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Quadratic Interpolatory Splines

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Summary. The projectional properties and associated global and local error bounds for quadratic interpolatory splines are studied, along with applications to the numerical solution of two-point boundary value problems via collocation.

1. Introduction

Our basic aims in this paper are to study the projectional properties of quadratic interpolatory splines, as well as to determine error bounds for quadratic spline interpolation. In § 3, a result of Marsden [13] is established, namely, that the linear interpolating projection operator P_Δ on $C^{-1}[a, b]$, the space of all bounded functions on $[a, b]$, is bounded for any partition Δ of $[a, b]$. In § 4, global error bounds for quadratic spline interpolation are developed. Then, in § 5, local error bounds for quadratic spline interpolation are developed. A numerical example is then given in this section to illustrate these local interpolation error bounds. Finally, an application of quadratic splines in § 6 to the numerical approximation by collocation of solutions of particular two-point boundary value problems is given, and the resulting error in the uniform norm is shown to be $\mathcal{O}(h^4)$ for uniform partitions.

2. Notation

For $-\infty < a < b < +\infty$ and for any positive integer $N \geqq 2$, let

$$\Delta: a = x_0 < x_1 < x_2 < \cdots < x_N = b \tag{2.1}$$

denote a partition of $[a, b]$ with knots x_i. The collection of all such partitions of $[a, b]$ is called $\mathscr{P}(a, b)$. We define

$$\bar{\Delta} = \max\{(x_{i+1} - x_i): 0 \leqq i \leqq N-1\} \quad \text{and} \quad \underline{\Delta} = \min\{x_{i+1} - x_i): 0 \leqq i \leqq N-1\}$$

for each partition of the form (2.1). For any real number σ with $\sigma \geqq 1$, $\mathscr{P}_\sigma(a, b)$ then denotes the subset of $\mathscr{P}(a, b)$ of all partitions Δ for which $\bar{\Delta} \leqq \sigma \underline{\Delta}$. In particular, $\mathscr{P}_1(a, b)$ is the collection of all *uniform* partitions of $[a, b]$.

If π_m denotes the collection of all real algebraic polynomials of degree at most m, then for any nonnegative integer n, the *polynomial spline space* $\mathrm{Sp}(n, \Delta)$ is defined as usual by

$$\begin{aligned} \mathrm{Sp}(n, \Delta) = \{s(x): s \in C^{n-1}[a, b],\ s(x) \in \pi_n \text{ for } x \in (x_i, x_{i+1}), \\ i = 0, 1, \ldots, N-1\}, \end{aligned} \tag{2.2}$$

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where, for the case $n=0$, $C^{-1}[a, b]$ denotes the space of all bounded functions on $[a, b]$. We remark that $\mathrm{Sp}(n, \Delta)$ is a finite-dimensional linear subspace of $C^{n-1}[a, b]$. In particular, we shall be concerned here with the *quadratic spline space* $\mathrm{Sp}(2, \Delta) \subset C^1[a, b]$, which can be seen to be of dimension $N+2$. For additional notation, we set $h_i = x_i - x_{i-1}$, $i=1, 2, \ldots, N$, and, for $g(x)$ defined on $[a, b]$, we set $g_i = g(x_i)$, $i=0, 1, \ldots, N$, and $g_{i+1/2} = g\left(x_i + \frac{h_{i+1}}{2}\right) = g\left(\frac{x_i + x_{i+1}}{2}\right)$ $i=0, 1, \ldots, N-1$.

Given any $s \in \mathrm{Sp}(2, \Delta)$, the restriction of s on any subinterval $[x_{i-1}, x_i]$ determined by Δ is simply the unique quadratic polynomial interpolating the values s_{i-1}, $s_{i-1/2}$, and s_i, respectively in the points x_{i-1}, $x_{i-1/2}$, and x_i. Comparing the restrictions of s on $[x_{i-1}, x_i]$ and on $[x_i, x_{i+1}]$, and using the fact (cf. (2.2)) that s is continuously differentiable at x_i, it is easily verified (cf. Marsden [13]) that

$$a_i s_{i-1} + 3 s_i + c_i s_{i+1} = 4 a_i s_{i-1/2} + 4 c_i s_{i+1/2}, \qquad 1 \leqq i \leqq N-1, \tag{2.3}$$

where

$$a_i \equiv \frac{h_{i+1}}{h_i + h_{i+1}}; \qquad c_i \equiv \frac{h_i}{h_i + h_{i+1}} = 1 - a_i, \qquad 1 \leqq i \leqq N-1, \tag{2.4}$$

so that $0 < a_i$, $c_i < 1$, and $a_i + c_i = 1$, $1 \leqq i \leqq N-1$. In a completely similar way, one can show that

$$\begin{gathered} c_i D s_{i-1} + 3 D s_i + a_i D s_{i+1} = 8(s_{i+1/2} - s_{i-1/2})/(h_i + h_{i+1}), \qquad 1 \leqq i \leqq N-1, \\ 3 D s_0 + D s_1 = 8(s_{1/2} - s_0)/h_1; \; 3 D s_N + D s_{N-1} = 8(s_N - s_{N-1/2})/h_N, \end{gathered} \tag{2.5}$$

where $D s_j \equiv \frac{d s(x_j)}{d x}$. The identities of (2.3) and (2.5) will be used in developing error bounds for quadratic spline interpolation.

3. Sp(2, Δ) Interpolation

Given any $f \in C^{-1}[a, b]$, let $s \in \mathrm{Sp}(2, \Delta)$ be its (unique) interpolant defined by the $N+2$ conditions

$$s_0 = f_0, \qquad s_{i+1/2} = f_{i+1/2}, \qquad 0 \leqq i \leqq N-1, \qquad s_N = f_N. \tag{3.1}$$

From (2.3) and (3.1), this implies that the values $\{s_i\}_{i=1}^{N-1}$ satisfy the $N-1$ linear equations

$$a_i s_{i-1} + 3 s_i + c_i s_{i+1} = 4 a_i f_{i-1/2} + 4 c_i f_{i+1/2}, \qquad 1 \leqq i \leqq N-1, \tag{3.2}$$

with $s_0 = f_0$ and $s_N = f_N$. With (2.4), the associated $(N-1) \times (N-1)$ coefficient matrix for the s_i's from (3.2) is evidently strictly diagonally dominant and hence nonsingular (cf. [17, p. 23]). This proves that the values $\{s_i\}_{i=1}^{N-1}$ are uniquely determined. But, as s is the unique quadratic interpolation on $[x_{i-1}, x_i]$ of the values s_{i-1}, $s_{i-1/2}$, and s_i in the points x_{i-1}, $x_{i-1/2}$, and x_i, then s is also uniquely determined on $[a, b]$.

Our interest now is in the projectional properties of the interpolants of elements in $C^{-1}[a, b]$. The following remarkable result is due to Marsden [13], and, because its proof is both short and elementary, we include it below.

Theorem 3.1. Given any $f \in C^{-1}[a, b]$ and given any $\Delta \in \mathscr{P}(a, b)$, let s be the unique interpolant of f in $\mathrm{Sp}(2, \Delta)$, in the sense of (3.1). Then,

$$\|s\|_{L_\infty[a,b]} \leqq 2\|f\|_{L_\infty[a,b]}. \tag{3.3}$$

Proof. We first show (by a typical diagonal dominance argument) that

$$\max\{|s_i| : 0 \leqq i \leqq N\} \leqq 2\|f\|_{L_\infty[a,b]}. \tag{3.4}$$

From (2.3), $3s_i = 4a_i f_{i-1/2} + 4c_i f_{i+1/2} - a_i s_{i-1} - c_i s_{i+1}$, $1 \leqq i \leqq N-1$. Taking absolute values and using the fact (cf. (2.4)) that $a_i + c_i = 1$, then

$$3|s_i| \leqq 4\|f\|_{L_\infty[a,b]} + \max\{|s_j| : 0 \leqq j \leqq N\}, \quad 1 \leqq i \leqq N-1. \tag{3.5}$$

Let $|s_j| = \max\{|s_i| : 0 \leqq i \leqq N\}$. If $j=0$ or $j=N$, then from (3.1),

$$\max\{|s_j| : 0 \leqq j \leqq N\} = |s_j| = |f_j| \leqq \|f\|_{L_\infty[a,b]},$$

which certainly implies (3.4). If, on the other hand, j satisfies $1 \leqq j \leqq N-1$, then choosing $i=j$ in (3.5) gives $2\max\{|s_j| : 0 \leqq j \leqq N\} \leqq 4\|f\|_{L_\infty[a,b]}$, the desired result of (3.4).

With (3.4), we now establish (3.3). Consider any subinterval $[x_{i-1}, x_i]$ determined by Δ. On this subinterval, the interpolant s is a quadratic polynomial, explicitly given by

$$s(x) = \frac{2(x - x_{i-1/2})}{h_i^2}\{s_{i-1}\cdot(x - x_i) + s_i\cdot(x - x_{i-1})\} + \frac{4f_{i-1/2}\cdot(x - x_{i-1})(x_i - x)}{h_i^2},$$

for $x \in [x_{i-1}, x_i]$. Then, using (3.4) and (3.1), a short calculation shows that

$$|s(x)| \leqq 4\|f\|_{L_\infty[a,b]}\left\{\frac{|x - x_{i-1/2}|}{h_i} + \frac{(x - x_{i-1})(x_i - x)}{h_i^2}\right\} \leqq 2\|f\|_{L_\infty[a,b]}$$

for all $x \in [x_{i-1}, x_i]$. As this holds for an arbitrary subinterval of Δ, then (3.3) is valid. Q.E.D.

If $\mathscr{P}_\Delta : C^{-1}[a, b] \to \mathrm{Sp}(2, \Delta)$ denotes the linear projection mapping of any $f \in C^{-1}[a, b]$ into its unique $\mathrm{Sp}(2, \Delta)$-interpolant $s \equiv P_\Delta f$ in the sense of (3.1), the result of Theorem 3.1 directly gives us the following

Corollary 3.2. For any partition $\Delta \in \mathscr{P}(a, b)$, the projection mapping P_Δ satisfies

$$\|P_\Delta\|_\infty \equiv \sup\{\|P_\Delta f\|_{L_\infty[a,b]} : \|f\|_{L_\infty[a,b]} \leqq 1\} \leqq 2. \tag{3.6}$$

Following Marsden [13] in the periodic case, it can also be shown that the upper of (3.6) is *sharp*, i.e., for any ε with $0 < \varepsilon < 2$, there is a partition Δ of $[a, b]$ for which $\|P_\Delta\|_\infty > 2 - \varepsilon$.

The interesting feature of the above corollary is that the quadratic spline interpolation mappings P_Δ are uniformly bounded, *independent* of any assumption on the partitions Δ, unlike the case for many cubic spline interpolation mappings (cf. [1, 4, 5, 11, 12], and [14]). On the other hand, de Boor [19] has recently shown that the analogue of Corollary 3.2 holds for cubic splines, provided that interpolation occurs at each of the average of three successive knots.

4. Global Error Bounds for Quadratic Spline Interpolation

We derive now global error bounds for quadratic spline interpolation in two distinct ways. The first method couples the projectional properties of quadratic spline interpolation with powerful results from the quasi-interpolation theory of de Boor and Fix [2], and we view this combination as an important and useful tool for deriving such error bounds. The second method in contrast is based on a straight-forward computation in conjunction with either (2.3) or (2.5), and, while the results from this approach are somewhat different, they have a definite appeal of their own. For notation, let P_Δ, as in § 3, denote the linear projection mapping of $C^{-1}[a, b]$ onto $\mathrm{Sp}(2, \Delta)$, and as usual, let $\omega(g, \delta, I) \equiv \sup\{|g(x) - g(y)| : |x-y| \leqq \delta \text{ and } x, y \in I\}$ denote the L_∞-modulus of continuity of g with respect to the interval I. If $I = [a, b]$, we write simply $\omega(g, \delta)$ for $\omega(g, \delta, [a, b])$.

For the first method, because P_Δ is a (bounded linear) projector on $\mathrm{Sp}(2, \Delta)$, i.e., $w = P_\Delta w$ for any $w \in \mathrm{Sp}(2, \Delta)$, it follows that, for any $f \in C^{-1}[a, b]$ and for any $w \in \mathrm{Sp}(2, \Delta)$,

$$\begin{aligned}\|f - P_\Delta f\|_{L_\infty[a,b]} &= \|f - w - P_\Delta f + P_\Delta w\|_{L_\infty[a,b]} = \|(I - P_\Delta)(f - w)\|_{L_\infty[a,b]} \\ &\leqq (1 + \|P_\Delta\|_\infty) \|f - w\|_{L_\infty[a,b]}.\end{aligned}$$

Thus, as this inequality is valid for any $w \in \mathrm{Sp}(2, \Delta)$, it follows from (3.6) that

$$\|f - P_\Delta f\|_{L_\infty[a,b]} \leqq 3 \inf\{\|f - w\|_{L_\infty[a,b]} : w \in \mathrm{Sp}(2, \Delta)\}. \tag{4.1}$$

We now use a special case of results from de Boor and Fix [2]. Recalling the definition of $\mathrm{Sp}(n, \Delta)$ in (2.2), it follows from [2] that if $f \in C^k(I)$ and $I \subseteq [a,b]$ and with $0 \leqq k \leqq n$, then

$$\inf\{\|D^j(f - w)\|_{L_\infty[I]} : w \in \mathrm{Sp}(n, \Delta)\} \leqq c_{j,k} (\bar{\Delta})^{k-j} \omega(D^k f, \bar{\Delta}(I), I), \quad 0 \leqq j \leqq k, \tag{4.2}$$

where the constants $c_{j,k}$ are independent of f, and independent of the partition Δ if $0 \leqq j \leqq \left[\frac{n+1}{2}\right]$, while for $\left[\frac{n+1}{2}\right] < j \leqq n$, the constants $c_{j,k}$ depend only on the local mesh ratio M of (4.3) below, determined from a somewhat larger interval covering I. Specifically, if knots x_{τ_1} and x_{τ_2} of the partition Δ are chosen so that $[x_{\tau_1}, x_{\tau_2}] \supseteq I$, then M and $\bar{\Delta}(I)$ are defined by

$$\begin{aligned} M &\equiv \max\left\{\left(\frac{x_{i+1} - x_i}{x_{j+1} - x_j}\right) : |i - j| = 1 \text{ and } \tau_1 - n \leqq i, j \leqq \tau_2 + n - 1\right\}, \\ \bar{\Delta}(I) &\equiv \max\{x_{j+1} - x_j : \tau_1 \leqq j < \tau_2\}. \end{aligned} \tag{4.3}$$

Thus, coupling the inequality in (4.1) with the inequality in (4.2) for the case $n = 2$, $j = 0$, and $I = [a, b]$, directly yields

Theorem 4.1. Given any $f \in C^k[a, b]$ with $0 \leqq k \leqq 2$, and given any $\Delta \in \mathscr{P}(a, b)$, then

$$\|f - P_\Delta f\|_{L_\infty[a,b]} \leqq C_{0,k} (\bar{\Delta})^k \omega(D^k f, \bar{\Delta}), \tag{4.4}$$

where the constants $C_{0,k}$ are independent of f and Δ.

To similarly derive global derivative arror bounds for $f - P_\Delta f$, the above technique using (4.2) can again be used. Specifically, suppose we wish to bound

$\|D(f-P_\Delta f)\|_{L_\infty[a,b]}$ for $f\in C^k[a,b]$, $1\leqq k\leqq 2$. Writing $f(x)=f(a)+\int_a^x g(t)\,dt$, so that $g(x)=Df(x)$, define $P_{\Delta,1}g$ as

$$(P_{\Delta,1}g)(x)\equiv D(P_\Delta f)(x).$$

It is easily verified that $P_{\Delta,1}$ is a linear projector in $C[a,b]$ with range $\mathrm{Sp}(1,\Delta)$, and one readily establishes as in Corollary 3.2, via a diagonal dominance argument based on (2.5), that $\|P_{\Delta,1}\|_\infty\leqq 2$ for any $\Delta\in\mathscr{P}(a,b)$. Thus, as in (4.1),

$$\|D(f-P_\Delta f)\|_{L_\infty[a,b]}=\|g-P_{\Delta,1}g\|_{L_\infty[a,b]}\leqq 3\inf\{\|g-w\|_{L_\infty[a,b]}:w\in\mathrm{Sp}(1,\Delta)\}.$$

Applying the inequality of (4.2) for the case $j=0$, $n=1$, and $I=[a,b]$ to the right-hand side of the above inequality directly gives

Theorem 4.2. Given any $f\in C^k[a,b]$ and $1\leqq k\leqq 2$, and given any $\Delta\in\mathscr{P}(a,b)$, then

$$\|D(f-P_\Delta f)\|_{L_\infty[a,b]}\leqq C_{1,k}(\bar{\Delta})^{k-1}\omega(D^kf,\bar{\Delta}),\tag{4.5}$$

where the constants $C_{1,k}$ are independent of f and Δ.

The second method for deriving error bounds for quadratic spline interpolation is based on (2.3) and (2.5). The most interesting case to consider is when $f\in C^2[a,b]$. Specifically, if $s=P_\Delta f$ and if $e(x)\equiv s(x)-f(x)$, it follows from (2.5) and (3.1) that

$$\begin{aligned}
&c_iDe_{i-1}+3De_i+a_iDe_{i+1}=8(f_{i+1/2}-f_{i-1/2})/(h_i+h_{i+1})-c_iDf_{i-1}\\
&\qquad\qquad -3Df_i-a_iDf_{i+1},\quad 1\leqq i\leqq N-1,\\
&3De_0+De_1=8(f_{1/2}-f_0)/h_1-3Df_0-Df_1,\\
&3De_N+De_{N-1}=8(f_N-f_{N-1/2})/h_N-3Df_N-Df_{N-1}.
\end{aligned}\tag{4.6}$$

With $f\in C^2[a,b]$, the right sides of (4.6) can be easily shown, using a finite Taylor series representation with remainder, to be bounded above in each case by $\bar{\Delta}\,\omega(D^2f,\bar{\Delta})$. A diagonal dominance argument, like that used in the proof of Theorem 3.1, then shows that

$$\max\{|De_i|:0\leqq i\leqq N\}\leqq\bar{\Delta}\cdot\omega(D^2f,\bar{\Delta})/2.\tag{4.7}$$

Next, consider $D^2e(x)=D^2s(x)-D^2f(x)$ on $[x_{i-1},x_i]$ for any $1\leqq i\leqq N$. Because $D^2e(x)$ is continuous on this interval, the mean value theorem gives the existence of an $\bar{x}\in(x_{i-1},x_i)$ for which $D^2e(\bar{x})=(De_i-De_{i-1})/h_i$. But as $D^2s(x)$ is necessarily a constant on this interval, it follows that $D^2e(x)+D^2f(x)=D^2s(x)=D^2s(\bar{x})=D^2e(\bar{x})+D^2f(\bar{x})$, i.e.,

$$D^2e(x)=D^2e(\bar{x})+D^2f(\bar{x})-D^2f(x)=(De_i-De_{i-1})/h_i+D^2f(\bar{x})-D^2f(x).\tag{4.8}$$

From this and (4.7), it follows that

$$|D^2e(x)|\leqq(\bar{\Delta}/h_i+1)\,\omega(D^2f,\bar{\Delta})\quad\text{for any}\quad x\in[x_{i-1},x_i],\tag{4.9}$$

from which it follows that (cf. (4.14))

$$\|D^2(f-P_\Delta f)\|_{L_\infty[a,b]}\leqq(\bar{\Delta}/\underline{\Delta}+1)\,\omega(D^2f,\bar{\Delta}).\tag{4.10}$$

To similarly determine a bound for $De(x)$ on $[x_{i-1}, x_i]$, let $p(x)$ be the linear function interpolating De_{i-1} and De_i respectively in the points x_{i-1} and x_i. By definition, $Dp(x) = D^2 e(\bar{x})$, and thus from (4.8) for any $x \in [x_{i-1}, x_i]$,

$$De(x) - p(x) = \int_{x_{i-1}}^{x} \{D^2 e(t) - Dp(t)\} dt = \int_{x_{i-1}}^{x} \{D^2 f(\bar{x}) - D^2 f(t)\} dt.$$

Hence,

$$\max\{|De(x)| : x_{i-1} \leqq x \leqq x_{i-1/2}\} \leqq \max\{|De_{i-1}|; |De_i|\} + \int_{x_{i-1}}^{x_{i-1/2}} |D^2 f(\bar{x}) - D^2 f(t)| dt \leqq (\bar{\Delta}/2)\,\omega(D^2 f, \bar{\Delta}) + (h_i/2)\,\omega(D^2 f, \bar{\Delta}),$$

where the second inequality makes use of (4.7). Similarly writing $De(x) - p(x) = \int_{x_i}^{x} \{D^2 f(\bar{x}) - D^2 f(t)\} dt$ shows that the above bound is valid also for $x_{i-1/2} \leqq x \leqq x_i$, i.e.,

$$|De(x)| \leqq (\bar{\Delta} + h_i)\,\omega(D^2 f, \bar{\Delta})/2 \quad \text{for any} \quad x \in [x_{i-1}, x_i]. \tag{4.11}$$

Finally, to determine a bound for $e(x)$ on $[x_{i-1}, x_i]$, note that $e(x_{i-1/2}) = 0$ by definition. Hence,

$$e(x) = \int_{x_{i-1\,2}}^{x} De(t)\, dt \quad \text{for any} \quad x \in [x_{i-1}, x_i],$$

whence

$$|e(x)| \leqq h_i(\bar{\Delta} + h_i)\,\omega(D^2 f, \bar{\Delta})/4 \quad \text{for any} \quad x \in [x_{i-1}, x_i]. \tag{4.12}$$

We summarize this in

Theorem 4.3. Given any $f \in C^2[a, b]$ and given any $\Delta \in \mathscr{P}(a, b)$, then

$$\left.\begin{cases} |(f - P_\Delta f)(x)| \leqq h_i(\bar{\Delta} + h_i)\,\omega(D^2 f, \bar{\Delta})/4 \\ |D(f - P_\Delta f)(x)| \leqq (\bar{\Delta} + h_i)\,\omega(D^2 f, \bar{\Delta})/2 \\ |D^2(f - P_\Delta f)(x) \leqq (\bar{\Delta}/h_i + 1)\,\omega(D^2 f, \bar{\Delta}) \end{cases}\right\} \quad \text{for any} \quad x \in [x_{i-1}, x_i]. \tag{4.13}$$

In particular,

$$\begin{cases} \|f - P_\Delta f\|_{L_\infty[a,b]} \leqq (\bar{\Delta})^2 \omega(D^2 f, \bar{\Delta})/2 \\ \|D(f - P_\Delta f\|_{L_\infty[a,b]} \leqq \bar{\Delta}\,\omega(D^2 f, \bar{\Delta}), \\ \|D^2(f - P_\Delta f)\|_{L_\infty[a,b]} \leqq (\bar{\Delta}/\underline{\Delta} + 1)\,\omega(D^2 f, \bar{\Delta}). \end{cases} \tag{4.14}$$

Note that the bounds of (4.14) are comparable to those of Theorems 4.1 and 4.2, but with explicit multiplicative constants. Moreover, the bounds of (4.13) show the dependence of the interpolation error on the *local* mesh spacing.

Basically, the same derivation in Theorem 4.3 of interpolation error bounds for quadratic spline interpolation for $f \in C^2[a, b]$ can be repeated for the cases when $f \in C^k[a, b]$, $k = 0, 1$, and the resulting global error bounds, like those of (4.14), can similarly be found in Marsden [13].

It is also worthwhile to remark that the error bounds of (4.13) and (4.14) of Theorem 4.3 are valid for other types of boundary interpolation as well. Indeed, Dr. Christian Reinsch has shown[1] that if the unique quadratic spline interpolant

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1 Personal communication.

$s \in \mathrm{Sp}(2, \Delta)$ of $f \in C^2[a, b]$, is defined by (cf. (3.1))

$$s_{i+1/2} = f_{i+1/2}, \qquad 0 \leqq i \leqq N-1, \tag{4.15}$$

and any one of the following boundary conditions:

$$\begin{aligned} &\text{Type I:} && s_0 = f_0, && s_N = f_N, \\ &\text{Type II:} && Ds_0 = Df_0, && Ds_N = Df_N, \\ &\text{Type III:} && s_0 = s_N, && Ds_0 = Ds_N, \end{aligned} \tag{4.16}$$

then the error bounds of (4.13) and (4.14) of Theorem 4.3 remain valid.

5. Local Rates of Convergence

We now develop local rates of convergence for quadratic spline interpolants. The argument parallels techniques found in Swartz and Varga [16] and Kammerer and Reddien [6], in that a given bounded function f is approximated by a smooth function g, and then, using global interpolation error bounds for g developed in § 4, along with the exponential damping properties of the off-diagonal entries of the inverse of the coefficient matrix associated with (2.3), local interpolation error are then obtained for f.

For notation to be used throughout this section, we shall consider any bounded function f defined on $[a, b]$, i.e., $f \in C^{-1}[a, b]$, with $f \in C^k[a', b']$, where $0 \leqq k \leqq 2$ and where $[a', b'] \equiv I' \subsetneqq [a, b]$. We shall then focus our attention on a fixed closed interval $[\alpha, \beta] \subset [a', b']$, where, for a given δ with $0 < \delta < 1/2$,

$$\begin{aligned} &\alpha = a' + \delta(b' - a') \quad \text{whenever} \quad a' \neq a, \quad \alpha = a \quad \text{otherwise, and} \\ &\beta = b' - \delta(b' - a') \quad \text{whenever} \quad b' \neq b, \quad \beta = b \quad \text{otherwise.} \end{aligned} \tag{5.1}$$

We then consider partitions $\Delta \in \mathscr{P}(a, b)$ for which

$$2\bar{\Delta} \leqq \delta. \tag{5.2}$$

With f a given function in $C^{-1}[a, b]$ with $f \in C^k[a', b']$ where $0 \leqq k \leqq 2$, let g denote any fixed function such that

$$\begin{aligned} &\text{i)} \ g \in C^k[a, b], \\ &\text{ii)} \ g \equiv f \quad \text{on} \quad [a', b'] \equiv I', \\ &\text{iii)} \ \omega(D^k g, t) \equiv \omega(D^k f, t, I') \quad \text{for any} \quad 0 < t \leqq (b' - a'). \end{aligned} \tag{5.3}$$

For example, with $g \equiv f$ on $[a', b']$ from (5.3 ii), such a function can be obtained simply by defining g on $[a, a']$ to be the unique element in π_k with $D^j g(a') = D^j f(a')$, $0 \leqq j \leqq k$, with an analogous definition for g in $[b', b]$.

If $[t]$ denotes as usual the greatest integer less than or equal to t, we state

Theorem 5.1. Given any $f \in C^{-1}[a, b]$ with $f \in C^k[a', b']$ where $0 \leqq k \leqq 2$, let $P_\Delta f$ denote the quadratic spline interpolant of f in $\mathrm{Sp}(2, \Delta)$, in the sense of (3.1). Then, for any $\Delta \in \mathscr{P}(a, b)$ with $2\bar{\Delta} \leqq \delta$,

$$\|f - P_\Delta f\|_{L_\infty[\alpha,\beta]} \leqq C_{0,k} (\bar{\Delta})^k \omega(D^k f, \bar{\Delta}(I'), I') + \frac{45 \|g - f\|_{L_\infty[a,b]}}{8 \cdot 3^{[\delta/\bar{\Delta}]}}, \tag{5.4}$$

where g satisfies (5.3), and where the constants $C_{0,k}$ are those of Theorem 4.1.

As an immediate consequence of Theorem 5.1, we have

Corollary 5.2. With the hypotheses of Theorem 5.1, for every $\varepsilon > 0$, there exists an $\eta > 0$ such that for any $\Delta \in \mathscr{P}(a, b)$ with $\bar{\Delta} < \eta$,

$$\|f - P_\Delta f\|_{L_\infty[\alpha,\beta]} \leqq (C_{0,k} + \varepsilon)(\bar{\Delta})^k \omega(D^k f, \bar{\Delta}(I'), I').$$

In particular, if $D^k f$ is constant on I', then

$$\|f - P_\Delta f\|_{L_\infty[\alpha,\beta]} \leqq \frac{45\|g - f\|_{L_\infty[a,b]}}{8 \cdot 3^{[\delta/\bar{\Delta}]}}.$$

We next state

Theorem 5.3. Given any $f \in C^{-1}[a, b]$ with $f \in C^k[a', b']$ where $1 \leqq k \leqq 2$, then for any $\Delta \in \mathscr{P}_\sigma(a, b)$ with $2\bar{\Delta} \leqq \delta$,

$$\|D^j(f - P_\Delta f)\|_{L_\infty[\alpha,\beta]} \leqq C_{j,k}(\bar{\Delta})^{k-j} \omega(D^k f, \bar{\Delta}(I'), I') + \gamma_{j,k} \frac{\|g - f\|_{L_\infty[a,b]}}{3^{[\delta/\bar{\Delta}]}} \tag{5.5}$$

for $1 \leqq j \leqq k$, where the constants $C_{j,k}$ are those of Theorem 4.2 and 4.3, and the constants $\gamma_{j,k}$ are independent of f but dependent on σ.

With the hypotheses of Theorems 5.1 and 5.3 and the fixed function g satisfying (5.3), the triangle inequality gives us for $0 \leqq j \leqq k$ that

$$\begin{aligned} \|D^j(f - P_\Delta f)\|_{L_\infty[\alpha,\beta]} \leqq \|D^j(f - g)\|_{L_\infty[\alpha,\beta]} + \|D^j(g - P_\Delta g)\|_{L_\infty[\alpha,\beta]} \\ + \|D^j(P_\Delta(g - f))\|_{L_\infty[\alpha,\beta]}. \end{aligned} \tag{5.6}$$

From the definition of g in (5.3 ii), the first term on the right side of (5.6) is necessarily zero, while the second term can be bounded above from the results of § 4 by

$$\begin{aligned} \|D^j(g - P_\Delta g)\|_{L_\infty[\alpha,\beta]} &\leqq \|D^j(g - P_\Delta g)\|_{L_\infty[a,b]} \\ &\leqq C_{j,k}(\bar{\Delta})^{k-j} \omega(D^k g, \bar{\Delta}) = C_{j,k}(\bar{\Delta})^{k-j} \omega(D^k f, \bar{\Delta}(I'), I'), \end{aligned}$$

the last equality following from (5.3 iii). Thus, it remains to bound above the last term on the right of (5.6). If $s \equiv P_\Delta(g - f)$, then s satisfies the hypotheses of the following lemma with $M = \|g - f\|_{L_\infty[\alpha,\beta]}$. The results of (5.8)–(5.10) of Lemma 5.4 then establish Theorem 5.1 and 5.3.

Lemma 5.4. Given a positive constant M, and given $\Delta \in \mathscr{P}(a, b)$ satisfying (5.2), let $s \in \mathrm{Sp}(2, \Delta)$ satisfy:

i) $s_{i-1/2} = 0$ whenever $x_{i-1/2} \in [a', b']$; $s_i = 0$ whenever $x_i \in [a', b']$ and $i = 0$ or $i = N$, (5.7)

ii) $|s_{i-1/2}| \leqq M$ for all $1 \leqq i \leqq N$, and $|s_0| \leqq M$, $|s_N| \leqq M$.

Then,

$$\|s\|_{L_\infty[\alpha,\beta]} \leqq \frac{45}{8} M \left(\frac{1}{3}\right)^{[\delta/\bar{\Delta}]}. \tag{5.8}$$

Moreover, if $\Delta \in \mathscr{P}_\sigma(a, b)$, then

$$\|Ds\|_{L_\infty[\alpha,\beta]} \leqq \frac{45}{2\underline{\Delta}} \sigma M \left(\frac{1}{3}\right)^{[\delta/\bar{\Delta}]}, \tag{5.9}$$

and

$$\|D^2 s\|_{L_\infty[\alpha,\beta]} \leqq \frac{45}{(\underline{\Delta})^2} \sigma^2 M \left(\frac{1}{3}\right)^{[\delta/\bar{\Delta}]}. \tag{5.10}$$

Proof. The values s_i at the interior knots x_i, $1 \leqq i \leqq N-1$, are governed by the $(N-1)$ linear equations of (2.3), i.e.,

$$a_i s_{i-1} + 3 s_i + c_i s_{i+1} = 4 a_i s_{i-1/2} + 4 c_i s_{i+1/2}, \quad 1 \leqq i \leqq N-1.$$

Equivalently, in matrix notation, these equations can be expressed as

$$A\, \boldsymbol{s} = \boldsymbol{k},$$

where $\boldsymbol{s} \equiv (s_1, s_2, \ldots, s_{N-1})^T$, and where $\boldsymbol{k} \equiv (k_1, k_2, \ldots, k_{N-1})^T$, with $k_1 = 4 a_1 s_{1/2} + 4 c_1 s_{3/2} - a_1 s_0$, $k_i - 4 a_i s_{i-1/2} + 4 c_i s_{i+1/2}$, $1 < i < N-1$, and $k_{N-1} = 4 a_{N-1} s_{N-3/2} + 4 c_{N-1} s_{N-1/2} = c_{N-1} s_N$. As previously noted, A is a strictly diagonally dominant matrix, and is hence nonsingular. Thus, $\boldsymbol{s} = A^{-1} \boldsymbol{k}$, and on writing $A^{-1} \equiv (b_{i,j})$, then

$$s_i = \sum_{j=1}^{N-1} b_{i,j} k_j, \quad 1 \leqq i \leqq N-1. \tag{5.11}$$

Because of the explicit tridiagonal form of A, a result of Kershaw [7] gives us that

$$|b_{i,j}| \leqq \tfrac{3}{8} \left(\tfrac{1}{3}\right)^{|i-j|}, \quad 1 \leqq i, \quad j \leqq N-1. \tag{5.12}$$

Next, the assumption that $2\bar{\Delta} \leqq \delta$ allows us to deduce that

$$a' \leqq x_{i_0 - 1/2} < x_{i_0 + 1/2} < \cdots < x_{i_0 + r + 1/2} \leqq b'$$

with $i_0 > 1$, $r \geqq 1$, and $i_0 + r + 1/2 < N-1$. Because of the hypothesis of (5.7 i), it follows that $k_j = 0$ for all $i_0 \leqq j \leqq i_0 + r$. Moreover, $|k_i| \leqq 5M$ for all $1 \leqq i \leqq N-1$ from (5.7 ii) and (2.4). Hence, with the bound of (5.12), we have from (5.11) that

$$|s_i| \leqq \frac{15M}{8} \left\{ \sum_{i < i_0} \frac{1}{3^{|i-j|}} + \sum_{i > i_0 + r} \frac{1}{3^{|i-j|}} \right\}, \quad 1 \leqq i \leqq N-1.$$

In particular, consider any $x_i \in [\alpha - \bar{\Delta}, \beta + \bar{\Delta}]$. On summing the above geometric series, it readily follows that

$$|s_i| \leqq \frac{45M}{8} \frac{1}{3^{[\delta/\bar{\Delta}]}} \quad \text{for any} \quad x_i \in [\alpha - \bar{\Delta}, \beta + \bar{\Delta}].$$

Finally, using an argument similar to that used in the end of the proof of Theorem 3.1 shows that

$$\|s\|_{L_\infty[\alpha,\beta]} \leqq \frac{45M}{8} \cdot \frac{1}{3^{[\delta/\bar{\Delta}]}},$$

which establishes (5.8).

Assume now that $\Delta \in \mathscr{P}_\sigma(a, b)$. On each subinterval $[x_{i-1}, x_i]$, $1 \leqq i \leqq N$, the derivative of s is evidently linear, and therefore takes on its extreme values at the endpoints. By direct computation, it can be shown that

$$\|Ds\|_{L_\infty[x_{i-1}, x_i]} = \frac{1}{(x_i - x_{i-1})} \max\{|3 s_{i-1} - 4 s_{i-1/2} + s_i|;\ |s_{i-1} - 4 s_{i-1/2} + 3 s_i|\}.$$

Thus, with the inequality of (5.8), this gives

$$\|Ds\|_{L_\infty[\alpha,\beta]} \leqq \frac{45}{2\bar{\Delta}} M \sigma \left(\frac{1}{3}\right)^{[\delta/\bar{\Delta}]},$$

the desired result of (5.9). As the second derivative of s is piecewise constant on (x_{i-1}, x_i), direct computation shows that

$$D^2 s(x) = \frac{4(s_{i-1} - 2s_{i-1/2} + s_i)}{(x_i - x_{i-1})^2}, \qquad x \in (x_{i-1}, x_i).$$

Again, using the inequality of (5.8) gives the result of (5.10):

$$\|D^2 s\|_{L_\infty[\alpha,\beta]} \leqq \frac{45 M \sigma^2}{(\bar{\Delta})^2} \left(\frac{1}{3}\right)^{[\delta/\bar{\Delta}]}. \qquad \text{Q.E.D.}$$

The local interpolating error bounds given in Theorem 5.3 for $\|D^j(f - P_\Delta f)\|_{L_\infty[\alpha,\beta]}$, $1 \leqq j \leqq 2$, requires the assumption that $\Delta \in \mathscr{P}_\sigma(a, b)$, and as such are weaker than the corresponding result of de Boor and Fix [2] (cf. (4.2)). However, because of the existence of Nord's well-known counterexample [15] for cubic spline interpolation, this assumption, $\Delta \in \mathscr{P}_\sigma(a, b)$, seems to be unavoidable.

For partitions $\Delta \in \mathscr{P}[a, b]$ which are uniform on $[a', b']$, that is, $h = x_i - x_{i-1}$ whenever both x_i and x_{i-1} are in $[a', b']$, the above local error bounds can be improved by a factor of $\bar{\Delta}$ at various points of $[\alpha, \beta]$. Similar high-order pointwise interpolation error bounds have been obtained globally for quadratic splines by Marsden [13], and for cubic spline interpolants by Birkhoff and de Boor [18], Swartz [20], and Lucas [9].

Theorem 5.5. Given any $f \in C^{-1}[a, b]$ with $f \in C^4[a', b']$, and given any partition $\Delta \in \mathscr{P}[a, b]$ which is uniform of size h on $[a', b']$ and satisfies $2\bar{\Delta} \leqq \delta$, the quadratic spline interpolant s of f in $\mathrm{Sp}(2, \Delta)$ in the sense of (3.1) satisfies

$$\begin{aligned} &\text{i)} \ |f(x_i) - s(x_i)| = \mathcal{O}(h^4) && \text{for} \ x_i \in [\alpha, \beta], \\ &\text{ii)} \ |D(f(x_i + \lambda h) - s(x_i + \lambda h))| = \mathcal{O}(h^3) && \text{for} \ x_i + \lambda h \in [\alpha, \beta], \\ &\text{iii)} \ |D^2(f(x_i + \tfrac{1}{2}h) - s(x_i + \tfrac{1}{2}h))| = \mathcal{O}(h^2) && \text{for} \ x_i + \tfrac{1}{2}h \in [\alpha, \beta], \end{aligned} \tag{5.13}$$

where $\lambda \equiv (3 \pm \sqrt{3})/6$.

Proof. From (3.2), the errors $e_i \equiv f_i - s_i$ at the interior knots x_i, $1 \leqq i \leqq N-1$, are seen to satisfy the $N-1$ linear equations

$$a_i e_{i-1} + 3e_i + c_i e_{i+1} = a_i f_{i-1} - 4a_i f_{i-1/2} + 3f_i - 4c_i f_{i+1/2} + c_i f_{i+1}, \tag{5.14}$$

for $1 \leqq i \leqq N-1$. Note that the associated coefficient matrix for the e_i's, determined from the left-hand side of (5.14), is identical to the matrix A defined in the proof of Lemma 5.4. Furthermore, because Δ is uniform on $[a', b']$ and $f \in C^4[a', b']$, Taylor series expansions of f about any $x_i \in [a' + h, b' - h]$ show that the corresponding equations of (5.14) reduce to

$$\tfrac{1}{2} e_{i-1} + 3e_i + \tfrac{1}{2} e_{i-1} = \tfrac{1}{2} f_{i-1} - 2f_{i-1/2} + 3f_i - 2f_{i+1/2} + \tfrac{1}{2} f_{i+1} = \mathcal{O}(h^4). \tag{5.15}$$

Inequality (5.13 i) can now be obtained by utilizing, as in the proof of Lemma 5.4, the exponential damping of the off-diagonal entries of A^{-1}.

To establish (5.13 ii), set $x = x_i + \lambda h$, for $x_i \in [a', b' - h]$ and $0 \leqq \lambda \leqq 1$. Then, by direct computations,

$$Ds(x_i + \lambda h) = \frac{2}{h} \left\{ \left(2\lambda - \frac{3}{2}\right) s_i - 2(2\lambda - 1) s_{i+1/2} + \left(2\lambda - \frac{1}{2}\right) s_{i+1} \right\},$$

and from Taylor series expansions about $x=x_i+\lambda h$,

$$D\big(f(x_i+\lambda h)-s(x_i+\lambda h)\big)=\frac{2}{h}\left\{\left(2\lambda-\frac{3}{2}\right)e_i-2(2\lambda-1)e_{i+1/2}+(2\lambda-1/2)e_{i+1}\right\}$$
$$+\frac{h^2}{3!}\left\{3\lambda^2-3\lambda+\frac{1}{2}\right\}D^3f(x_i+\lambda h)+\mathcal{O}(h^3).$$

But $3\lambda^2-3\lambda+\frac{1}{2}=0$ if and only if $\lambda=(3\pm\sqrt{3})/6$. Assigning either of these values of λ and making use of (5.13 i) gives the desired result (5.13 ii). Inequality (5.13 iii) follows readily from the identity

$$D^2f_{i+1/2}-D^2s_{i+1/2}=4h^{-2}\{e_{i+1}-2e_{i+1/2}+e_i\}-\frac{h^2}{48}f^{(4)}(\xi),$$

for some $\xi\in(x_i,x_{i+1})$, $x_i, x_{i+1}\in[a',b']$. Q.E.D.

In order to illustrate numerically some of the results of this section, we interpolate the function $f\in C^{-1}[0,1]$, defined by

$$f(x)=\begin{cases}\sin 2\pi x, & 0\leqq x\leqq 0.5,\\ -1, & 0.5<x\leqq 1,\end{cases} \tag{5.16}$$

by its unique interpolant, $P_h f$, in $\mathrm{Sp}(2,\Delta_h)$, using uniform partitions Δ_h of size h of $[0,1]$. Column 2 of Table 5.1 contains the maximum interpolation error over the subinterval $[0, 0.25]$. Assuming that $\|f-P_hf\|_{L_\infty[0,0.25]}$ behaves, as a function of h, like Ch^β, then one can estimate the exponent β by successively computing

$$\beta=\ln\left\{\frac{\|f-P_{h_1}f\|_{L_\infty[0,\,0.25]}}{\|f-P_{h_2}f\|_{L_\infty[0,\,0.25]}}\right\}\Big/\ln(h_1/h_2).$$

These computations are contained in column 3 of Table 5.1. Note that Theorem 5.1 gives us that this exponent β theoretically tends to 3, as $h\to 0$. Column 4 of Table 5.1 contains the interpolation error at $x=0.25$, a common nodal point, and column 5 gives the associated observed exponent of h for the numbers in column 4. From Theorem 5.5, we know that these numbers β in column 5 theoretically tend to 4, as $h\to 0$. Finally, since f takes on the constant value -1 on the interval $(1/2, 1]$, Corollary 5.2 implies that $\|f-P_h f\|_{L_\infty[0.70,1]}\leqq K3^{-[1/(4h)]}$. The last column of Table 5.1 shows the interpolation error $|f(x)-P_hf(x)|$ at $x=0.75$ is at least $\mathcal{O}(3^{-[1/(4h)]})$.

Table 5.1

h	$\|f-P_hf\|_{L_\infty[0.25]}$	β	$\vert f(0.25)-P_hf(0.25)\vert$	β	$\vert f(0.75)-P_hf(0.75)\vert$
1/16	$0.561\cdot 10^{-3}$	—	$0.561\cdot 10^{-3}$	—	$0.494\cdot 10^{-3}$
1/32	$0.612\cdot 10^{-4}$	3.19	$0.120\cdot 10^{-4}$	5.50	$0.402\cdot 10^{-6}$
1/48	$0.180\cdot 10^{-4}$	3.01	$0.230\cdot 10^{-5}$	4.08	$0.340\cdot 10^{-9}$
1/64	$0.760\cdot 10^{-5}$	3.01	$0.726\cdot 10^{-6}$	4.00	$0.292\cdot 10^{-12}$
1/128	$0.948\cdot 10^{-6}$	3.00	$0.454\cdot 10^{-7}$	4.00	$< 10^{-15}$

6. A High-Order Collocation Method

Based on the interpolation scheme of § 3, we now develop a specific collocation method for nonlinear two-point boundary value problems. For simplicity, consider the second-order problem

$$D^2 u(x) = f(x, u(x)), \quad a < x < b, \tag{6.1}$$

with homogeneous boundary conditions

$$u(a) = u(b) = 0. \tag{6.2}$$

Recalling the definition of $\mathrm{Sp}(4, \Delta)$ in (2.2), let $\mathrm{Sp}_0(4, \Delta)$ denote the set of all polynomial splines in $\mathrm{Sp}(4, \Delta)$ which satisfy (6.2), and let $\{\Delta_n\}_{n=1}^{\infty}$ be any set of partitions of $[a, b]$ for which $\bar{\Delta}_n \to 0$ as $n \to \infty$. For each n, an approximate solution u_n in $\mathrm{Sp}_0(4, \Delta_n)$ to (6.1)–(6.2) is then defined by means of

$$D^2 u_n(x) = P_n f(x, u_n(x)), \quad a < x < b, \tag{6.3}$$

where P_n is the projection operator $P_{\Delta_n}: C^{-1}[a, b] \to \mathrm{Sp}(2, \Delta_n)$, defined in § 3. Because u_n, if it exists, necessarily satisfies (6.2), and, from the definition of P_n, satisfies the differential Eq. (6.1) at the points a and b, as well as at the midpoint of each subinterval $[x_i^{(n)}, x_{i+1}^{(n)}]$ of Δ_n, then (6.3) clearly defines a collocation **method.**

To settle the existence, uniqueness, and convergence properties of these collocation approximations $\{u_n\}$, we now make use of results of Lucas and Reddien [10]. Specifically, since $\{P_n\}$ is, from Corollary 3.2, a bounded sequence of projections, we directly have from Theorem 3 of [10]

Theorem 6.1. Let $\{\Delta_n\}_{n=1}^{\infty}$ be a sequence of partitions of $[a, b]$ such that $\bar{\Delta}_n \to 0$ as $n \to \infty$, and let $P_n: C^{-1}[a, b] \to \mathrm{Sp}(2, \Delta_n)$ be the associated interpolating projections defined in § 3. Let $u_0 \in C^k[a, b]$, $2 \leqq k \leqq 4$, be a solution of (6.1)–(6.2), and, in some ε-neighborhood of the curve $\mathscr{C} \equiv \{(x, u_0(x)): a \leqq x \leqq b\} \subset \mathbb{R}^2$, let f and f_u be defined and continuous. Further, assume that the linear equation $D^2 u(x) - f_u(x, u_0(x)) \cdot u(x) = 0$, $a < x < b$, with boundary conditions (6.2), has only the trivial solution $u(x) \equiv 0$. Then, in some sphere

$$\mathscr{S} = \{u \in C^2[a, b]: \|D^2(u - u_0)\|_{L_\infty[a,b]} \leqq \varrho, \ \varrho > 0\},$$

the collocation approximations u_n of (6.3) exist and are unique for all n sufficiently large. Moreover,

$$\|D^j(u_0 - u_n)\|_{L_\infty[a,b]} = \mathcal{O}\left((\bar{\Delta}_n)^{k-2}\omega(D^k u_0; \bar{\Delta}_n)\right), \quad 0 \leqq j \leqq 2, \tag{6.4}$$

as $n \to \infty$.

Corollary 6.2. With the hypotheses of Theorem 6.1, assume that $u_0 \in C^5[a, b.]$ Then,

$$\|D^j(u_0 - u_n)\|_{L_\infty[a,b]} = \mathcal{O}\left((\bar{\Delta}_n)^3\right), \quad 0 \leqq j \leqq 2, \tag{6.5}$$

as $n \to \infty$.

We next establish a rather surprising result which improves on Corollary 6.2, namely if $u_0 \in C^6[a, b]$ and if the partitions $\{\Delta_n\}$ are all uniform, then $\|u_0 - u_n\|_{L_\infty[a,b]} = \mathcal{O}\left((\bar{\Delta}_n)^4\right)$. To denote such uniform partitions of $[a, b]$, we write

$$\Delta_n: a = x_0^{(n)} < x_1^{(n)} < \cdots < x_{N_n}^{(n)} = b,$$

where $x_i^{(n)} = a + i h_n$, $0 \leqq i \leqq N_n$, and where $\bar{\Delta}_n = h_n = (b - a)/N_n$.

To establish this result, we first treat the linear problem

$$D^2 u(x) + q(x) \cdot u(x) = g(x), \quad a < x < b, \tag{6.6}$$

with boundary conditions (6.2). The collocation approximation u_n of (6.6) is, from (6.3), defined by

$$D^2 u_n(x) = P_n\{g(x) - q(x) \cdot u_n(x)\}, \quad a < x < b. \tag{6.7}$$

Theorem 6.3. Assume that (6.6)–(6.2) has a unique solution u_0, that the functions q and g of (6.6) are of class $C^4[a, b]$, and that the partitions $\{\Delta_n\}_{n=1}^{\infty}$ are all uniform. Then, the collocation approximations u_n of (6.7) satisfy

$$\|u_0 - u_n\|_{L_\infty[a,b]} = \mathcal{O}(h_n^4), \quad \text{as} \quad n \to \infty. \tag{6.8}$$

To prove Theorem 6.3, we first establish two lemmas.

Lemma 6.4. With the hypotheses of Theorem 6.3, the functions $v_n(x) \equiv D^2 u_n(x) + q(x) \cdot u_n(x)$ satisfy

$$\begin{aligned} \tfrac{1}{2} v_n(x_{i-1}^{(n)}) + 3 v_n(x_i^{(n)}) + \tfrac{1}{2} v_n(x_{i+1}^{(n)}) &= 2\{v_n(x_{i-1/2}^{(n)}) + v_n(x_{i-1/2}^{(n)})\} \\ &\quad + \mathcal{O}(h_n^4), \quad 1 \leqq i \leqq N_n - 1. \end{aligned} \tag{6.9}$$

Proof. Using Theorem 6.1, we note that u_n of (6.7) is well defined for all n sufficiently large, and that $\{D^j u_n\}$, $0 \leqq j \leqq 2$, evidently form bounded sets of functions since they converge. Next, it follows from the case $k = 1$ of Theorem 4.2 that

$$\|D[(g - q u_n) - P_n(g - q u_n)]\|_{L_\infty[a,b]} \leqq 2 C_{1,1} \|D(g - q u_n)\|_{L_\infty[a,b]},$$

so that, from the triangle inequality,

$$\|D P_n(g - q u_n)\|_{L_\infty[a,b]} \leqq (1 + 2 C_{1,1}) \|D(g - q u_n)\|_{L_\infty[a,b]}.$$

Because $\{D^j u_n\}$, $0 \leqq j \leqq 2$, are bounded, as are $\{D^j g\}$ and $\{D^j q\}$, $0 \leqq j \leqq 4$, because of the hypothesis $g, q \in C^4[a, b]$, then from the above inequality, so is the sequence $\{D P_n(g - q u_n)\}$. However, with the definition of u_n in (6.7), this implies that the sequence $\{D^3 u_n\}$ is bounded. But, repeating the above argument and using instead Theorem 4.3 similarly shows that $\{D^4 u_n\}$ is also bounded. Hence, this implies that sequence $\{D^4(q u_n)\}$ is necessarily bounded, a fact which we shall later use.

Returning to the proof of Lemma 6.4, we have $v_n \equiv D^2 u_n + q u_n$. Setting $s_n = P_n v_n$, then by definition

$$s_n = P_n(D^2 u_n + q u_n) = D^2 u_n + P_n(q u_n)$$

since $D^2 u_n \in \mathrm{Sp}(2, \Delta_n)$ and P_n is a projector on $\mathrm{Sp}(2, \Delta_n)$. Thus,

$$s_n - v_n = P_n(q u_n) - (q u_n). \tag{6.10}$$

By a careful application of Taylor's Theorem to the right-hand side of (5.15), (5.13i) of Theorem 5.5 can be extended to functions f in $C^3[a, b]$ which are of class C^4 on each closed subinterval defined by the uniform partition, where the bound in (5.13i) depends on $\|D^4 f\|_{L_\infty[a,b]}$. Applying this result to $q u_n$ and recalling that $\{D^4(q u_n)\}$ is bounded, then

$$P_n(q u_n)(x_i^{(n)}) = (q u_n)(x_i^{(n)}) = \mathcal{O}(h_n^4), \quad 0 \leqq i \leqq N_n,$$

and thus, from (6.10),

$$s_n(x_i^{(n)}) - v_n(x_i^{(n)}) = \mathcal{O}(h_n^4), \quad 0 \leqq i \leqq N_n. \tag{6.11}$$

As s_n is by definition the quadratic spline interpolant of v_n, it follows from (2.3), (2.4), and (3.1) that

$$\tfrac{1}{2}s_n(x_{i-1}^{(n)})+3s_n(x_i^{(n)})+\tfrac{1}{2}s_n(x_{i+1}^{(n)})=2\{v_n(x_{i-1/2}^{(n)})+v_n(x_{i+1/2}^{(n)})\},\qquad 1\leqq i\leqq N_n-1.$$

Substituting (6.11) into the above expression then gives the desired result of (6.9). Q.E.D.

Lemma 6.5. With the hypotheses of Theorem 6.3, the functions $v_n(x)\equiv D^2u_n(x)+q(x)\cdot u_n(x)$ satisfy

$$|g(x_i^{(n)})-v_n(x_i^{(n)})|=\mathcal{O}(h_n^4),\qquad 0\leqq i\leqq N_n. \tag{6.12}$$

Proof. Since u_n satisfies $D^2u_n=P_n\{qu_n\}=P_ng$ from (6.7), it follows that $v_n(x_{i+1/2}^{(n)})=g(x_{i+1/2}^{(n)})$ for $0\leqq i\leqq N_n-1$, and that $v_n(a)=g(a)$ and $v_n(b)=g(b)$. Using (6.9) of Lemma 6.4, then

$$\tfrac{1}{2}v_n(x_{i-1}^{(n)})+3v_n(x_i^{(n)})+\tfrac{1}{2}v_n(x_{i+1}^{(n)})=2\{g(x_{i-1/2}^{(n)})+g(x_{i+1/2}^{(n)})\}+\mathcal{O}(h_n^4),\quad 1\leqq i\leqq N_n-1.$$

Then, arguing as in the proof of (5.13i) of Theorem 5.5 gives the desired result of (6.12). Q.E.D.

Having established Lemma 6.4 and 6.5, we return to the proof of Theorem 6.3. Let G be the Green's function associated with (6.6)–(6.2), i.e., for any $g\in C[a,b]$, the unique solution u_0 of (6.6)–(6.2) is given by

$$u_0(x)=\int_a^b G(x,s)g(s)\,ds,$$

where $G(x,s)$ is defined on the closed square $a\leqq x,\ s\leqq b$. Since the collocation approximations u_n by definition satisfy $D^2u_n+qu_n=v_n$, then similarly

$$u_n(x)=\int_a^b G(x,s)v_n(s)\,ds.$$

For any $x\in[a,b]$, then $x\in[x_{j_0}^{(n)},x_{j_0+1}^{(n)}]$ for suitable j_0. Thus, we can write

$$\begin{aligned}u_n(x)-u_0(x)&=\int_a^b G(x,s)\big(v_n(s)-g(s)\big)ds\\&=\sum_{i\leqq j_0}\int_{x_{i-1}^{(n)}}^{x_i^{(n)}} G(x,s)\big(v_n(s)-g(s)\big)ds+\int_{x_{j_0}^{(n)}}^{x_{j_0+1}^{(n)}} G(x,s)\big(v_n(s)-g(s)\big)ds\\&\quad+\sum_{i>j_0}\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} G(x,s)\big(v_n(s)-g(s)\big)ds.\end{aligned}\tag{6.13}$$

It is known that G is continuous on $[a,b]\times[a,b]$, and, because $q\in C^4[a,b]$, there exists a constant K such that

$$\sup_{a\leqq s\leqq b}\left[\sup\left\{\left|\frac{\partial^4}{\partial s^4}G(x,s)\right|: a\leqq x<s;\ s<x\leqq b\right\}\right]\leqq K.$$

Now, recalling that $g \in C^4[a, b]$, and that v_n is of class C^4 on each $[x_i^{(n)}, x_{i+1}^{(n)}]$, then Simpson's rule implies for $i \neq j_0$ that

$$\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} G(x, s)\left(v_n(s) - g_n(s)\right) ds = \frac{h_n}{6}\Big\{G(x, x_i^{(n)}) \cdot \left(v_n(x_i^{(n)}) - g(x_i^{(n)})\right) + 4\,G(x, x_{i+1/2}^{(n)}) \cdot \left(v_n(x_{i+1/2}^{(n)}) - g(x_{i+1/2}^{(n)})\right) + G(x, x_{i+1}^{(n)}) \cdot \left(v_n(x_{i+1}^{(n)}) - g(x_{i+1}^{(n)})\right)\Big\} + \mathcal{O}(h_n^5). \tag{6.14}$$

Using Lemma 6.5 and the fact that $v_n(x_{i+1/2}^{(n)}) = g(x_{i+1/2}^{(n)})$, it follows that

$$\left|\int_{x_i^{(n)}}^{x_{i+1}^{(n)}} G(x, s)\left(v_n(s) - g(s)\right) ds\right| = \mathcal{O}(h_n^5), \qquad i \neq j_0. \tag{6.15}$$

For the case $i = j_0$, it suffices to use the continuity of G and the fact that $v_n - g$ can be expressed as $v_n - g = D^2(u_n - u_0) + q(u_n - u_0)$. Thus, from Corollary 6.2, $\|v_n - g\|_{L_\infty[a,b]} = \mathcal{O}(h_n^3)$, so that

$$\left|\int_{x_{j_0}^{(n)}}^{x_{j_0+1}^{(n)}} G(x,s)\left(v_n(s) - g(s)\right) ds\right| = \mathcal{O}(h_n^4). \tag{6.16}$$

Coupling the results of (6.16) and (6.15) with the expression in (6.13) yields the desired result (6.8) that $\|u_n - u_0\|_{L_\infty[a,b]} = \mathcal{O}(h_n^4)$. Q.E.D.

We next extend the results of Theorem 6.3 to the nonlinear problem of (6.1)–(6.2). The arguments to be given are based in part upon a similar development in de Boor and Swartz [3]. For notation, again let u_0 be a solution of (6.1)–(6.2), and let $\mathcal{N}$ denote the closure of an ε-neighborhood of the curve $\mathcal{C} \equiv \{(x, u_0(x)) : a \leq x \leq b\} \subset \mathbb{R}^2$.

Theorem 6.6. In addition to the hypotheses of Theorem 6.1, assume that the solution u_0 of (6.1)–(6.2) is in $C^6[a, b]$, with $f_u(x, u_0(x)) \in C^4[a, b]$, and with $\frac{\partial^2 f}{\partial u^2}(x, u)$ continuous in $\mathcal{N}$. If all partitions in the sequence $\{\Delta_n\}$ are uniform partitions of $[a, b]$, then in some sphere

$$\mathcal{S} = \{u \in C^2[a, b] : \|D^2(u - u_0)\|_{L_\infty[a,b]} \leq \varrho,\ \varrho > 0\}, \tag{6.17}$$

the collocation approximations u_n of (6.3) exist and are unique for all n sufficiently large, and

$$\|u_n - u_0\|_{L_\infty[a,b]} = \mathcal{O}(h_n^4), \qquad \text{as} \quad n \to \infty.$$

Proof. From Theorem 6.1, we are guaranteed, for all n sufficiently large, of the existence and uniqueness of the approximations u_n near u_0. Moreover, for n sufficiently large, $u_n \in \mathcal{N}$, and from Taylor's Theorem,

$$f(x, u_n) = f(x, u_0) + f_u(x, u_0) \cdot (u_n - u_0) + E(u_n),$$

where $E(u_n) = \mathcal{O}(\|u_n - u_0\|^2_{L_\infty[a,b]}) = \mathcal{O}(h_n^6)$, using Corollary 6.2. Applying P_n to the above expression and using (6.1) and (6.3), we can write

$$D^2 u_n - P_n\{f_u(x, u_0) \cdot u_n\} = P_n D^2 u_0 - P_n\{f_u(x, u_0) \cdot u_0\} + P_n E(u_n). \tag{6.18}$$

Now, let $z_n(x)$ be the solution of

$$D^2 z_n - P_n\{f_u(x, u_0) \cdot z_n\} = P_n D^2 u_0 - P_n\{f_u(x, u_0) \cdot u_0\}, \tag{6.19}$$

and let $z(x)$ be the solution of

$$D^2 z - f_u(x, u_0) \cdot z = D^2 u_0 - f_u(x, u_0) \cdot u_0, \tag{6.20}$$

both having the boundary conditions (6.2). The hypotheses of Theorem 6.1 imply that (6.20) has the unique solution $z = u_0$, and that z_n exists and is unique for all n sufficiently large. Because all the hypotheses of Theorem 6.3 are satisfied, we thus have from (6.8) that $\|u_0 - z_n\|_{L_\infty[a,b]} = \|z - z_n\|_{L_\infty[a,b]} = \mathcal{O}(h_n^4)$. Next, subtracting (6.19) from (6.18) yields

$$D^2(u_n - z_n) - P_n\{f_u(x, u_0) \cdot (u_n - z_n)\} = P_n E(u_n). \tag{6.21}$$

Now, define w_n to be the solution of

$$D^2 w_n - f_u(x, u_0) \cdot w_n = E(u_n), \tag{6.22}$$

with the boundary conditions (6.2). Because $E(u_n) = \mathcal{O}(h_n^6)$, a Green's function argument applied to (6.22) readily establishes that $\|w_n\|_{L_\infty[a,b]} = \mathcal{O}(h_n^6)$. Then, write $u_n - u_0 = \{(u_n - z_n) - w_n\} + w_n + \{z_n - u_0\}$. The last two terms have both been shown to be $\mathcal{O}(h_n^4)$. Thus, if we can show that $\|(u_n - z_n) - w_n\|_{L_\infty[a,b]} = \mathcal{O}(h_n^4)$, we will have established the desired result of (6.17). This, however, will follow from the next lemma. Note from (6.21) that $u_n - z_n$ is by definition (cf. (6.3)) the collocation approximation of w_n in (6.22).

Lemma 6.7. Let $\{\Delta_n\}$ be a sequence of partitions of $[a, b]$ with $\bar{\Delta}_n \to 0$ as $n \to \infty$, and let $P_n: C^{-1}[a, b] \to \mathrm{Sp}(2, \Delta_n)$ be the interpolation projection of § 3. For a given $q \in C^0[a, b]$, assume that the homogeneous problem $D^2 u(x) + q(x) \cdot u(x) = 0$, $a < x < b$, with $u(a) = u(b) = 0$, has only the trivial solution $u(x) \equiv 0$. If $\{g_n\}$ is a sequence of continuous functions on $[a, b]$ satisfying $\|g_n\|_{L_\infty[a,b]} = \mathcal{O}((\bar{\Delta}_n)^\alpha)$, $\alpha > 0$, then

$$\|w_n - \hat{w}_n\|_{L_\infty[a,b]} = \mathcal{O}((\bar{\Delta}_n)^\alpha), \tag{6.23}$$

where w_n is the solution of

$$D^2 w_n(x) + q(x) w_n(x) = g_n(x), \quad a < x < b, \quad w_n(a) = w_n(b) = 0, \tag{6.24}$$

while $\hat{w}_n$ is the solution of

$$D^2 \hat{w}_n(x) + P_n(q(x) \cdot \hat{w}_n(x)) = P_n(g_n(x)); \quad \hat{w}_n \in \mathrm{Sp}(4, \Delta_n). \tag{6.25}$$

Proof. Let $\tilde{G}(x, s)$ be the spezific Green's function associated with $D^2 w(x) = 0$, $a < x < b$, and (6.2), i.e., given $v \in C^0[a, b]$, if $D^2 w = v$ with $w(a) = w(b) = 0$, then $w(x) = \int_a^b \tilde{G}(x, s) v(s)\, ds \equiv [\tilde{G} v](x)$. With this notation, the boundary value problems (6.24) and (6.25) are respectively equivalent to

$$v_n + T v_n = g_n, \quad D^2 w_n = v_n, \quad w_n(a) = w_n(b) = 0, \tag{6.26}$$

and

$$\hat{v}_n + P_n T \hat{v}_n = P_n g_n, \quad D^2 \hat{w}_n = \hat{v}_n, \quad \hat{w}_n(a) = \hat{w}_n(b) = 0, \tag{6.27}$$

where $Tv \equiv q\tilde{G}[v]$ for all $v \in C^0[a, b]$. Applying P_n to (6.26), subtracting the result from (6.27), and then subtracting v_n from both sides leads to the equation

$$(I + P_n T)(\hat{v}_n - v_n) = (P_n - I)\, v_n. \tag{6.28}$$

Now, the bounded linear operator $(I+T)^{-1}$ exists, since by hypothesis, (6.26) has a unique solution for any $g_n \in C^0[a, b]$. Then, it follows from Theorem 4.1 that the completely continuous operators $P_n T$ converge in norm to T as $n \to \infty$, and thus, there exist an n_0 and a $K > 0$ such that for all $n \geqq n_0$, $(I + P_n T)^{-1}$ exists and $\|(I + P_n T)^{-1}\| \leqq K$. Thus, from (6.28) and Corollary 3.2.

$$\|\hat{v}_n - v_n\|_{L_\infty[a,b]} \leqq 3K \|v_n\|_{L_\infty[a,b]} \quad \text{for all} \quad n \geqq n_0.$$

But, the hypothesis that $\|g_n\| = \mathcal{O}((\bar{\Delta}_n)^\alpha)$ coupled with $(I+T)\hat{v}_n = g_n$ from (6.26) implies that $\|v_n\| = \mathcal{O}((\bar{\Delta}_n)^\alpha)$. Hence, the above inequality becomes $\|v_n - v_n\|_{L_\infty[a,b]} = \mathcal{O}((\bar{\Delta}_n)^\alpha)$. But, by definition, $\hat{w}_n(x) - w_n(x) = \int_a^b \tilde{G}(x, s) \cdot (\hat{v}_n(s) - v_n(s))\, ds$, and the desired inequality of (6.23) immediately follows. Q. E. D.

The collocation method described in this section for second-order two-point boundary value problems is intermediate to those usually studied. In using smooth quadratic splines, the collocation equations are as easy to define, and their accuracy is moreover comparable $(\mathcal{O}(h^4))$ to that obtained from collocation for smooth cubic splines. When B-splines are used as a basis for quadratic splines, the resulting matrix problem generated by the collocation method of (6.3) will be essentially five-diagonal.

To give concrete illustrations of Theorems 6.3 and 6.6, consider the numerical approximations of the following two-point boundary value problems

$$D^2 u(x) - 4u(x) = 4\cosh(1), \qquad 0 < x < 1, \tag{6.29}$$

and

$$D^2 u(x) = e^{u(x)}, \qquad 0 < x < 1, \tag{6.30}$$

both subject to the homogeneous boundary conditions

$$u(0) = u(1) = 0. \tag{6.31}$$

The function $u(x) = \cosh(2x - 1) - \cosh(1)$ is the unique solution of (6.29)–(6.31), while $u(x) = -\ln 2 + 2\ln\{c \sec[0.5c(x - 0.5)]\}$ with $c = \sqrt{2}\cos(c/4) \doteqdot 1.3360556949$ is the unique solution of (6.30)–(6.31). Both solutions are of class $C^\infty[0, 1]$, so that the error bounds (6.17) of Theorem 6.6 are applicable. Observed numerical errors and observed rates of convergence for these two problems for uniform partitions of [0, 1] are given in Tables 6.1 and 6.2, respectively. For comparison, we remark that the solutions of these two problems have also been approximated numerically by collocation with piecewise quintics in de Boor and Swartz [3].

Table 6.1. Observed errors and rates associated with (6.29)–(6.31)

h	$\|u_0 - u_n\|_{L_\infty[0,1]}$	β
1/5	$0.355 \cdot 10^{-4}$	—
1/7	$0.926 \cdot 10^{-5}$	3.99
1/9	$0.339 \cdot 10^{-5}$	3.99
1/18	$0.212 \cdot 10^{-6}$	3.99
1/36	$0.132 \cdot 10^{-7}$	4.005

Table 6.2. Observed errors and rates associated with (6.30)–(6.31)

h	$\|u_0 - u_n\|_{L_\infty[0,1]}$	β
1/4	$0.550 \cdot 10^{-5}$	—
1/8	$0.341 \cdot 10^{-6}$	4.01
1/16	$0.213 \cdot 10^{-7}$	4.00
1/32	$0.134 \cdot 10^{-8}$	3.99

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