[a,b]}, & \text{if} \quad k < j \leqslant 2m - 1, \end{cases} \tag{1.2}$$

where ω_∞ denotes the usual L_∞-modulus of continuity. If $f \in W_p{}^k[a, b]$ with $1 \leqslant k \leqslant 2m$, and $2 \leqslant p \leqslant \infty$, one can deduce from (1.2) (cf. [11, Corollary 7.5]) that

$$\begin{aligned} &Kh^{k-j+(1/q)-(1/p)} \| D^k f \|_{L_p[a,b]} \\ &\geqslant \begin{cases} \| D^j(f - s)\|_{L_q[a,b]}, & 0 \leqslant j \leqslant k - 1, \quad p \leqslant q \leqslant \infty, \\ \| D^j s \|_{L_q[a,b]}, & \text{if} \quad k - 1 < j \leqslant 2m - 1, \quad p \leqslant q \leqslant \infty. \end{cases} \end{aligned} \tag{1.3}$$

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