

Convergence of Padé Approximants to e^{-z} on Unbounded Sets

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DEDICATED TO PROFESSOR G. G. LORENTZ ON THE OCCASION
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1. INTRODUCTION

The basic aims of this paper are to study the convergence in the uniform norm of particular Padé approximants to e^{-z} on certain unbounded sets in the complex plane. After some preliminary results are developed in Section 2, we consider in Section 3 the convergence of Padé approximants to e^{-z} on the ray $\{z = x + iy : x \geq 0, y = 0\}$. In Theorem 3.1, we give a necessary and sufficient condition for the uniform convergence of a sequence of Padé approximants to e^{-z} on this set, while in Theorem 3.2, we give a sufficient condition for the geometric convergence of a sequence of Padé approximants to e^{-z} on this set. Also, an application of these results to the problem of constrained Chebyshev rational approximations to e^{-x} on $[0, +\infty)$ is included in this section.

In Section 4, the geometric convergence of the particular Padé approximants $\{R_{0,n}(z)\}_{n=0}^{\infty}$ to e^{-z} on unbounded parabolic-like sets in the complex plane is derived in Theorem 4.1, while in Theorem 4.3, it is shown that the particular Padé approximants $\{R_{n-1,n}(z)\}_{n=1}^{\infty}$ and $\{R_{n-2,n}(z)\}_{n=2}^{\infty}$ converge uniformly to e^{-z} on the sectors $S_{\delta} \equiv \{z = re^{i\theta} : |\theta| \leq (\pi/2) - \delta\}$, for any $0 < \delta \leq (\pi/2)$.

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2. NOTATION AND PRELIMINARY RESULTS

We shall make of the following notation. Let π_m denote the set of all complex polynomials in the variable z having degree at most m , and let $\pi_{\nu,n}$ denote the set of all complex rational functions $r_{\nu,n}(z)$ of the form

$$r_{\nu,n}(z) = \frac{q_{\nu,n}(z)}{p_{\nu,n}(z)} \text{ where } q_{\nu,n} \in \pi_{\nu}, p_{\nu,n} \in \pi_n, p_{\nu,n}(0) = 1.$$

Then, given any function $f(z) = \sum_{k=0}^{\infty} a_k z^k$ analytic in a neighborhood of $z = 0$, and given any nonnegative integers ν and n , the (ν, n) -th Padé approximant to $f(z)$ is defined as that element $R_{\nu,n} \in \pi_{\nu,n}$ for which the following expression,

$$f(z) - R_{\nu,n}(z) = \mathcal{O}(|z|^m) \text{ as } |z| \rightarrow 0, \tag{2.1}$$

is valid with the largest integer m . In the case that $f(z) = e^{-z}$, the (ν, n) -th Padé approximant $R_{\nu,n}(z) \equiv Q_{\nu,n}(z)/P_{\nu,n}(z)$ of e^{-z} is explicitly given by (cf. [12, p. 433; 15, p. 269])

$$Q_{\nu,n}(z) \equiv \sum_{k=0}^{\nu} \frac{(\nu+n-k)! \nu! (-z)^k}{(\nu+n)! k! (\nu-k)!}, \tag{2.2}$$

and

$$P_{\nu,n}(z) \equiv \sum_{k=0}^n \frac{(\nu+n-k)! n! z^k}{(\nu+n)! k! (n-k)!}. \tag{2.3}$$

It is further known that (cf. [11; 12, p. 436; 14]), for finite z ,

$$\epsilon_{\nu,n}(z) \equiv R_{\nu,n}(z) - e^{-z} = \frac{(-1)^{\nu} z^{n+\nu+1}}{(n+\nu)! e^z P_{\nu,n}(z)} \int_0^1 e^{tz} t^{\nu} (1-t)^n dt. \tag{2.4}$$

In particular, this expression shows that (2.1) is always valid with

$$m = n + \nu + 1,$$

when $f(z) = e^{-z}$. Moreover, as $P_{\nu,n}(x) \geq 1$ from (2.3) for all $x \geq 0$, it also follows from (2.4) that the error, $\epsilon_{\nu,n}(x)$, for the (ν, n) -th Padé approximant to e^{-x} , is of one sign for all $x \geq 0$. It is convenient to define the numbers $\eta_{\nu,n}$ as

$$\eta_{\nu,n} \equiv \sup\{|\epsilon_{\nu,n}(x)| : x \geq 0\} = \|R_{\nu,n} - e^{-x}\|_{L_{\infty}[0,\infty]}. \tag{2.5}$$

We begin with

PROPOSITION 2.1. $\eta_{\nu,\nu} = 1$ for all integers $\nu \geq 0$.

Proof. First, assume $\nu = 2j, j \geq 0$. From (2.4), $\epsilon_{2j,2j}(x) \geq 0$ for all $x \geq 0$. Next, it is clear upon comparing coefficients in (2.2)-(2.3) that $R_{2j,2j}(x) \leq 1$ for all $x \geq 0$, with $R_{2j,2j}(x) \rightarrow 1$ as $x \rightarrow +\infty$. Hence,

$$0 \leq \epsilon_{2j,2j}(x) = R_{2j,2j}(x) - e^{-x} \leq 1 - e^{-x} \leq 1 \text{ for all } x \geq 0,$$

with $\epsilon_{2j,2j}(x) \rightarrow 1$ as $x \rightarrow +\infty$. Thus it follows that $\eta_{2j,2j} = 1$.

Assuming $\nu = 2j + 1, j \geq 0$, note that $Q_{2j+1,2j+1}(x) = P_{2j+1,2j+1}(-x)$ for any real x . Thus, we can write $R_{2j+1,2j+1}(x) = P_{2j+1,2j+1}(-x)/P_{2j+1,2j+1}(x)$, or equivalently,

$$R_{2j+1,2j+1}(x) = \frac{\{P_{2j+1,2j+1}(x) + P_{2j+1,2j+1}(-x)\}}{P_{2j+1,2j+1}(x)} - 1 \text{ for all } x.$$

But from (2.3), $\{P_{2j+1,2j+1}(x) + P_{2j+1,2j+1}(-x)\} \equiv \tilde{P}_{2j}(x)$, a polynomial of degree $2j$, is a positive sum of even powers of x with constant term 2, so that trivially $\tilde{P}_{2j}(x) \geq 1$ for all $x \geq 0$. Next, by comparing coefficients, it is easy to verify from (2.3) that $P_{2j+1,2j+1}(x) \leq e^x$ for all $x \geq 0$. Thus, from (2.4),

$$\begin{aligned} 0 &\leq -\epsilon_{2j+1,2j+1}(x) = e^{-x} - R_{2j+1,2j+1}(x) = e^{-x} - \frac{\tilde{P}_{2j}(x)}{P_{2j+1,2j+1}(x)} + 1 \\ &\leq e^{-x} - \frac{1}{e^x} + 1 = 1, \text{ for } x \geq 0, \end{aligned}$$

with $-\epsilon_{2j+1,2j+1}(x) \rightarrow 1$ as $x \rightarrow \infty$. Thus, $\eta_{2j+1,2j+1} = 1$.

Q.E.D.

We now state an identity in (2.6), which can be obtained by directly appealing to the definitions of $Q_{\nu,n}$ and $P_{\nu,n}$ in (2.2) and (2.3).

LEMMA 2.2. For any $\nu \geq 0, n \geq 1$,

$$\frac{d}{dx} [e^x Q_{\nu,n}(x) - P_{\nu,n}(x)] = \frac{n}{(n+\nu)} [e^x Q_{\nu,n-1}(x) - P_{\nu,n-1}(x)]. \quad (2.6)$$

With these results and with the definition of $\eta_{\nu,n}$ in (2.5), we now prove

THEOREM 2.3. For any nonnegative integers ν and n with $n > \nu$,

$$\eta_{\nu,n} \leq \frac{n}{(2n+\nu)} \eta_{\nu,n-1}. \quad (2.7)$$

Thus,

$$\eta_{\nu,n} \leq \prod_{j=1}^{n-\nu} \left(\frac{\nu+j}{3\nu+2j} \right) \leq \frac{1}{2^{n-\nu}}, \quad \text{for all } 0 \leq \nu < n. \quad (2.8)$$

Proof. Using (2.4), we see that $(-1)^\nu \epsilon_{\nu,n}(x) \geq 0$ for all $x \geq 0$. Because n exceeds ν by hypothesis then $\epsilon_{\nu,n}(x) \rightarrow 0$ as $x \rightarrow +\infty$. Hence, there exists a $\xi > 0$ for which $(-1)^\nu \epsilon_{\nu,n}(\xi) = \eta_{\nu,n}$, and $\epsilon'_{\nu,n}(\xi) = 0$. Now, from (2.4), we can write

$$(-1)^\nu \epsilon_{\nu,n}(x) \cdot e^x \cdot P_{\nu,n}(x) = (-1)^\nu \{e^x Q_{\nu,n}(x) - P_{\nu,n}(x)\}.$$

Thus, on differentiating the above expression and evaluating the result at $x = \xi$, we obtain since $\epsilon'_{\nu,n}(\xi) = 0$ that

$$(-1)^\nu \epsilon_{\nu,n}(\xi) e^\xi [P_{\nu,n}(\xi) + P'_{\nu,n}(\xi)] = (-1)^\nu \frac{d}{dx} [e^x Q_{\nu,n}(x) - P_{\nu,n}(x)]|_{x=\xi}.$$

Hence, from (2.6) of Lemma 2.2,

$$(-1)^\nu \epsilon_{\nu,n}(\xi) e^\xi [P_{\nu,n}(\xi) + P'_{\nu,n}(\xi)] = \frac{(-1)^\nu n}{(n + \nu)} [e^\xi Q_{\nu,n-1}(\xi) - P_{\nu,n-1}(\xi)]. \quad (2.9)$$

Now, from (2.3), it is easy to verify that $P'_{\nu,n}(x) = (n/(n + \nu)) P_{\nu,n-1}(x)$ for all x , and that $P_{\nu,n}(x) \geq P_{\nu,n-1}(x)$ for all $x \geq 0$. Thus,

$$(-1)^\nu \epsilon_{\nu,n}(\xi) e^\xi [P_{\nu,n}(\xi) + P'_{\nu,n}(\xi)] \geq \left(\frac{2n + \nu}{n + \nu}\right) (-1)^\nu \epsilon_{\nu,n}(\xi) e^\xi P_{\nu,n-1}(\xi).$$

Using (2.9), this implies that

$$\frac{(-1)^\nu n}{(n + \nu)} [e^\xi Q_{\nu,n-1}(\xi) - P_{\nu,n-1}(\xi)] \geq \left(\frac{2n + \nu}{n + \nu}\right) (-1)^\nu \epsilon_{\nu,n}(\xi) e^\xi P_{\nu,n-1}(\xi),$$

or

$$0 \leq (-1)^\nu \epsilon_{\nu,n}(\xi) \leq \frac{(-1)^\nu n}{(2n + \nu)} [R_{\nu,n-1}(\xi) - e^{-\xi}] = \frac{(-1)^\nu n}{(2n + \nu)} \epsilon_{\nu,n-1}(\xi).$$

Since $(-1)^\nu \epsilon_{\nu,n}(\xi) = \eta_{\nu,n}$ and since $(-1)^\nu \epsilon_{\nu,n-1}(\xi) \leq \eta_{\nu,n-1}$, then

$$\eta_{\nu,n} \leq \left(\frac{n}{2n + \nu}\right) \eta_{\nu,n-1},$$

the desired result of (2.7). By induction on the above inequality, it follows that

$$\eta_{\nu,n} \leq \left\{ \prod_{j=1}^{n-\nu} \left(\frac{\nu + j}{3\nu + 2j}\right) \right\} \eta_{\nu,\nu} = \prod_{j=1}^{n-\nu} \left(\frac{\nu + j}{3\nu + 2j}\right), \quad (2.10)$$

the last expression following from Proposition 2.1. But as each term in the above product is at most $\frac{1}{2}$, then we obtain the desired result of (2.8).

Q.E.D.

We remark that the inequality of (2.8) reduces in the case $\nu = 0$ to

$$\eta_{0,n} \leq 1/2^n,$$

which was first established in Cody, Meinardus, and Varga [2].

For the special case $\nu = n - 1$, we note that the inequality of (2.7), coupled with Proposition 2.1, gives simply

$$\eta_{n-1,n} \leq \left(\frac{n}{3n-1} \right), \quad \text{for all } n \geq 1,$$

which implies only the boundedness of the sequence $\{\eta_{n-1,n}\}_{n=1}^{\infty}$. Actually, for our later use in Theorem 3.1, we need that $\eta_{n-1,n}$ tends to zero as $n \rightarrow \infty$, but we prove the following stronger result.

PROPOSITION 2.4. *There exist positive constants A_1 and A_2 such that*

$$\frac{A_1}{n} \leq \eta_{n-1,n} \leq \frac{A_2 \ln n}{n} \quad \text{for all } n > 1. \quad (2.11)$$

Proof. With definitions in (2.2) and (2.3), it is easy to show, by comparing coefficients, that

$$|R_{n-1,n}(x)| = \left| \frac{Q_{n-1,n}(x)}{P_{n-1,n}(x)} \right| \leq \frac{Q_{n-1,n}(-x)}{P_{n-1,n}(x)} \leq \left(1 + \frac{x}{2n-1} \right)^{-1}$$

for all $x \geq 0$, $n \geq 1$. Thus, from (2.4),

$$|\epsilon_{n-1,n}(x)| \leq |R_{n-1,n}(x)| + e^{-x} \leq e^{-x} + \left(1 + \frac{x}{2n-1} \right)^{-1}, \quad x \geq 0. \quad (2.12)$$

On the other hand, the integral representation in (2.4) gives us that

$$|\epsilon_{n-1,n}(x)| = \frac{x^{2n}}{(2n-1)! e^x P_{n-1,n}(x)} \int_0^1 e^{tx} t^{n-1} (1-t)^n dt, \quad x \geq 0,$$

which can be written in the form

$$|\epsilon_{n-1,n}(x)| = \frac{x^{2n}}{(2n-1)! P_{n-1,n}(x)} \int_0^1 e^{-tx} t^n (1-t)^{n-1} dt, \quad x \geq 0.$$

A simple calculation shows that the above integrand, considered as a function of $t \in [0, 1]$ is maximized when $t = u_n(x)$, where

$$0 < u_n(x) \equiv \frac{2n}{(2n-1+x) + ((2n-1+x)^2 - 4nx)^{1/2}} < 1, \quad n > 1. \quad (2.13)$$

Thus, $|\epsilon_{n-1,n}(x)|$ can be bounded above by

$$|\epsilon_{n-1,n}(x)| \leq \frac{x^{2n}(u_n(x))^n(1-u_n(x))^{n-1}}{(2n-1)!e^{x \cdot u_n(x)}P_{n-1,n}(x)}, \quad x \geq 0.$$

Next, it follows from (2.3) that $P_{n-1,n}(x) \geq ((n-1)!x^n/(2n-1)!)$ for all $x \geq 0$, so that

$$|\epsilon_{n-1,n}(x)| \leq \frac{[x \cdot u_n(x)]^n(1-u_n(x))^{n-1}}{(n-1)!e^{x \cdot u_n(x)}}, \quad x \geq 0,$$

and since $e^{x \cdot u_n(x)} \geq (x \cdot u_n(x))^n/n!$ and since $e^{-u_n(x)} \geq 1 - u_n(x) > 0$, then the above inequality implies that

$$|\epsilon_{n-1,n}(x)| \leq ne^{-(n-1) \cdot u_n(x)}, \quad x \geq 0. \tag{2.14}$$

Consequently, from (2.12) and (2.14),

$$|\epsilon_{n-1,n}(x)| \leq \min \left\{ ne^{-(n-1) \cdot u_n(x)}; e^{-x} + \left(1 + \frac{x}{2n-1}\right)^{-1} \right\}, \quad x \geq 0. \tag{2.15}$$

Now, let $\alpha_n = n^2/(6 \ln n)$, $n > 1$. For all $x \geq \alpha_n$, it is clear that

$$e^{-x} + \left(1 + \frac{x}{2n-1}\right)^{-1} \leq e^{-\alpha_n} + \left(1 + \frac{\alpha_n}{2n-1}\right)^{-1} \leq A \frac{\ln n}{n}, \quad x \geq \alpha_n, \tag{2.16}$$

for some positive constant A independent of n . Next, using (2.13), for $0 \leq x \leq \alpha_n$,

$$u_n(x) \geq \frac{2n}{2(2n-1+x)} \geq \frac{n}{2n-1+\alpha_n} \geq \frac{3 \ln n}{n-1},$$

for all n sufficiently large. Hence,

$$ne^{-(n-1)u_n(x)} \leq ne^{-3 \ln n} = \frac{1}{n^2} \quad \text{for } 0 \leq x \leq \alpha_n. \tag{2.17}$$

Consequently, using (2.15)–(2.17) and the definition of $\eta_{v,n}$ in (2.5), then

$$\eta_{n-1,n} \leq A_2(\ln n)/n$$

for all $n > 1$.

To obtain the first inequality of (2.11), we first write $P_{n-1,n}(x)$ in the form

$$P_{n-1,n}(x) = \frac{n!}{(2n-1)!} x^n \sum_{m=0}^n \frac{(n-1+m)!}{(n-m)! m! x^m}, \quad x \neq 0.$$

Because

$$\frac{(n-1+m)!}{(n-m)!m!} = \frac{n(n^2-1)\cdots[n^2-(m-1)^2]}{m!} \leq \frac{n^{2m-1}}{m!}, \quad 0 \leq m \leq n,$$

the above sum can be bounded above by

$$P_{n-1,n}(x) \leq \frac{n!x^n}{n \cdot (2n-1)!} \sum_{m=0}^n \frac{(n^2/x)^m}{m!} < \frac{n!x^n}{n(2n-1)!} e^{n^2/x}, \quad x > 0.$$

Now, let $x = 2n^2$. Since $e^{1/2} < 2$, this implies that

$$P_{n-1,n}(2n^2) < \frac{n!2^{n+1}n^{2n-1}}{(2n-1)!}. \quad (2.18)$$

To obtain a similar lower bound for $Q_{n-1,n}(x)$, we first write $Q_{n-1,n}(x)$ in the form

$$(-1)^{n-1} Q_{n-1,n}(x) = \frac{(n-1)!x^{n-1}}{(2n-1)!} \sum_{m=0}^{n-1} \frac{(n+m)!(-1)^m}{m!(n-1-m)!x^m}, \quad x \neq 0.$$

For $x = 2n^2$, the above sum is an alternating sum with strictly decreasing terms, so that $(-1)^{n-1} Q_{n-1,n}(2n^2)$ exceeds the sum of the first two terms:

$$(-1)^{n-1} Q_{n-1,n}(2n^2) > \frac{(n-1)!(2n^2)^{n-1} \cdot (n^2+1)}{2n \cdot (2n-1)!}. \quad (2.19)$$

Thus, from (2.18) and (2.19),

$$\frac{(-1)^{n-1} Q_{n-1,n}(2n^2)}{P_{n-1,n}(2n^2)} > \frac{(n^2+1)}{8n^3},$$

so that $(-1)^{n-1} \epsilon_{n-1,n}(2n^2) > (n^2+1)/8n^3 - (-1)^{n-1} e^{-2n^2}$. It is thus clear that

$$(-1)^{n-1} \epsilon_{n-1,n}(2n^2) \geq \frac{A_1}{n},$$

which implies the first inequality of (2.11).

Q.E.D.

3. THE CONVERGENCE OF PADÉ APPROXIMANTS TO e^{-x} ON $[0, +\infty)$

Based on the results of the previous section, we now establish the convergence of particular Padé approximants to e^{-x} on the infinite segment $[0, +\infty)$. Actually, we are interested in two kinds of convergence, namely, the *uniform* convergence and, more particularly, the *geometric* convergence

Thus, $|\epsilon_{n-1,n}(x)|$ can be bounded above by

$$|\epsilon_{n-1,n}(x)| \leq \frac{x^{2n}(u_n(x))^n (1 - u_n(x))^{n-1}}{(2n - 1)! e^{x \cdot u_n(x)} P_{n-1,n}(x)}, \quad x \geq 0.$$

Next, it follows from (2.3) that $P_{n-1,n}(x) \geq ((n - 1)! x^n / (2n - 1)!)$ for all $x \geq 0$, so that

$$|\epsilon_{n-1,n}(x)| \leq \frac{[x \cdot u_n(x)]^n (1 - u_n(x))^{n-1}}{(n - 1)! e^{x \cdot u_n(x)}}, \quad x \geq 0,$$

and since $e^{x \cdot u_n(x)} \geq (x \cdot u_n(x))^n / n!$ and since $e^{-u_n(x)} \geq 1 - u_n(x) > 0$, then the above inequality implies that

$$|\epsilon_{n-1,n}(x)| \leq n e^{-(n-1) \cdot u_n(x)}, \quad x \geq 0. \tag{2.14}$$

Consequently, from (2.12) and (2.14),

$$|\epsilon_{n-1,n}(x)| \leq \min \left\{ n e^{-(n-1) \cdot u_n(x)}; e^{-x} + \left(1 + \frac{x}{2n - 1} \right)^{-1} \right\}, \quad x \geq 0. \tag{2.15}$$

Now, let $\alpha_n = n^2 / (6 \ln n)$, $n > 1$. For all $x \geq \alpha_n$, it is clear that

$$e^{-x} + \left(1 + \frac{x}{2n - 1} \right)^{-1} \leq e^{-\alpha_n} + \left(1 + \frac{\alpha_n}{2n - 1} \right)^{-1} \leq A \frac{\ln n}{n}, \quad x \geq \alpha_n, \tag{2.16}$$

for some positive constant A independent of n . Next, using (2.13), for $0 \leq x \leq \alpha_n$,

$$u_n(x) \geq \frac{2n}{2(2n - 1 + x)} \geq \frac{n}{2n - 1 + \alpha_n} \geq \frac{3 \ln n}{n - 1},$$

for all n sufficiently large. Hence,

$$n e^{-(n-1) u_n(x)} \leq n e^{-3 \ln n} = \frac{1}{n^2} \quad \text{for } 0 \leq x \leq \alpha_n. \tag{2.17}$$

Consequently, using (2.15)–(2.17) and the definition of $\eta_{\nu,n}$ in (2.5), then

$$\eta_{n-1,n} \leq A_2 (\ln n) / n$$

for all $n > 1$.

To obtain the first inequality of (2.11), we first write $P_{n-1,n}(x)$ in the form

$$P_{n-1,n}(x) = \frac{n!}{(2n - 1)!} x^n \sum_{m=0}^n \frac{(n - 1 + m)!}{(n - m)! m! x^m}, \quad x \neq 0.$$

Because

$$\frac{(n-1+m)!}{(n-m)!m!} = \frac{n(n^2-1)\cdots[n^2-(m-1)^2]}{m!} \leq \frac{n^{2m-1}}{m!}, \quad 0 \leq m \leq n,$$

the above sum can be bounded above by

$$P_{n-1,n}(x) \leq \frac{n!x^n}{n \cdot (2n-1)!} \sum_{m=0}^n \frac{(n^2/x)^m}{m!} < \frac{n!x^n}{n(2n-1)!} e^{n^2/x}, \quad x > 0.$$

Now, let $x = 2n^2$. Since $e^{1/2} < 2$, this implies that

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To obtain a similar lower bound for $Q_{n-1,n}(x)$, we first write $Q_{n-1,n}(x)$ in the form

$$(-1)^{n-1} Q_{n-1,n}(x) = \frac{(n-1)!x^{n-1}}{(2n-1)!} \sum_{m=0}^{n-1} \frac{(n+m)!(-1)^m}{m!(n-1-m)!x^m}, \quad x \neq 0.$$

For $x = 2n^2$, the above sum is an alternating sum with strictly decreasing terms, so that $(-1)^{n-1} Q_{n-1,n}(2n^2)$ exceeds the sum of the first two terms:

$$(-1)^{n-1} Q_{n-1,n}(2n^2) > \frac{(n-1)!(2n^2)^{n-1} \cdot (n^2+1)}{2n \cdot (2n-1)!}. \quad (2.19)$$

Thus, from (2.18) and (2.19),

$$\frac{(-1)^{n-1} Q_{n-1,n}(2n^2)}{P_{n-1,n}(2n^2)} > \frac{(n^2+1)}{8n^3},$$

so that $(-1)^{n-1} \epsilon_{n-1,n}(2n^2) > (n^2+1)/8n^3 - (-1)^{n-1} e^{-2n^2}$. It is thus clear that

$$(-1)^{n-1} \epsilon_{n-1,n}(2n^2) \geq \frac{A_1}{n},$$

which implies the first inequality of (2.11). Q.E.D.

3. THE CONVERGENCE OF PADÉ APPROXIMANTS TO e^{-x} ON $[0, +\infty)$

Based on the results of the previous section, we now establish the convergence of particular Padé approximants to e^{-x} on the infinite segment $[0, +\infty)$. Actually, we are interested in two kinds of convergence, namely, the *uniform* convergence and, more particularly, the *geometric* convergence

of sequences of Padé approximants to e^{-x} on $[0, +\infty)$. We first treat uniform convergence in

THEOREM 3.1. *The sequence $\{R_{\nu(n),n}\}_{n=1}^{\infty}$ of Padé approximants converges uniformly to e^{-x} on $[0, \infty)$ if and only if $\nu(n) < n$ for all n sufficiently large.*

Proof. Assume first that $\nu(n) < n$ for all $n \geq n_0$. From (2.7) and (2.8), we have that

$$\eta_{\nu(n),n} \leq \eta_{\nu(n),\nu(n)+1}, \quad n \geq n_0, \tag{3.1}$$

and that

$$\eta_{\nu(n),n} \leq \frac{1}{2^{n-\nu(n)}}, \quad n \geq n_0. \tag{3.2}$$

How, given any $\epsilon > 0$, there is, from Proposition 2.4, an $n_1(\epsilon)$ such that $\eta_{n,n+1} < \epsilon$ for all $n > n_1(\epsilon)$. We may assume that $n_1(\epsilon) \geq n_0$. Next, choose $n_2(\epsilon) > n_1(\epsilon)$ such that $2^{-n_2(\epsilon)+n_1(\epsilon)} < \epsilon$. Consider then any $n \geq n_2(\epsilon)$. If $0 \leq \nu(n) \leq n_1(\epsilon)$, then using (3.2),

$$\eta_{\nu(n),n} \leq \frac{1}{2^{n-\nu(n)}} \leq \frac{1}{2^{n_2(\epsilon)-n_1(\epsilon)}} < \epsilon.$$

On the other hand, suppose that $n_1(\epsilon) < \nu(n) \leq n - 1$. With the inequality of (3.1) and the fact that $\eta_{n,n+1} < \epsilon$ for all $n > n_1(\epsilon)$, then $\eta_{\nu(n),n} < \epsilon$. Thus, for any $n \geq n_2(\epsilon)$ and for any $\nu(n)$ with $\nu(n) < n$, we have that $\eta_{\nu(n),n} < \epsilon$.

Conversely, assume that $\{\eta_{\nu(n),n}\}_{n=1}^{\infty}$ converges to zero as $n \rightarrow \infty$. Since $\eta_{\nu,n}$ is finite only if $\nu \leq n$, and since $\eta_{\nu,\nu} = 1$ for all $\nu \geq 0$ from Proposition 2.1, then evidently $\nu(n) < n$ for all n sufficiently large. Q.E.D.

To establish a sufficient condition for the geometric convergence of certain Padé approximants to e^{-x} on $[0, +\infty)$, we need only use (2.8) of Theorem 2.3 to prove

THEOREM 3.2. *If $\limsup_{n \rightarrow \infty} \{\prod_{j=1}^{n-\nu(n)} (\nu(n) + j)/(3\nu(n) + 2j)\}^{1/n} = \alpha < 1$, then the sequence of Padé approximants $\{R_{\nu(n),n}(x)\}_{n=1}^{\infty}$ converges geometrically in the uniform norm to e^{-x} on $[0, +\infty)$, i.e.,*

$$\limsup_{n \rightarrow \infty} (\eta_{\nu(n),n})^{1/n} \leq \alpha < 1. \tag{3.3}$$

As a special case, if $\limsup_{n \rightarrow \infty} (\nu(n)/n) = \beta < 1$, then

$$\limsup_{n \rightarrow \infty} (\eta_{\nu(n),n})^{1/n} \leq \frac{1}{2^{1-\beta}} < 1. \tag{3.4}$$

While the result of Theorem 3.2 establishes a sufficient condition for the geometric convergence of the Padé approximants $\{R_{\nu(n),n}(x)\}_{n=1}^{\infty}$ to e^{-x} on $[0, +\infty)$, it is not known whether this condition is also necessary. On the other hand, from the lower bound in Proposition 2.4, i.e.,

$$\frac{A_1}{n} \leq \eta_{n-1,n}, \quad \text{for all } n \geq 1,$$

it is clear that the particular Padé approximants $\{R_{\nu(n),n}(x)\}_{n=1}^{\infty}$ with $\nu(n) = n - 1$, for which $\limsup_{n \rightarrow \infty} (\nu(n)/n) = 1$, cannot possess geometric convergence to e^{-x} on $[0, +\infty)$. More generally, it can be shown that no sequence of the form $\{R_{n-\mu,n}\}_{n=\mu}^{\infty}$, $\mu \geq 1$ fixed, converges geometrically on this ray.

It is interesting to note that the result of Theorem 3.2 has applications to the problem of constrained Chebyshev rational approximations to e^{-x} on $[0, +\infty)$. We use the following notation. Let $\hat{\pi}_m$ be the set of all real polynomials of degree at most m , let $\hat{\pi}_{\nu,n}$ be the set of all real rational functions $r_{\nu,n}(x)$ of the form $r_{\nu,n}(x) = q_{\nu,n}(x)/p_{\nu,n}(x)$, where $q_{\nu,n} \in \hat{\pi}_{\nu}$, and $p_{\nu,n} \in \hat{\pi}_n$, and $p_{\nu,n}(0) = 1$, and, for any nonnegative integer k with $0 \leq k \leq n + \nu + 1$, let $\hat{\pi}_{\nu,n}^{(k)}$ be the subset of those $r_{\nu,n}$ in $\hat{\pi}_{\nu,n}$ for which

$$e^{-x} - r_{\nu,n}(x) = \mathcal{O}(|x|^k), \quad x \text{ real, } |x| \rightarrow 0.$$

Then, for any nonnegative integers n , ν , and k with $0 \leq \nu \leq n$ and with $0 \leq k \leq n + \nu + 1$, the constrained Chebyshev constants $\lambda_{\nu,n}^{(k)}$ for e^{-x} on $[0, +\infty)$ are defined as

$$\lambda_{\nu,n}^{(k)} = \inf \left\{ \sup_{0 \leq x < \infty} |e^{-x} - r_{\nu,n}(x)| : r_{\nu,n} \in \hat{\pi}_{\nu,n}^{(k)} \right\}. \quad (3.5)$$

For the special case $k = 0$, these (unconstrained) Chebyshev constants for e^{-x} have been studied in [2], Newman [8], and Schönhage [13]. Note that because the (ν, n) -the Padé approximant $R_{\nu,n}$ is real, i.e., $R_{\nu,n} \in \hat{\pi}_{\nu,n}$, the special case $k = n + \nu + 1$ is, from (2.5) such that $\lambda_{\nu,n}^{(n+\nu+1)} = \eta_{\nu,n}$.

Recently, J. D. Lawson [5] has considered the particular constrained Chebyshev constants $\lambda_{n,n}^{(n+1)}$ for e^{-x} on $[0, +\infty)$, and, from his computed values of $\lambda_{n,n}^{(n+1)}$ for $2 \leq n \leq 5$, one would naturally suspect the geometric convergence of these constants to zero. That this is theoretically so can be seen to be a special case of

THEOREM 3.3. Assume that the sequence of nonnegative integers

$$\{k(n)\}_{n=0}^{\infty},$$

satisfying $0 \leq k(n) \leq 2n + 1$ for every $n \geq 0$, has the property that

$$\limsup_{n \rightarrow \infty} \left(\frac{k(n) - (n + 1)}{n} \right) = \alpha < 1, \tag{3.6}$$

and define $\delta = \max(0, \alpha)$. Then, for any sequence of nonnegative integers $\{m(n)\}_{n=0}^{\infty}$ satisfying $\max\{0; k(n) - (n + 1)\} \leq m(n) \leq n$,

$$\frac{1}{1280} \leq \liminf_{n \rightarrow \infty} \{\lambda_{m(n),n}^{(k(n))}\}^{1/n} \leq \limsup_{n \rightarrow \infty} \{\lambda_{m(n),n}^{(k(n))}\}^{1/n} \leq \frac{1}{2^{1-\delta}}. \tag{3.7}$$

Proof. First, set $\nu(n) = \max\{0; k(n) - (n + 1)\}$, so that $0 \leq \nu(n) \leq n$. From the integral representation in (2.4), the Padé approximant $R_{\nu(n),n}(x)$ to e^{-x} evidently satisfies

$$|R_{\nu(n),n}(x) - e^{-x}| = \mathcal{O}(|x|^{n+\nu(n)+1}), \quad |x| \rightarrow 0,$$

for each $n \geq 0$, and hence, from the definition of $\nu(n)$,

$$|R_{\nu(n),n}(x) - e^{-x}| = \mathcal{O}(|x|^{k(n)}), \quad |x| \rightarrow 0,$$

for each $n \geq 0$. Thus, $R_{\nu(n),n} \in \hat{\pi}_{\nu(n),n}^{(k(n))} \subset \hat{\pi}_{m(n),n}^{(k(n))}$ for any integer $m(n)$ with $\nu(n) \leq m(n) \leq n$. Hence, by definition,

$$\lambda_{m(n),n}^{(k(n))} \leq \eta_{\nu(n),n}, \quad \text{for each } n \geq 0.$$

But using (3.4), we have that

$$\{\lambda_{m(n),n}^{(k(n))}\}^{1/n} \leq \frac{1}{2^{1-(\nu(n)/n)}},$$

so that applying the hypothesis of (3.6) establishes the last inequality of (3.7). On the other hand, Newman [8] has shown that for any polynomials $p, q \in \hat{\pi}_n$,

$$\sup_{0 \leq x < \infty} \left| e^{-x} - \frac{p(x)}{q(x)} \right| > \frac{1}{(1280)^{n+1}},$$

which establishes the first inequality of (3.7).

Q.E.D.

We remark that a stronger result, analogous to (3.7), can be similarly established from the inequality (2.10).

4. THE CONVERGENCE OF PARTICULAR PADÉ APPROXIMANTS TO e^{-z} ON UNBOUNDED REGIONS

In this section, we shall be concerned with the convergence, in the uniform norm, of particular Padé approximants to e^{-z} on unbounded sets in the complex plane which are symmetric with respect to the positive ray $0 \leq x < \infty$. To begin, let $s_n(z) \equiv \sum_{k=0}^n z^k/k!$ denote the familiar n -th partial sum of e^z . Then, it is clear from (2.2)–(2.3) that the $(0, n)$ -th Padé approximant $R_{0,n}(z)$ of e^{-z} is given by

$$R_{0,n}(z) = \frac{1}{s_n(z)}. \quad (4.1)$$

Thus, the poles of the Padé approximant $R_{0,n}$ are the zeros of s_n . It is further known that the parabolic region T in the complex plane, defined by

$$T \equiv \{z = x + iy : x \geq 0 \quad \text{and} \quad |y| \leq dx^{1/2}\}, \quad (4.2)$$

where

$$d < 0.863\,369\,712, \quad (4.3)$$

contains no zeros of any s_n , i.e., $1/s_n$ is analytic in T for all n sufficiently large. That such a parabolic region with this property could exist was first indicated by the numerical results of Iverson [4], and the existence of this region was later established¹ by Newman and Rivlin [9].

The special case $\nu = 0$ of (2.8) of Theorem 2.3, coupled with (4.1), implies that

$$\left\| e^{-x} - \frac{1}{s_n(x)} \right\|_{L_\infty[0, \infty]} \leq \frac{1}{2^n} \quad (4.4)$$

for all $n \geq 0$, and moreover, from Theorem 1 of Meinardus and Varga [6], we have that

$$\lim_{n \rightarrow \infty} \left\{ \left\| e^{-x} - \frac{1}{s_n(x)} \right\|_{L_\infty[0, \infty]} \right\}^{1/n} = \frac{1}{2}. \quad (4.5)$$

It is natural to ask if the sequence $\{1/s_n\}_{n=1}^\infty$ converges geometrically to e^{-x} on some larger set in the complex plane, especially when we know that $1/s_n$ is analytic in the parabolic region T of (4.2), for all n . That this is so is

¹ Strictly speaking, the above-mentioned property of T , as stated in [9], does not follow completely from results of [9], but depends additionally on a subsequent note by Newman and Rivlin [10].

established in the following result. For added notation, if S is any set in the complex plane and f is defined on S , we write

$$\|f\|_{L_\infty(n)} \equiv \sup\{|f(z)| : z \in S\}.$$

THEOREM 4.1. *Let g be a positive continuous function on $[0, +\infty)$ which satisfies*

$$\lim_{x \rightarrow +\infty} \frac{g(x)}{(x)^{1/2}} = d^*, \quad d^* \geq 0, \tag{4.6}$$

and let $G \equiv \{z = x + iy : x \geq 0 \text{ and } |y| \leq g(x)\}$. If (cf. (4.2))

$$d^* < d \left(\frac{(2)^{1/2} - 1}{(2)^{1/2} + 1} \right), \quad \text{e.g.,} \quad d^* < 0.184 \ 130 \ 824, \tag{4.7}$$

then the sequence $\{1/s_n\}_{n=1}^\infty$ converges geometrically to e^{-z} on G . In particular, if d^* of (4.6) is positive, then

$$\limsup_{n \rightarrow \infty} \left\| \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(G)} \right\|^{1/n} \leq \frac{1}{2} \left(\frac{d + d^*}{d - d^*} \right)^2 < 1, \tag{4.8}$$

while if $d^* = 0$, then

$$\lim_{n \rightarrow \infty} \left\| \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(G)} \right\|^{1/n} = \frac{1}{2}. \tag{4.9}$$

Proof. By way of construction, it is possible from (4.7) to choose positive numbers d_0 and d_1 such that $d^* < d_0 < d_1 < d$ and such that $\frac{1}{2}\{(d_1 + d_0)/(d_1 - d_0)\}^2 < 1$. With these positive numbers, the sets T_i are defined:

$$T_i \equiv \{z = x + iy : x \geq 0 \quad \text{and} \quad |y| \leq d_i x^{1/2}\}, \quad i = 0, 1,$$

and hence $T_0 \subset T_1$. Next, it is clear from (4.6) that there is a finite $\sigma \geq 0$ such that the subset $G_\sigma \equiv \{z = x + iy : z \in G \text{ and } x \geq \sigma\}$ of G satisfies

$$G_\sigma \subset T_0.$$

Next, since the zeros of the s_n 's have no finite limit point, i.e., if $\{z_j^{(n)}\}_{j=1}^n$ denote the zeros of s_n , then $\lim_{n \rightarrow \infty} \{\min_{1 \leq j \leq n} |z_j^{(n)}|\} = +\infty$, then for all n sufficiently large, say $n \geq n_0$, each s_n is free of zeros in the sets T_0 , T_1 , and G .

Continuing our construction, for each $t \geq 0$ and each $\beta > 0$, let $m(t, \beta)$ be the interval $[t - \beta t^{1/2}, t + \beta t^{1/2}]$ of the real axis. For $t \geq \beta^2$, $m(t, \beta)$ lies entirely on the nonnegative axis. Next, for each $\mu > 1$, let $m_\mu(t, \beta)$ denote the level curve of $m(t, \beta)$ in the complex plane, i.e., $m_\mu(t, \beta)$ is an ellipse given by

$$m_\mu(t, \beta) \equiv \left\{ z = x + iy : \frac{(x - t)^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}, \tag{4.10}$$

where

$$a = a(t, \beta, \mu) = \frac{\beta t^{1/2}}{2} (\mu + \mu^{-1}), \quad \text{and} \quad b = b(t, \beta, \mu) = \frac{\beta t^{1/2}}{2} (\mu - \mu^{-1}). \quad (4.11)$$

For each $t \geq \beta^2$, we seek the largest value of $\mu \geq 1$ such that $m_\mu(t, \beta) \subseteq T_1$. This value of μ , which we call $A_1 = A_1(t, \beta, d_1)$ is obtained when $m_\mu(t, \beta)$ is tangent to the parabola $y^2 = d_1^2 x$ which defines T_1 . In particular, as is readily shown, for $\beta^2 \leq t \leq \beta^2 M$ where $M \equiv 1 + d_1^2/2\beta^2$, A_1 is obtained by making $m_\mu(t, \beta)$ tangent to T_1 at the origin, and A_1 is given in this case by

$$A_1 = \frac{t^{1/2}}{\beta} + \left(\frac{t}{\beta^2} - 1\right)^{1/2}, \quad \beta^2 \leq t \leq \beta^2 M. \quad (4.12)$$

For $t > \beta^2 M$, $m_\mu(t, \beta)$ will have exactly two points of intersection with T_1 , i.e.,

$$(x - t)^2/a^2 + d_1^2 x/b^2 = 1$$

will have exactly one (nonnegative) root for x , precisely when the discriminant of the above quadratic in x , equals zero:

$$\{2t - a^2 d_1^2/b^2\}^2 - 4(t^2 - a^2) = 0,$$

or equivalently, solving for b^2 ,

$$b^2 = \frac{d_1^2}{2} \left\{ t + (t^2 - a^2)^{1/2} \right\}.$$

Thus, with (4.11), the largest value of $\mu \geq 1$, i.e., A_1 , for which $m_\mu(t, \beta) \subseteq T_1$ satisfies

$$\left(A_1 - \frac{1}{A_1}\right)^2 = 2 \left(\frac{d_1}{\beta}\right)^2 \left\{ 1 + \left(1 - \frac{1}{4u^2} \left(A_1 + \frac{1}{A_1}\right)^2\right)^{1/2} \right\}, \quad u^2 \equiv \frac{t}{\beta^2} > M, \quad (4.12')$$

which gives rise to a polynomial equation in A_1 of degree 6. It is apparent from (4.12) and (4.12') that $A_1 = A_1(t, \beta, d_1)$ is in reality a function of $u \equiv t^{1/2}/\beta$ and d_1/β , and we also write $A_1 = A_1(u, d_1/\beta)$. It is also clear from (4.12') that A_1 is a continuous strictly increasing function of u . Next, to obtain an upper bound for A_1 , one sees geometrically that forcing the ellipse $m_\mu(t, \beta)$ to intersect the curve $y = d_1 t^{1/2}$ in the particular point (t, b) must give an upper bound for A_1 . Thus, $b = d_1 t^{1/2} = \beta t^{1/2}(\hat{u} - 1/\hat{u})/2$ implies $A_1 < \hat{u}$, or equivalently

$$A_1 \left(u, \frac{d_1}{\beta}\right) < \left(\frac{d_1}{\beta}\right) + \left(1 + \left(\frac{d_1}{\beta}\right)^2\right)^{1/2} \text{ for all } u \geq 1.$$

Hence, $A_1(u, d_1/\beta)$ is bounded, for fixed d_1/β , as $u \rightarrow +\infty$. Using this fact, it follows from (4.12') that the above upper bound is asymptotically sharp:

$$\lim_{u \rightarrow +\infty} A_1\left(u, \frac{d_1}{\beta}\right) = \alpha_1\left(\frac{d_1}{\beta}\right) \equiv \left(\frac{d_1}{\beta}\right) + \left(1 + \left(\frac{d_1}{\beta}\right)^2\right)^{1/2}. \quad (4.13)$$

Similarly, if $A_0(t, \beta, d_0) = A_0(u, d_0/\beta)$ denotes the largest value of $\mu \geq 1$ such that $m_\mu(t, \beta)$ is contained in T_0 for all $t \geq \beta^2$, the argument above directly gives

$$\lim_{u \rightarrow +\infty} A_0\left(u, \frac{d_0}{\beta}\right) = \alpha_0\left(\frac{d_0}{\beta}\right) \equiv \left(\frac{d_0}{\beta}\right) + \left(1 + \left(\frac{d_0}{\beta}\right)^2\right)^{1/2}. \quad (4.13')$$

Next, it is straightforward to deduce from (4.13) and (4.13') that

$$\lim_{\beta \rightarrow +\infty} \left\{ \frac{\alpha_1(d_1/\beta) \alpha_0(d_0/\beta) - 1}{\alpha_1(d_1/\beta) - \alpha_0(d_0/\beta)} \right\} = \frac{d_1 + d_0}{d_1 - d_0} < (2)^{1/2}, \quad (4.14)$$

the last inequality following from our choice of d_0 and d_1 .

Now, with the inequality of (4.4), we have

$$\begin{aligned} \left| \frac{1}{s_{n+1}(x)} - \frac{1}{s_n(x)} \right| &\leq \left| \frac{1}{s_{n+1}(x)} - e^{-x} \right| + \left| e^{-x} - \frac{1}{s_n(x)} \right| \\ &\leq \frac{1}{2^{n+1}} + \frac{1}{2^n} = \frac{3}{2^{n+1}}, \end{aligned}$$

for any $x \geq 0$ and any $n \geq 0$. In particular, for any $t \geq \beta^2$ (so that $m(t, \beta)$ lies entirely on the nonnegative axis),

$$\left| \frac{1}{s_{n+1}(x)} - \frac{1}{s_n(x)} \right| \leq \frac{3}{2^{n+1}}, \quad x \in m(t, \beta), t \geq \beta^2, \quad n \geq 0.$$

In addition, we know that the rational function $(1/s_{n+1} - 1/s_n) \in \pi_{n+1, 2n+1}$ has, for any $n \geq n_0$ all its poles outside of T_1 . Then, applying *Walsh's Lemma* (cf. [16; Eq. (41), p. 250]) to this rational function on the set $m(t, \beta)$ yields

$$\left| \frac{1}{s_{n+1}(z)} - \frac{1}{s_n(z)} \right| \leq \frac{3}{2^{n+1}} \left\{ \frac{A_1(t, \beta, d_1) A_0(t, \beta, d_0) - 1}{A_1(t, \beta, d_1) - A_0(t, \beta, d_0)} \right\}^{2n+1},$$

for all $z \in \bar{m}_{\mu_0}(t, \beta)$, $t \geq \beta^2$, $n \geq n_0$, where $\bar{m}_\mu(t, \beta)$ denotes all points z on or inside $m_\mu(t, \beta)$ i.e.,

$$\bar{m}_\mu(t, \beta) \equiv \left\{ z = x + iy : \frac{(x-t)^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}.$$

Hence, given any $\epsilon > 0$ sufficiently small, so that

$$((d_1 + d_0)/(d_1 - d_0) + \epsilon) < (2)^{1/2},$$

it follows from (4.13)–(4.14) that there is a $\tilde{\beta}$ and a \tilde{u} sufficiently large so that

$$\left| \frac{1}{s_{n+1}(z)} - \frac{1}{s_n(z)} \right| \leq \frac{3}{2^{n+1}} \left\{ \frac{d_1 + d_0}{d_1 - d_0} + \epsilon \right\}^{2^{n+1}}, \quad (4.15)$$

for all $n \geq n_0 + 1$, for all $z \in \bar{m}_{A_0}(t, \tilde{\beta})$, and for all $t \geq \tilde{\beta}^2 \tilde{u}^2$. Thus, since

$$\left| \frac{1}{s_{n+r}(z)} - \frac{1}{s_n(z)} \right| \leq \sum_{j=0}^{r-1} \left| \frac{1}{s_{n+j+1}(z)} - \frac{1}{s_{n+j}(z)} \right|$$

for any $r \geq 1$, then applying the inequality of (4.15) in the above sum and summing the resultant geometric series gives

$$\left| \frac{1}{s_{n+r}(z)} - \frac{1}{s_n(z)} \right| \leq \frac{3\gamma^{2^{n+1}}}{2^{n+1}} \left\{ \frac{2}{2 - \gamma^2} \right\}, \quad \gamma \equiv \left[\frac{d_1 + d_0}{d_1 - d_0} + \epsilon \right].$$

Consequently, letting $r \rightarrow \infty$,

$$\left| e^{-z} - \frac{1}{s_n(z)} \right| \leq \frac{3\gamma^{2^{n+1}}}{2^{n+1}} \left\{ \frac{2}{2 - \gamma^2} \right\}, \quad z \in \bar{m}_{A_0}(t, \beta), \quad t \geq \beta^2 u^2, \quad n \geq n_0. \quad (4.16)$$

Now, by construction, the closed ellipses $\bar{m}_{A_0}(t, \beta)$ trace out the set T_0 , i.e., for every $\beta > 0$,

$$\bigcup_{t \geq \beta^2} \{\bar{m}_{A_0}(t, \beta)\} = T_0.$$

Hence, the set $\bigcup_{t \geq \beta^2 u^2} \{\bar{m}_{A_0}(t, \beta)\}$ can be expressed as $T_0 - C$, where $C = C(\epsilon)$ is some compact set in the complex plane. Thus, (4.16) can be equivalently expressed as

$$\left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(T_0 - C)} \leq \frac{3\gamma^{2^{n+1}}}{2^{n+1}} \left\{ \frac{2}{2 - \gamma^2} \right\}, \quad n \geq n_0.$$

Recalling that the set G of Theorem 4.1 is a subset of $T_0 - C$ with the exception of some compact set C' , this implies that

$$\limsup_{n \rightarrow \infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(G - C')} \right\}^{1/n} \leq \frac{1}{2} \left[\frac{d_1 + d_0}{d_1 - d_0} + \epsilon \right]^2. \quad (4.17)$$

On the other hand, for any compact set C ,

$$\lim_{n \rightarrow \infty} \left\{ \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_\infty(C)} \right\}^{1/n} = 0. \quad (4.18)$$

To see this, define $0 < \delta \equiv \inf\{|e^{+z}| : z \in C\}$, and $\rho \equiv \sup\{|z| : z \in C\}$. Because of the uniform convergence of s_n to e^z on C , then $\delta/2 \leq |s_n(z)|$ for all $z \in C$, all $n \geq n_1$. Thus, for $n \geq \max\{\rho - 2, n_1\}$,

$$\begin{aligned} \left| e^{-z} - \frac{1}{s_n(z)} \right| &= \frac{|s_n(z) - e^z|}{|e^z \cdot s_n(z)|} \leq \frac{2}{\delta^2} |s_n(z) - e^z| = \frac{2}{\delta^2} \left| \sum_{k=n+1}^{\infty} z^k/k! \right| \\ &\leq \frac{2}{\delta^2} \sum_{k=n+1}^{\infty} \rho^k/k! \leq \frac{2(n+2)\rho^{n+1}}{\delta^2(n+1)!(n+2-\rho)}, \end{aligned}$$

for all $z \in C$. Thus, using Stirling's formula, (4.18) follows. Hence, combining (4.17) and (4.18), we deduce that

$$\limsup_{n \rightarrow \infty} \left\| \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_{\infty}(G)} \right\|^{1/n} \leq \frac{1}{2} \left[\frac{d_1 + d_0}{d_1 - d_0} + \epsilon \right]^2. \tag{4.19}$$

Thus, letting both $\epsilon \rightarrow 0$ in (4.19) yields

$$\limsup_{n \rightarrow \infty} \left\| \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_{\infty}(G)} \right\|^{1/n} \leq \frac{1}{2} \left(\frac{d_1 + d_0}{d_1 - d_0} \right)^2.$$

Finally letting $d_1 \rightarrow d$ and $d_0 \rightarrow d^*$ in the above expression then establishes

$$\limsup_{n \rightarrow \infty} \left\| \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_{\infty}(G)} \right\|^{1/n} \leq \frac{1}{2} \left(\frac{d + d^*}{d - d^*} \right)^2 < 1,$$

the desired result of (4.8). Of course, if $d^* = 0$, then

$$\limsup_{n \rightarrow \infty} \left\| \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_{\infty}(G)} \right\|^{1/n} \leq \frac{1}{2}.$$

But as $[0, +\infty)$ is a subset of G , it follows from (4.5) and the above inequality, that

$$\frac{1}{2} \leq \limsup_{n \rightarrow \infty} \left\| \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_{\infty}(G)} \right\|^{1/n} \leq \limsup_{n \rightarrow \infty} \left\| \left\| e^{-z} - \frac{1}{s_n} \right\|_{L_{\infty}(G)} \right\|^{1/n} \leq \frac{1}{2},$$

whence $\liminf_{n \rightarrow \infty} \left\| \left\| e^{-z} - 1/s_n \right\|_{L_{\infty}(G)} \right\|^{1/n} = \frac{1}{2}$, the desired result of (4.9).
 Q.E.D.

As a special case of Theorem 4.1, we have

COROLLARY 4.2. *For any semi-infinite strip*

$$I_{\tau} \equiv \{z = x + iy : x \geq 0, |y| \leq \tau\},$$

where $0 \leq \tau < \infty$,

$$\lim_{n \rightarrow \infty} \left\{ \left\| e^{-z} - \frac{1}{S_n} \right\|_{L_\infty(I_\tau)} \right\}^{1/n} = \frac{1}{2}.$$

It is again natural to ask if the geometric convergence of (4.8)–(4.9) of Theorem 4.1 holds for similar unbounded domains in the complex plane, for other Padé approximations of e^{-z} . Such a result, which would extend Theorem 3.2 to larger sets in the complex plane, of course depends on a precise knowledge of the location of the poles of other Padé approximations of e^{-z} , which seems not to be known in the general case. On the other hand, the uniform convergence of Padé approximants to e^{-z} on $[0, +\infty)$ of Theorem 3.1 can be similarly extended to larger sets in the complex plane for particular Padé approximations, as we now show.

THEOREM 4.3. *Given any δ with $0 < \delta \leq \pi/2$, the sequences*

$$\{R_{n-1,n}(z)\}_{n=1}^\infty \quad \text{and} \quad \{R_{n-2,n}(z)\}_{n=2}^\infty$$

converge uniformly to e^{-z} on the sector $S_\delta \equiv \{z = re^{i\theta} : |\theta| \leq \pi/2 - \delta\}$.

Proof. It was originally shown by Birkhoff and Varga [1] that all the Padé approximants $R_{n,n}(z)$ of e^{-z} are analytic in the right-half plane $\operatorname{Re} z \geq 0$, and are bounded in modulus there by unity. More recently, Ehle [3] has extended both of these results to $\{R_{n-1,n}(z)\}_{n=1}^\infty$ and $\{R_{n-2,n}(z)\}_{n=2}^\infty$. Dealing for definiteness with $\{R_{n-1,n}(z)\}_{n=1}^\infty$, we thus have that each

$$f_n(z) = e^{-z} - R_{n-1,n}(z)$$

is analytic in the open first quadrant $S \equiv \{z = x + iy : x > 0 \text{ and } y > 0\}$, and that $\sup\{|f_n(z)| : z \in S\} \leq 2$, for all $n \geq 1$. Since the boundary of S consists of the rays $\gamma_1 \equiv \{z = x + iy : x \geq 0, y = 0\}$ and

$$\gamma_2 \equiv \{z = x + iy : x = 0, y \geq 0\},$$

the harmonic measure $w(z)$ of γ_1 with respect to S , defined as a function which is harmonic and bounded in S and for which $w(z) = 1$ for all $z \in \operatorname{int} \gamma_1$ and $w(z) = 0$ for all $z \in \operatorname{int} \gamma_2$, is obviously given by

$$w(z) = 1 - \frac{2}{\pi} \arg z. \quad (4.20)$$

Then, by the *Nevanlinna Two-Constants Theorem* (cf. [7, p. 41]), if

$$M_i = \sup\{|f_n(z)| : z \in \operatorname{int} \gamma_i\}, \quad i = 1, 2,$$

then

$$|f_n(z)| \leq M_1^{w(z)} \cdot M_2^{1-w(z)}, \quad \text{for all } z \in S. \quad (4.21)$$

Strictly speaking the Two-Constants Theorem is stated for *bounded* domains. Therefore, the validity of (4.21) follows by considering an appropriate conformal mapping of S .

Now since $M_1 = \eta_{n-1,n}$ (cf. (2.5)), and $M_2 \leq 2$, it follows from (4.20) and (4.21) that

$$|f_n(z)| \leq \eta_{n-1,n}^{1-(2/\pi) \arg z} \cdot 2^{(2/\pi) \arg z}, \quad \text{for all } z \in S.$$

Thus, as $\arg(z) < \pi/2$ in S ,

$$|f_n(z)| \leq 2\eta_{n-1,n}^{1-(2/\pi) \arg(z)}, \quad \text{all } z \in S.$$

Now, from Proposition 2.4, there exists an $n_0 > 0$ such that $\eta_{n-1,n} < 1$ for all $n \geq n_0$. Thus, restricting z to be in the sector $S_\delta^+ \equiv \{z = re^{i\theta} : 0 \leq \theta \leq \pi/2 - \delta\}$ where $0 < \delta \leq \pi/2$, then

$$|f_n(z)| \leq 2 \cdot \eta_{n-1,n}^{(2/\pi)\delta}, \quad \text{all } z \in S_\delta^+,$$

and, as the same result evidently holds for the reflected sector $S_\delta^- = \{z = re^{i\theta} : -(\pi/2 - \delta) \leq \theta \leq 0\}$, we have

$$\|e^{-z} - R_{n-1,n}\|_{L_\infty(S_\delta)} \leq 2\eta_{n-1,n}^{(2/\pi)\delta}.$$

Thus, since $\eta_{n-1,n} \rightarrow 0$ as $n \rightarrow \infty$ from Proposition 2.4, then $\{R_{n-1,n}(z)\}_{n=1}^\infty$ converges uniformly to e^{-z} on S_δ , the same conclusion being true also for $\{R_{n-2,n}(z)\}_{n=2}^\infty$. Q.E.D.

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