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On Approximation by Polynomials Increasing to the Right of the Interval*

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We obtain sufficient conditions on a real valued function f, continuous on $[0, +\infty)$, to insure that, for some nonnegative integer n, there is a nonnegative number $r(n)$ so that for any $r \geqslant r(n)$, the polynomial of best approximation to f on $[0, r]$ from π_n is increasing and nonnegative on $[r, +\infty)$. Here, π_n denotes the set of all real polynomials of degree n or less. The proofs of Theorems 1 and 2 use only properties of Lagrange interpolation while that of Theorem 3 employs results on the location of interpolation points in Chebyshev approximation.

1. INTRODUCTION

In this paper, we obtain sufficient conditions on a real-valued function f, continuous on $[0, +\infty)$, to insure that, for some nonnegative integer n, there is a nonnegative number $r(n)$ so that for any $r \geqslant r(n)$, the polynomial of best approximation to f on $[0, r]$ from π_n is increasing and nonnegative on $[r, +\infty)$. (Here, π_n denotes the set of all real polynomials of degree n or less.)

The interest in this problem stems from a method of proof used in the study of rational Chebyshev approximation to reciprocals of entire functions as in [2], [3], and [4]. In fact, the proof of Theorem 5 in [3] uses the special

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case, Corollary 1, of our Theorem 1. Namely, f is assumed there to be an entire function all of whose Maclaurin coefficients are nonnegative.

However, we also feel that this is an interesting problem in its own right. The proofs of Theorems 1 and 2 use only properties of Lagrange interpolation while Theorem 3 uses results of Rowland [5] and [6] on the location of the interpolation points in Chebyshev approximation.

2. Construction and Main Results

We begin by stating two lemmas whose proofs are elementary. The proof of Lemma 1 follows easily from a finite Taylor series representation for f with remainder.

Lemma 1. *Assume* $f \in C^m[0, +\infty)$, $m \geqslant 0$, *satisfies* $f^{(m)}(x) \geqslant B_m > 0$ *for all* $x \geqslant 0$. *Then, for each integer* j *with* $0 \leqslant j \leqslant m$, *the quantity* $r_j(f) \equiv \inf\{t \geqslant 0: f^{(j)}(x) > 0$ *on* $(t, +\infty)\}$ *is finite.*

The proof of Lemma 2 below follows easily from expanding the interpolating polynomial in Newton interpolation form, and using known properties of divided differences.

Lemma 2. *Assume* $f \in C^m[0, +\infty)$, $m \geqslant 0$. *Then, the following are equivalent*:

(i) $f^{(m)}(x) \geqslant B_m > 0$ *for all* $x \geqslant 0$;

(ii) *any polynomial* $p_m \in \pi_m$ *which interpolates* f *in any* $m + 1$ *finite points (counting multiplicities) of* $[0, +\infty)$ *satisfies* $p_m^{(m)}(x) \geqslant B_m > 0$ *for all* x.

We remark that if p_m interpolates $f \in C^m[0, +\infty)$ in a finite point $x_0 \geqslant 0$ with multiplicity k, $1 \leqslant k \leqslant m + 1$, then, as is customary, this means that $(f - p_m)^{(i)}(x_0) = 0$ for all $0 \leqslant i \leqslant k - 1$.

Next, given $f \in C^m[0, +\infty)$ with $f^{(m)}(x) \geqslant B_m > 0$ for all $x \in [0, +\infty)$, let the finite numbers $r_j(f)$, $0 \leqslant j \leqslant m$, be defined as in Lemma 1, and define the nonnegative quantity C_m by

$$C_m \equiv - \min_{0 \leqslant j \leqslant m} [\min_{x \geqslant 0} f^{(j)}(x); 0] \geqslant 0. \tag{1}$$

Note that if all $r_j(f)$, $0 \leqslant j \leqslant m$, are zero, so is C_m. We then consider the particular polynomials $H_{j,m} \in \pi_j$, defined for $0 \leqslant j \leqslant m$ by

$$H_{j,m}(x) \equiv \frac{B_m x^j}{m!} - C_m \sum_{i=0}^{j-1} \frac{x^i}{(m - j + i)!}, \quad j > 0; \quad H_{0,m}(x) \equiv \frac{B_m}{m!}. \tag{2}$$

LEMMA 3. *There exists a least nonnegative real number* $\sigma_{j,m}$ *such that* $H_{j,m}^{(i)}(x) > 0$ *for all* $x \in (\sigma_{j,m}, +\infty)$ *and all* $0 \leqslant i \leqslant j$. *Moreover,* $\sigma_{j,m} = \max_{0 \leqslant i \leqslant j} \sigma_{i,m}$, *and*

$$\sigma_{j,m} \leqslant \begin{cases} \max(m-1; jmC_m/B_m); & C_m > 0 \\ 0 & ; C_m = 0 \end{cases}.$$

Proof. Of course, we can choose $\sigma_{j,m} = 0$ if $C_m = 0$. In particular, we set $\sigma_{0,m} = 0$. Fixing $j > 0$, if $C_m > 0$, then by Descartes' rule of signs, $H_{j,m}(x)$ has exactly one positive zero, which we define as $\sigma_{j,m}$. A classical result of Cauchy (cf. [1, p. 95]) states that all zeros λ of $H_{j,m}$ satisfy $|\lambda| \leqslant \sigma_{j,m}$. But, if the zeros λ of $H_{j,m}$ all lie in the disk $\{z: |z| \leqslant \sigma_{j,m}\}$, so does their convex hull. Consequently, by the Gauss–Lucas Theorem (cf. [1, p. 14]), all the zeros μ of any $H_{j,m}^{(i)}$, $i \geqslant 0$, also satisfy $|\mu| \leqslant \sigma_{j,m}$. Thus, $H_{j,m}^{(i)}(x)$ is of one sign on $(\sigma_{j,m}, +\infty)$ and hence positive there for each $0 \leqslant i \leqslant j$.

To obtain the upper bound for $\sigma_{j,m}$ when $C_m > 0$, first note that

$$\frac{x^{j-1}}{(m-1)!} \geqslant \frac{x^{j-i}}{(m-i)!} \qquad \text{for} \quad 1 \leqslant i \leqslant j$$

is valid for any $x \geqslant m-1$. Thus, on adding the above inequalities,

$$\frac{jx^{j-1}}{(m-1)!} \geqslant \sum_{i=0}^{j-1} \frac{x^i}{(m-j+i)!}.$$

Hence, $H_{j,m}(x) \geqslant (B_m x^j/m!) - jC_m x^{j-1}/(m-1)! = (x^{j-1}/m!)[B_m x - jmC_m]$ for any $x \geqslant m-1$ and $C_m > 0$. Since the term in brackets is nonnegative for $x \geqslant jmC_m/B_m$, then $H_{j,m}(x) \geqslant 0$ for all $x \geqslant \max\{m-1, (jmC_m/B_m)\}$. But as $H_{j,m}(x) \geqslant 0$ for $x \geqslant 0$ only if $x \geqslant \sigma_{j,m}$, then

$$\sigma_{j,m} \leqslant \begin{cases} \max(m-1; jmC_m/B_m); & C_m > 0 \\ 0 & ; C_m = 0 \end{cases}.$$

Finally, it follows from (2) that

$$H_{j,m}(x) = xH_{j-1,m}(x) - C_m/(m-j)!.$$

Since $H_{j-1,m}(\sigma_{j-1,m}) = 0$, then $H_{j,m}(\sigma_{j-1,m}) = -C_m/(m-j)! \leqslant 0$, whence $\sigma_{j-1,m} \leqslant \sigma_{j,m}$. Thus, $\sigma_{j,m} = \max_{0 \leqslant i \leqslant j} \sigma_{i,m}$. Q.E.D.

This brings us to

THEOREM 1. *Assume* $f \in C^m[0, +\infty)$, $m \geqslant 0$, *satisfies* $f^{(m)}(x) \geqslant B_m > 0$ *for all* $x \geqslant 0$, *and set* $r(m) = r_m^* + \sigma_{m,m}$, *where* $r_m^* = \max_{0 \leqslant j \leqslant m}(r_j(f))$, *the quantities* $r_j(f)$ *and* $\sigma_{m,m}$ *being defined respectively in Lemmas 1 and 3.*

Then, for any $r \geqslant r(m)$, *any polynomial* $p_m \in \pi_m$ *which interpolates* f *in* $m + 1$ *points (counting multiplicities) of* $[0, r]$ *satisfies*

$$p_m^{(i)}(x) > 0 \qquad \text{for all} \quad x > r, \quad \text{all} \quad 0 \leqslant i \leqslant m. \tag{3}$$

Moreover, the quantity $r(m)$ *can be bounded above by*

$$r(m) \leqslant r_m{}^* + \begin{Bmatrix} \max(m-1;\, m^2 C_m/B_m); & C_m > 0 \\ 0 & ; C_m = 0 \end{Bmatrix}. \tag{4}$$

Before proving Theorem 1, several comments can be made. First, for *any* $r \geqslant 0$, if p_m interpolates f in $m + 1$ points of $[0, r]$, it follows from Lemma 2 that $p_m^{(m)} \geqslant B_m > 0$. Thus, applying Lemma 1 to p_m gives us the existence of the least nonnegative real numbers $r_j(p_m) \geqslant 0$ such that $p_m^{(j)}(x) > 0$ on $(r_j(p_m), +\infty)$ for each $0 \leqslant j \leqslant m$. Hence, $p_m^{(i)}(x) > 0$ on $(R_m, +\infty)$ for all $0 \leqslant i \leqslant m$, where $R_m \equiv \max_{0 \leqslant j \leqslant m} r_j(p_m)$, but it is in general possible that $R_m > r$. If $R_m > r$, then the desired strictly increasing nature (3) of the interpolant p_m takes place *not* immediately to the right of the interpolation interval $[0, r]$, but on $(R_m, +\infty)$. Thus, the major point in Theorem 1 is that it is possible to select $r \geqslant 0$ sufficiently large such that $R_m \leqslant r$, regardless of the interpolation points in $[0, r]$. Another point of Theorem 1 is that an upper bound for $r(m)$ is given in (4).

Proof. For $r \geqslant r(m)$, consider any $p_m \in \pi_m$ which interpolates f in any $m + 1$ points (counting multiplicities) in $[0, r]$, and label these interpolation points

$$0 \leqslant x_m \leqslant x_{m-1} \leqslant \cdots \leqslant x_1 \leqslant x_0 \leqslant r.$$

Using the convention throughout that $\prod_{i=0}^{k} (x - x_i) = 1$, whenever $k < 0$, we first express p_m in Newton interpolation series form, i.e.,

$$p_m(x) = \sum_{j=0}^{m} a_j \prod_{i=0}^{j-1} (x - x_i) \qquad \text{with} \quad a_j \equiv f[x_0, x_1, \ldots, x_j], \tag{5}$$

where $f[x_0, x_1, \ldots, x_j]$, $j > 1$, is the divided difference of f in the points $x_0, x_1, \ldots, x_j$, and $f[x_0] \equiv f(x_0)$. For any $1 \leqslant j \leqslant m$, it is well-known that $f[x_0, x_1, \ldots, x_j] = f^{(j)}(\xi_j)/j!$, where $x_j < \xi_j < x_0$ if $x_j < x_0$, and $\xi_j = x_0$ if $x_j = x_0$. Next, for each i with $0 \leqslant i \leqslant m$, we analyze the $i + 1$ "right-most" terms of p_m, by defining

$$p_{i,m}(x) = a_m \prod_{j=1}^{i} (x - x_{m-j}) + a_{m-1} \prod_{j=2}^{i} (x - x_{m-j}) + \cdots + a_{m-i}, \tag{6}$$

so that $p_{i,m} \in \pi_i$ for each $0 \leqslant i \leqslant m$, and also $p_{m,m}(x) \equiv p_m(x)$ (cf. (5)). In addition, it follows from (6) that, for $m > 0$,

$$p_{i,m}(x) = p_{i-1,m}(x) \cdot (x - x_{m-i}) + a_{m-i}, \qquad 1 \leqslant i \leqslant m, \tag{7}$$

so that (6) and (7) in essence represent a Horner-like method for recursively defining $p_m(x)$.

By hypothesis, $p_{0,m}(x) = a_m = f[x_0, x_1, ..., x_m] \geqslant f^{(m)}(\xi_m)/m! \geqslant B_m/m!$. Thus, from the definition in (2), then

$$p_{0,m}(x) \geqslant H_{0,m}(x) = B_m/m! > 0 \qquad \text{for all} \quad x > r. \tag{8}$$

Next, for $m > 0$, consider $p_{1,m}(x) = p_{0,m}(x) \cdot (x - x_{m-1}) + a_{m-1}$, where $a_{m-1} = f[x_0, x_1, ..., x_{m-1}] = f^{(m-1)}(\xi_{m-1})/(m-1)!$, with $x_{m-1} \leqslant \xi_{m-1} \leqslant x_0$. Two cases arise.

Case 1. $r_m^* \leqslant x_{m-1}$.

In this case, it follows from the definition of r_m^* in the statement of Theorem 1 that $a_{m-1} \geqslant 0$. Thus, using (8), we deduce that $p_{1,m}(x) \geqslant p_{0,m}(x) \cdot (x - x_{m-1}) > 0$ for all $x > r$, and that $p_{1,m}^{(1)}(x) = p_{0,m}(x) > 0$ for all $x > r$, i.e.,

$$p_{1,m}^{(i)}(x) > 0 \qquad \text{for all} \quad x > r \quad \text{and for all} \quad 0 \leqslant i \leqslant 1.$$

Case 2. $x_{m-1} < r_m^*$.

In this case, for $x > r$, then $x - x_{m-1} > r - r_m^* \geqslant \sigma_{m,m}$. Next, if $\xi_{m-1} \geqslant r_{m-1}(f)$, then $a_{m-1} \geqslant 0$ and we can surely bound a_{m-1} below by $a_{m-1} \geqslant -C_m/(m-1)!$. If $\xi_{m-1} < r_{m-1}(f)$, the previous inequality still holds from the definition of C_m. Thus, from (8),

$$\begin{aligned} p_{1,m}(x) &= p_{0,m}(x) \cdot (x - x_{m-1}) + a_{m-1} \geqslant (B_m \sigma_{m,m})/m! + a_{m-1} \\ &\geqslant (B_m \sigma_{m,m})/m! - C_m/(m-1)! \equiv H_{1,m}(\sigma_{m,m}) > 0, \end{aligned}$$

the last inequality following from the fact that $\sigma_{1,m} \leqslant \sigma_{m,m}$ (cf. Lemma 3), and the definition of $\sigma_{1,m}$. Upon differentiating, we further see, as in the previous case, that

$$p_{1,m}^{(i)}(x) > 0 \qquad \text{for all} \quad x > r \quad \text{and} \quad 0 \leqslant i \leqslant 1.$$

The inductive step is now clear. Assuming for $m > 0$ that

$$p_{j-1,m}^{(i)}(x) > 0 \qquad \text{for all} \quad x > r \quad \text{and all} \quad 0 \leqslant i \leqslant j-1, \tag{9}$$

and that

$$p_{j-1,m}(x) \geqslant H_{j-1,m}(\sigma_{m,m}) \qquad \text{for all} \quad x > r \quad \text{if} \quad x_{m-(j-1)} < r_m^*, \tag{10}$$

we then consider $p_{j,m}(x) = p_{j-1,m}(x) \cdot (x - x_{m-j}) + a_{m-j}$. Recalling that $a_{m-j} = f[x_0, x_1, ..., x_{m-j}] = f^{(m-j)}(\xi_{m-j})/(m-j)!$, two cases similarly arise.

Case 1. $r_m^* \leqslant x_{m-j}$.

In this case, it again follows from the definition of r_m^* that $a_{m-j} \geqslant 0$. Hence, $p_{j,m}(x) \geqslant p_{j-1,m}(x) \cdot (x - x_{m-j}) > 0$ for all $x > r$ from $i = 0$ of (9). Moreover, since in general

$$p_{j,m}^{(i)}(x) = p_{j-1,m}^{(i)}(x) \cdot (x - x_{m-j}) + i p_{j-1,m}^{(i-1)}(x) \qquad \text{for any} \quad i \geqslant 1, \quad (11)$$

it follows from the inductive hypothesis of (9) that

$$p_{j,m}^{(i)}(x) > 0 \qquad \text{for all} \quad x > r \quad \text{and all} \quad 0 \leqslant i \leqslant j.$$

Case 2. $x_{m-j} < r_m^*$.

In this case, for $x > r$ then $x - x_{m-j} > r - r_m^* \geqslant \sigma_{m,m}$. With the inductive hypothesis of (10), then for $x > r$,

$$\begin{aligned} p_{j,m}(x) &= p_{j-1,m}(x) \cdot (x - x_{m-j}) + a_{m-j} \\ &\geqslant H_{j-1,m}(\sigma_{m,m}) \cdot \sigma_{m,m} - C_m/(m-j)! \end{aligned}$$

but, as this last sum of two terms is, from (2), just $H_{j,m}(\sigma_{m,m})$, then

$$p_{j,m}(x) \geqslant H_{j,m}(\sigma_{m,m}) > 0 \qquad \text{for all} \quad x > r \qquad \text{if} \quad x_{m-j} < r_m^*.$$

Again, using (11) and the inductive hypothesis of (9), we see that

$$p_{j,m}^{(i)}(x) > 0 \qquad \text{for all} \quad x > r \quad \text{and all} \quad 0 \leqslant i \leqslant j.$$

This completes the induction. Thus, since $p_m(x) \equiv p_{m,m}(x)$, then

$$p_m^{(i)}(x) > 0 \qquad \text{for all} \quad x > r \quad \text{and all} \quad 0 \leqslant i \leqslant m. \qquad \text{Q.E.D.}$$

Of course, if $f \in C^m[0, +\infty)$ satisfies the hypothesis of Theorem 1 and moreover has $f^{(j)}(0) \geqslant 0$ for all $0 \leqslant j \leqslant m$, then $r(m)$ can be chosen to be zero in Theorem 1. This gives us the following.

COROLLARY 1. *Assume* $f \in C^m[0, +\infty)$, $m \geqslant 0$, *that* $f^{(m)}(x) \geqslant B_m > 0$ *for all* $x \geqslant 0$, *and that* $f^{(j)}(0) \geqslant 0$ *for all* $0 \leqslant j \leqslant m$. *Then, for any* $r \geqslant 0$, *any polynomial* $p_m \in \pi_m$ *which interpolates* f *in any* $m + 1$ *points* (*counting multiplicities*) *of* $[0, r]$ *satisfies* (3).

Next, given $f \in C^\infty[0, +\infty)$, suppose that there exists a strictly increasing sequence $\{n_k\}_{k=0}^\infty$ of nonnegative integers such that

for each nonnegative integer k, there exists a real number $B_{n_k} > 0$ such that $f^{(n_k)}(x) \geqslant B_{n_k} > 0$ for all $x \geqslant 0$. (12)

Clearly, we can apply Theorem 1 to f for each $m = n_k$, and thus, we deduce that, for each $k \geqslant 0$, there is a real number $r(n_k)$ such that for any $r \geqslant r(n_k)$, any polynomial $p_{n_k} \in \pi_{n_k}$ which interpolates f in any $n_k + 1$ points in $[0, r]$, satisfies

$$p_{n_k}^{(i)}(x) > 0 \qquad \text{for all} \quad x > r \quad \text{and all} \quad 0 \leqslant i \leqslant n_k. \tag{13}$$

Actually, what we are interested in is the manner in which the numbers $r(n_k)$ increase with k. Of course, Theorem 1 directly provides an upper bound for each $r(n_k)$, but, as we now show, the existence of lower derivatives strictly positive on $[0, \infty)$ allows us to sharpen the upper bound (4) of Theorem 1 for $r(n_k)$. Specifically, in analogy with (1) and (2), we introduce the notation

$$C_{n_s} \equiv - \min_{n_{s-1}+1 \leqslant j \leqslant n_s} [\min_{x \geqslant 0} f^{(j)}(x); 0] \geqslant 0, \qquad s = 0, 1, \ldots, \tag{14}$$

$$L_s \equiv n_s - (n_{s-1} + 1) \geqslant 0, \qquad s = 0, 1, \ldots, \tag{15}$$

and, for $0 \leqslant j \leqslant L_s$,

$$\begin{aligned} H_{j,n_s}(x) &\equiv \frac{B_{n_s} x^j}{(n_s)!} - C_{n_s} \sum_{i=0}^{j-1} \frac{x^i}{(n_s - j + i)!}, \qquad j > 0; \\ H_{0,n_s}(x) &\equiv \frac{B_{n_s}}{(n_s)!}, \end{aligned} \tag{16}$$

where we set $n_{-1} \equiv -1$.

We then have the following.

THEOREM 2. *Let f be a real-valued function in $C^\infty[0, +\infty)$, and let $\{n_k\}_{k=0}^{\infty}$ be a strictly increasing sequence of nonnegative integers for which* (12) *is valid. Then, for each nonnegative integer k, there is a real number $r(n_k)$ such that for any $r \geqslant r(n_k)$, any polynomial $p_{n_k} \in \pi_{n_k}$ which interpolates f in any $n_k + 1$ points* (*counting multiplicities*) *of* $[0, r]$ *satisfies*

$$p_{n_k}^{(i)}(x) > 0 \qquad \textit{for all} \quad x > r \quad \textit{and all} \quad 0 \leqslant i \leqslant n_k.$$

Moreover, with $r_{n_k}^* \equiv \max_{0 \leqslant j \leqslant n_k} (r_j(f))$, the quantities $r_j(f)$ being defined in Lemma 1, *then*

$$r(n_k) \leqslant r_{n_k}^* + \max_{0 \leqslant s \leqslant k} \begin{cases} n_s \max(1; L_s C_{n_s}/B_{n_s}); & C_{n_s} > 0 \\ 0 & ; C_{n_s} = 0 \end{cases}. \tag{17}$$

Proof. As previously mentioned, the existence of such an $r(n_k)$ is guaranteed by Theorem 1. It remains then to establish the upper bound for $r(n_k)$ in (17).

For any $r \geqslant r(n_k)$, consider any $p_{n_k} \in \pi_{n_k}$ which interpolates f in any $n_k + 1$ points (counting multiplicities) of $[0, r]$. As in the proof of Theorem 1, label these points

$$0 \leqslant x_{n_k} \leqslant x_{n_k-1} \leqslant \cdots \leqslant x_0 \leqslant r.$$

Expressing p_{n_k} in Newton interpolation series form,

$$p_{n_k}(x) = \sum_{j=0}^{n_k} a_j \prod_{i=0}^{j-1} (x - x_i) \qquad \text{with} \quad a_j \equiv f[x_0, x_1, ..., x_j],$$

we then write p_{n_k} in the form

$$p_{n_k}(x) = \sum_{s=0}^{k} Q_s(x), \qquad \text{where} \quad Q_s(x) \equiv \sum_{j=n_{s-1}+1}^{n_s} a_j \prod_{i=0}^{j-1} (x - x_i),$$

and where $n_{-1} \equiv -1$. Note that $Q_s \in \pi_{n_s}$. Then, writing Q_s as

$$Q_s(x) = \prod_{j=0}^{n_{s-1}} (x - x_j) \cdot R_s(x), \qquad \text{where} \quad R_s(x) \equiv \sum_{j=n_{s-1}+1}^{n_s} a_j \prod_{i=n_{s-1}+1}^{j-1} (x - x_i), \tag{18}$$

we note that $(\prod_{j=0}^{n_{s-1}} (x - x_j))^{(i)} > 0$ on $(r, +\infty)$ for all $0 \leqslant i \leqslant n_{s-1} + 1$. Next, with the definition of L_s in (15), it follows that $R_s \in \pi_{L_s}$, and hence, it suffices to show that $R_s^{(i)}(x) > 0$ for all $x \in (r, \infty)$ and for all $0 \leqslant i \leqslant L_s$, for, using the chain rule for differentiation of a product, this will imply that the product Q_s, as given in (18), then satisfies $Q_s^{(i)}(x) > 0$ for all $x > r$ and all $0 \leqslant i \leqslant n_s$.

To show that $R_s^{(i)}(x) > 0$ for all $x > r$ and for all $0 \leqslant i \leqslant L_s$, we now simply modify the proof of Theorem 1. Let σ_{j,n_s}, as guaranteed by Lemma 3, be the least nonnegative real number such that $H_{j,n_s}^{(i)}(x) > 0$ for all $x > \sigma_{j,n_s}$ and for all $0 \leqslant i \leqslant j$, where H_{j,n_s} is defined in (16). The proof of Lemma 3 easily shows that

$$\sigma_{j,n_s} \leqslant \begin{cases} n_s \max(1; jC_{n_s}/B_{n_s}); & C_{n_s} > 0 \\ 0 & ; C_{n_s} = 0 \end{cases},$$

and that $\sigma_{j,n_s} = \max_{0 \leqslant i \leqslant j} \sigma_{i,n_s}$. Thus,

$$\sigma_{L_s,n_s} = \max_{0 \leqslant i \leqslant L_s} \sigma_{i,n_s} \leqslant \begin{cases} n_s \max(1; L_s C_{n_s}/B_{n_s}); & C_{n_s} > 0 \\ 0 & ; C_{n_s} = 0 \end{cases}. \tag{19}$$

Then, directly applying the inductive proof of Theorem 1 to R_s, rather than to p_{n_s}, shows that we can set $r(n_k) = r_{n_k}^* + \max_{0 \leqslant s \leqslant k} (\sigma_{L_s}, n_s)$, and the desired result of (17) then follows from (19). Q.E.D.

3. Chebyshev Approximation

As mentioned earlier, if f satisfies the assumptions of Theorem 1, then the polynomial $p_m^* \in \pi_m$ of best Chebyshev approximation to f on $[0, r]$ from π_m surely satisfies the conclusion (3) of Theorem 1, since p_m^* interpolates f in at least $m + 1$ distinct points of $[0, r]$. In contrast with Theorem 1 which only uses interpolation by polynomials, the next result makes specific use of polynomials of best approximation.

Theorem 3. *Assume $f \in C^{m+1}[0, +\infty)$, $m \geqslant 0$, is such that $f^{(m+1)}(x)$ is positive and strictly increasing on $(0, +\infty)$, and set*

$$\tilde{r}(m) = 2r_m^* \Big/ \Big\{1 - \cos\Big(\frac{\pi}{2m+2}\Big)\Big\}, \tag{20}$$

where $r_m^* \equiv \max_{0 \leqslant j \leqslant m} (r_j(f))$, the quantities $r_j(f)$ being defined in Lemma 1. Then, for any $r \geqslant \tilde{r}(m)$, the polynomial $p_m^* \in \pi_m$ of best approximation to f on $[0, r]$ from π_m satisfies

$$p_m^{*(i)}(x) > 0 \qquad \text{for all} \quad x > r \quad \text{and all} \quad 0 \leqslant i \leqslant m. \tag{21}$$

Proof. Let $p_m^* \in \pi_m$ be the polynomial of best Chebyshev approximation to f on $[0, r]$ from π_m. Now, p_m^* interpolates f in at least $m + 1$ distinct points of $[0, r]$. But, the hypothesis that $f^{(m+1)}(x) > 0$ on $(0, \infty)$ gives, from Rowland [5, Corollary 2.3 and Theorem 2.2], that p_m^* interpolates f in *precisely* $m + 1$ distinct points of $[0, r]$. We label these points $\{x_j(r)\}_{k=0}^{m}$ where

$$0 < x_0(r) < x_1(r) < \cdots < x_m(r) \leqslant r.$$

Next, with the result of Rowland [6, Theorem 3.3], it directly follows, because $f^{(m+1)}(x)$ is by hypothesis positive and strictly increasing on $(0, r)$, that

$$w_k(r) \equiv \frac{r}{2}\Big\{1 + \cos\Big(\frac{(m + \frac{1}{2} - k)\pi}{m+1}\Big)\Big\} < x_k(r) \qquad \text{for all} \quad 0 \leqslant k \leqslant m.$$

In particular, the case $k = 0$ above gives

$$w_0(r) = \frac{r}{2}\Big\{1 - \cos\Big(\frac{\pi}{2m+2}\Big)\Big\} < x_k(r) \qquad \text{for all} \quad 0 \leqslant k \leqslant m.$$

With $r_m^* \equiv \max_{0 \leqslant j \leqslant m} (r_j(f))$, then $w_0(r) \geqslant r_m^*$ is equivalent to $r \geqslant 2r_m^*/\{1 - \cos(\pi/(2m + 2))\} \equiv \tilde{r}(m)$ from (20). Hence, for any $r \geqslant \tilde{r}(m)$,

$$r_m^* \leqslant w_0(r) < x_k(r) \qquad \text{for all} \quad 0 \leqslant k \leqslant m. \tag{22}$$

Now, if we express, as before, $p_m{}^*$ in Newton interpolation series form, i.e.,

$$p_m{}^*(x) = \sum_{j=0}^{m} a_j \prod_{i=0}^{j-1} (x - x_i(r)) \qquad \text{with} \quad a_j \equiv f[x_0(r),..., x_j(r)], \tag{23}$$

then $a_j = f^{(j)}(\xi_j)/j!$ with $x_0(r) < \xi_j < x_j(r)$ for all $1 \leqslant j \leqslant m$, and $a_0 = f(x_0(r))$. Hence, for any $r \geqslant \tilde{r}(m)$, it follows from (22) and the definition of $r_m{}^*$ that $a_j > 0$ for all $0 \leqslant j \leqslant m$. Thus, differentiating the expression for $p_m{}^*$ in (23) then yields

$$p_m^{*(i)}(x) > 0 \qquad \text{for all} \quad x > r \quad \text{and all} \quad 0 \leqslant i \leqslant m. \qquad \text{Q.E.D.}$$

4. Examples and Remarks

As in the transition from Theorem 1 to Theorem 2, we can apply Theorem 3 to the case where $f \in C^\infty[0, +\infty)$, and where $\{n_k\}_{k=0}^\infty$ is a strictly increasing sequence of nonnegative integers for which $f^{(n_k+1)}(x)$ is positive and strictly increasing on $(0, +\infty)$ for each $k = 0, 1,...$. If we further assume that f is such that $\sup_{j\geqslant 0} (r_j(f)) \leqslant R < \infty$, it follows from Theorem 3 that, for each $k \geqslant 0$, the polynomial $p_{n_k}^* \in \pi_{n_k}$ of best approximation to f on $[0, r]$ from π_{n_k} satisfies

$$p_{n_k}^{*(i)}(x) > 0 \qquad \text{for all} \quad x > r \quad \text{and all} \quad 0 \leqslant i \leqslant n_k$$

where $r \geqslant 2R/\{1 - \cos(\pi/(2n_k + 2))\} \geqslant \tilde{r}(n_k)$. In this situation, these upper bounds for $\tilde{r}(n_k)$ are $\mathcal{O}(n_k{}^2)$ as $k \to \infty$.

Similarly, if $f \in C^\infty[0, +\infty)$ is such that $\sup_{j\geqslant 0} (r_j(f)) \leqslant R < \infty$, and if $\sup_{k\geqslant 0} (L_k C_{n_k}/B_{n_k}) \leqslant D < \infty$, $D \geqslant 1$, then it follows from Theorem 2 that, for each $k \geqslant 0$, any polynomial $p_{n_k} \in \pi_{n_k}$ which interpolates f in any $n_k + 1$ points (counting multiplicities) of $[0, r]$ satisfies

$$p_{n_k}^{(i)}(x) > 0 \qquad \text{for all} \quad x > r \quad \text{and all} \quad 0 \leqslant i \leqslant n_k\,,$$

where $r \geqslant R + Dn_k \geqslant r(n_k)$. Thus, in this case, these upper bounds for $r(n_k)$ are $\mathcal{O}(n_k)$ as $k \to \infty$.

We now apply the above remarks to the specific function

$$f(x) = e^x + e^\alpha \cos x, \qquad \alpha < \ln(\sqrt{2}) + (3\pi/4) \doteqdot 2.703.$$

On differentiating f, one sees that Theorem 2 is applicable for the sequence $n_0 = 0$, $n_1 = 3$, $n_2 = 4$, $n_3 = 7$, $n_4 = 8$, etc. On the other hand, Theorem 3 will apply with the sequence $n_0 = 2$, $n_1 = 6$, $n_2 = 10$, etc. Thus, Theorem 2 and 3 are independent and interlace here.

REFERENCES

1. M. MARDEN, "The Geometry of the Zeros of a Polynomial in a Complex Variable," American Mathematical Society, Providence, Rhode Island, 1949.
2. G. MEINARDUS AND R. S. VARGA, Chebyshev rational approximation to certain entire functions in $[0, +\infty)$, *J. Approximation Theory* **3** (1970), 300–309.
3. G. MEINARDUS, A. R. REDDY, G. D. TAYLOR, AND R. S. VARGA, Converse theorems and extensions in Chebyshev rational approximation to certain entire functions in $[0, +\infty)$, *Trans. Amer. Math. Soc.* **170** (1972), 171–186.
4. J. A. ROULIER AND G. D. TAYLOR, "Rational Chebyshev approximation on $[0, +\infty)$, *J. Approximation Theory* **11** (1974), 208–215.
5. J. H. ROWLAND, On the structure of the error curve in Chebyshev approximation by polynomials, *SIAM J. Numer. Anal.* **4** (1967), 398–405.
6. J. H. ROWLAND, On the location of the deviation points in Chebyshev approximation by polynomials, *SIAM J. Numer. Anal.* **6** (1969), 118–126.

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